Algebroids
– A playground for Geometric Mechanics

Bieszczady mountains, on our way to
Krzemieniec/Кременець/Kremenec/Kremenáros

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INTRODUCTION

HAMILTONIAN MECHANICS

- ASSOCIATED WITH A PHASE SPACE $\mathcal{P}$ WHICH IS SYMPLECTIC $(\mathcal{P}, \omega)$ OR POISSON $(\mathcal{P}, \Lambda)$ MANIFOLD

- ω OR Λ ARE USED TO PRODUCE A VECTOR FIELD FROM A HAMILTONIAN FUNCTION $\mathcal{H} : \mathcal{P} \longrightarrow \mathbb{R}$

SYMPLECTIC $\quad X_H \lrcorner\, \omega = d\mathcal{H}$

POISSON $\quad d\mathcal{H} \lrcorner\, \Lambda = X_H$

IN DARBOUX COORDINATES $\quad \dot{q}^i = \dfrac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}^j = -\dfrac{\partial \mathcal{H}}{\partial q^j}$

USING POISSON BRACKET $\quad \dot{f} = \{f, \mathcal{H}\}$

LAGRANGIAN MECHANICS

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LAGRANGIAN MECHANICS

- „LIVES ON" THE TANGENT BUNDLE TM
- LAGRANGIAN FUNCTION $\mathcal{L}: TM \longrightarrow \mathbb{R}$ + VARIATIONAL CALCULUS GIVES EULER-LAGRANGE EQUATIONS AND DEFINITION OF MOMENTA

OR

- LAGRANGIAN FUNCITION + TULCZYJEW MAP $\alpha_M : TT^*M \longrightarrow T^*TM$ GIVES PHASE EQUATIONS ON T^*M

QUESTION: WHAT GEOMETRIC STRUCTURE IS „RESPONSIBLE" FOR LAGRANGIAN MECHANICS ? $\longrightarrow$ ALGEBROID

INTRODUCTION

HAMILTONIAN MECHANICS

- ASSOCIATED WITH A PHASE SPACE $\mathcal{P}$ WHIC[H] IS SYMPLECTIC $(\mathcal{P}, \omega)$ OR POISSON $(P, ...$
- ω ...

VECT...

FUNC...

SYMP...

POIS...

IN DARBOUX COORDINATES $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $\dot{p}^j = -\frac{\partial H}{\partial q^j}$

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LAGRANGIAN MECHANICS

- „LIVES ON" THE TANGENT BUNDLE TM

...$TM \longrightarrow \mathbb{R}$

...S EULER-

...FINITION

...RZYJEW MAP

GIVES PHASE EQUATIONS ON T^*M

QUESTION: WHAT GEOMETRIC STRUCTURE IS „RESPONSIBLE" FOR LAGRANGIAN MECHANICS? $\longrightarrow$ ALGEBROID

WHY?

QUANTUM MECHANICS IS USUALLY FORMULATED
USING A CONCEPT OF A HAMILTONIAN,
BUT FIELD THEORIES - CLASSICAL, QUANTUM, σ-MODELS ...
ARE BASED ON LAGRANGIAN FUNCTONS / FORMS

TULCZYJEW TRIPLE

M – CONFIGURATION MANIFOLD

TM – POSITIONS & VELOCITIES T^*M – POSITIONS & MOMENTA – PHASE SPACE

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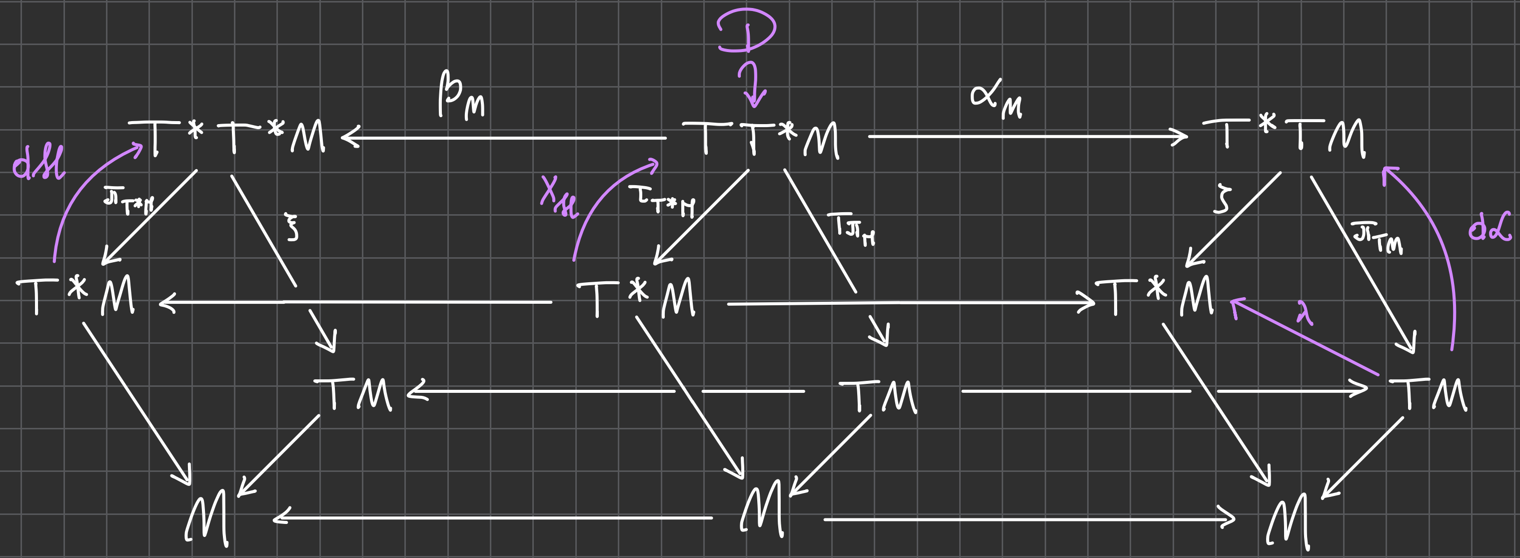


$$\mathcal{D} = X_{\mathcal{H}}(T^*M) = \beta_M^{-1}\left(d\mathcal{H}(T^*M)\right) \qquad \mathcal{D} = \alpha_M^{-1}\left(d\mathcal{L}(TM)\right)$$

BOTH CAN BE GENERALIZED

TULCZYJEW TRIPLE

M – CONFIGURATION MANIFOLD

TM – POSITIONS & VELOCITIES T^*M – POSITIONS & MOMENTA – PHASE SPACE

$$
\begin{tikzcd}
 & & & & \mathcal{D} \arrow[d] & & & & \\
 & T^*T^*M \arrow[dl, "\pi_{T^*M}"'] \arrow[ddr, "\xi"] & & & TT^*M \arrow[lll, "\beta_M"'] \arrow[rrr, "\alpha_M"] \arrow[dl, "\tau_{T^*M}"'] \arrow[ddr, "T\pi_M"] & & & T^*TM \arrow[dl, "\xi"'] \arrow[ddr, "\pi_{TM}"] & \\
T^*M \arrow[ur, bend left, "d\mathcal{H}"] \arrow[ddr] & & & T^*M \arrow[lll] \arrow[rrr] \arrow[ur, bend left, "X_{\mathcal{H}}"] \arrow[ddr] & & & T^*M \arrow[ddr] & & \\
 & & TM \arrow[dl] & & & TM \arrow[lll] \arrow[rrr] \arrow[dl] & & & TM \arrow[dl] \arrow[ull, "\lambda"'] \arrow[uul, bend right, "d\mathcal{L}"'] \\
 & M & & & M \arrow[lll] \arrow[rrr] & & & M &
\end{tikzcd}
$$

$$\mathcal{D} = X_{\mathcal{H}}(T^*M) = \beta_M^{-1}\left(d\mathcal{H}(T^*M)\right) \qquad \mathcal{D} = \alpha_M^{-1}\left(d\mathcal{L}(TM)\right)$$

BOTH CAN BE GENERALIZED

SOME LAGRANGIAN SUBMANIFOLD OF T^*TM

$$\mathcal{D} = \beta_M^{-1}(S_{\mathcal{H}}) \qquad \mathcal{D} = \alpha_M^{-1}(S_{\mathcal{L}})$$

SOME LAGRANGIAN SUBMANIFOLD OF T^*T^*M

EXAMPLES – MASSIVE AND MASSLESS RELATIVISTIC PARTICLES

$(+,-,-,-)$

IN THE FOLLOWING WE ASSUME THAT M IS AN AFFINE MINKOWSKI SPACE WITH CONSTANT METRIC η. ONE MAY WORK AS WELL WITH THE SPACE-TIME OF GENERAL RELATIVITY AS LONG AS WE TREAT PARTICLES AS TEST PARTICLES I.E. NOT BEING SOURCES OF THE GRAVITATIONAL FIELD.

$M \ni q \qquad TM = M\times V \qquad T^*M = M\times V^* \qquad \tilde{\eta}: V \longrightarrow V^* \qquad \tilde{\eta}(v) = \eta(v,\cdot) \qquad \|v\| = \sqrt{\eta(v,v)}$

$\eta(v,v)>0$

$$\begin{array}{ccccc}
T^*T^*M & \xleftarrow{\ \beta_M\ } & TT^*M & \xrightarrow{\ \alpha_M\ } & T^*TM \\
M\times V^*\times V^*\times V & & M\times V^*\times V\times V^* & & M\times V\times V^*\times V \\
(q,p,-\dot p,\dot q) & \longleftarrow\!\shortmid & (q,p,\dot q,\dot p) & \shortmid\!\longrightarrow & (q,\dot q,\dot p,p)
\end{array}$$

$\mathcal{H}: M\times V^*\times \mathbb{R}_+ \longrightarrow \mathbb{R}$

$\mathcal{H}(q,p,r) = r\left(\|p\|^2 - m^2\right)$

$$D = \{(\overset{\dot q}{\dot q}, p, \overset{\dot q}{r\,\tilde\eta^{-1}(p)}, \overset{\dot p}{0}) : \langle p, \tilde\eta^{-1}(p)\rangle = m^2 \quad r>0\}$$

A FREE PARTICLE OF MASS m

$L: M\times V_+ \longrightarrow \mathbb{R}$

$L(q,\dot q) = m\,\|\dot q\|$

$S_L = dL(M\times V_+)$

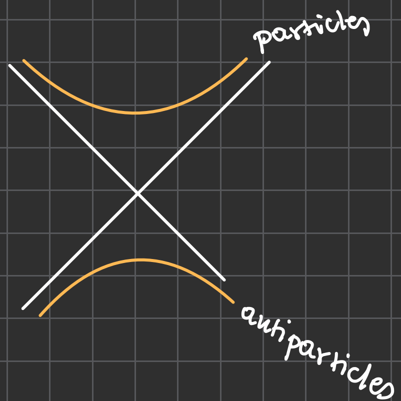
$$\frac{\partial L}{\partial q} = 0 = \dot p$$

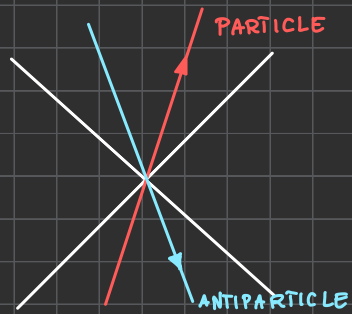
$$\left\{\left(q, \dot q, 0, m\frac{\tilde\eta(\dot q)}{\|\dot q\|}\right)\right\}$$

$\dot q$ TIMELIKE

$$p = \lambda(\dot q) = m\frac{\tilde\eta(\dot q)}{\|\dot q\|}$$

LEGENDRE MAP IS NOT INVERTIBLE





GENERATING FAMILY:

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d] & \mathbb{R} \\ Q & \end{tikzcd} \qquad \begin{tikzcd} (VF)^\circ \subset T^*F \arrow[d, "\rho"] \\ T^*Q \end{tikzcd} \qquad D_h = \rho\left(dh \cap (VF)^\circ\right)$$

+ CONDITIONS FOR h ENSURING THAT D_h IS ACTUALLY A SUBMANIFOLD

GENERATING FAMILY IN MECHANICS

$$\mathcal{H} : T^*M \times_M TM \longrightarrow \mathbb{R} \qquad \mathcal{H}(q,\dot q, p) = \langle p, \dot q\rangle - L(q,\dot q)$$

SOMETIMES GENERATING FAMILY CAN BE SIMPLIFIED

$$L(q,\dot q) = m\,\|\dot q\|$$

$$\mathcal{H}(q,p,\dot q) = \langle p,\dot q\rangle - L(q,\dot q) = \langle p,\dot q\rangle - m\,\|\dot q\|$$

$$\frac{\partial \mathcal{H}}{\partial \dot q} = 0 = p - \frac{m}{\|\dot q\|}\tilde\eta(\dot q) = p - m\,\tilde\eta\left(\frac{\dot q}{\|\dot q\|}\right) \implies \dot q = r\,\tilde\eta^{-1}(p)$$

VELOCITY IS PROPORTIONAL TO $\tilde\eta^{-1}(p)$ WITH POSITIVE COEFFICIENT!

$$\mathcal{H}(q,p,r) = \langle p, r\,\tilde\eta^{-1}(p)\rangle - m\sqrt{r^2 \underbrace{\langle p,\tilde\eta^{-1}(p)\rangle}_{m^2}} =$$

$$= r\,\|p\|^2 - r\,m^2 =$$

$$= r\left(\|p\|^2 - m^2\right)$$

MASSLESS PARTICLE

$$
\begin{array}{ccccc}
T^*T^*M & \xleftarrow{\ \beta_M\ } & TT^*M & \xrightarrow{\ \alpha_M\ } & T^*TM \\
M\times V^*\times V^*\times V & & M\times V^*\times V\times V^* & & M\times V\times V^*\times V \\
(q,p,-\dot p,\dot q) & \longleftarrow\!\shortmid & (q,p,\dot q,\dot p) & \shortmid\!\longrightarrow & (q,\dot q,\dot p,p)
\end{array}
$$

WORLD LINE OF A MASSLES PARTICLE
LIES ON THE LIGHT CONE

$$D=\{(q,p,\dot q,\dot p):\ \|p\|=0,\ \dot p=0\ \ \dot q = r\tilde\eta^{-1}(p)\}$$

NON-ZERO VECTORS

$$H: M\times (V^*)^\times\times \mathbb{R}_+ \longrightarrow \mathbb{R}$$

$$H(q,\dot q,y)=\tfrac{1}{2}y\,\eta(p,p)$$

$$L: M\times V^\times\times \mathbb{R}_+ \longrightarrow \mathbb{R}$$

$$L(q,\dot q,y)=\frac{1}{2y}\eta(\dot q,\dot q)$$

THESE TWO GENERATING FAMILIES ARE
RELATED BY THE LEGENDRE TRANSFORMATION

W.M. Tulczyjew, P. Urbański

ACTA PHYSICA POLONICA B

A Slow and Careful Legendre Transformation for Singular Lagrangians

vol. 30, p. 2909 (70 pages)

abstract • links/reference

Vol. **30** (1999), No. 10, pp. 2853 – 3027

LIE ALGEBROID

THERE EXISTS SEVERAL EQUIVALENT DEFINITIONS OF A LIE ALGEBROID ON A VECTOR BUNDLE $\tau : E \longrightarrow M$

① $[\cdot,\cdot] : Sec(\tau) \times Sec(\tau) \longrightarrow Sec(\tau)$,

$$\begin{array}{ccc} E & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M \end{array}$$

VB MORPHISM

ANTISYMMETRIC $[X,Y] = -[Y,X]$,

JACOBI IDENTITY $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$,

$[X, fY] = f[X,Y] + \rho(X)(f)Y$, $f \in C^\infty(M)$.

EXAMPLES: $\left(\begin{array}{c} TM \\ \downarrow \\ M \end{array}, [\cdot,\cdot], id_{TM} \right)$

$\left(\begin{array}{c} \mathfrak{g} \\ \downarrow \\ \{\cdot\} \end{array}, [\cdot,\cdot], 0 \right)$

FOR A PRINCIPAL BUNDLE $\begin{array}{c} P \circlearrowleft G \\ \pi \downarrow \\ M \end{array}$ WE DEFINE $AP = TP/G$

WITH SECTIONS BEING INVARIANT VECTOR FIELDS AND ρ COMING from $T\pi$

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$$[X, fY] = f[X,Y] + \rho(X)(f)Y \quad , \quad f \in C^\infty(M).$$

2. A $\underbrace{\text{LINEAR}}$ POISSON STRUCTURE Λ ON E^* $(\{\cdot,\cdot\})$

COMPATIBLE WTH A VB STRUCTURE
HOMOGENEOUS OF DEGREE -1 : $\mathcal{L}_{\nabla_{E^*}} \Lambda = -\Lambda$

CLOSED ON LINEAR FUNCTIONS

$$\imath_X : E^* \longrightarrow \mathbb{R}, \quad \imath_X(\varphi) = \langle \varphi, X \rangle$$

$$\{\imath_X, \imath_Y\} = \imath_{[X,Y]}$$

$$\{\imath_X, \pi^* f\} = \pi^*(\rho(X) f)$$

$\pi : E^* \longrightarrow M$

1. BRACKET OF $Sec(\tau)$ & ANCHOR

2. LINEAR POISSON STRUCTURE ON E^*

3. A HOMOLOGICAL DERIVATION $d_E : \mathcal{A}(E^*) \longrightarrow \mathcal{A}(E^*)$ OF DEGREE 1 IN THE GRASSMANN ALGEBRA $\mathcal{A}(E^*)$ OF E^*

$$d_E : \mathcal{A}^i(E^*) \longrightarrow \mathcal{A}^{i+1}(E^*),$$
$$d_E^2 = 0,$$
$$d_E(\alpha \wedge \beta) = (d^E \alpha) \wedge \beta + (-1)^a \alpha \wedge d_E(\beta), \quad \alpha \in \mathcal{A}^a(E^*).$$

RELATION WITH THE BRACKET

$f \in C^\infty(M) = \mathcal{A}^0(E^*)$, $d_E f \in \mathcal{A}^1(E^*)$ $\qquad \langle d_E f, X \rangle = \rho(X) f,$

$\alpha \in \mathcal{A}^1(E^*)$ $\qquad d_E \alpha(X, Y) = \rho(X)\langle \alpha, Y \rangle - \rho(Y)\langle \alpha, X \rangle - \langle \alpha, [X, Y] \rangle.$

① BRACKET OF $Sec(\tau)$ & ANCHOR

② LINEAR POISSON STRUCTURE ON E^*

③ A HOMOLOGICAL DERIVATION d_E OF DEGREE 1

④ A DVB MORPHISM $\varepsilon : T^*E \longrightarrow TE^*$ OVER THE IDENTITY ON E^*

$$
\begin{tikzcd}
 & T^*E \arrow[dl] \arrow[ddr, "\pi_E"] \arrow[rrr, "\varepsilon"] & & & TE^* \arrow[dl, "\tau_{E^*}"'] \arrow[ddr, "T\pi"] & \\
E^* \arrow[rrr, "id_{E^*}"] \arrow[ddr, "\pi"'] & & & E^* \arrow[ddr, "\pi"'] & & \\
 & & E \arrow[dl, "\tau"] \arrow[rrr, "\rho"] & & & TM \arrow[dl, "\tau"] \\
 & M \arrow[rrr] & & & M &
\end{tikzcd}
$$

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E^* \arrow[rrr, "\mathrm{id}_{E^*}"] \arrow[ddr, "\pi"'] & & & E^* \arrow[ddr, "\pi"] & & \\
& T^*M \arrow[rrr, "\rho^*"] & E \arrow[rrr, "\rho"] \arrow[dl, "\tau"] & & E^* & TM \arrow[dl, "\tau"] \\
& M \arrow[rrr] & & & M &
\end{tikzcd}
$$

RELATION WITH LINEAR POISSON STRUCTURE

$$
\begin{tikzcd}
T^*E \arrow[rr, "\varepsilon"] \arrow[dr, "\mathcal{R}_E"'] & & TE^* \\
& T^*E^* \arrow[ur, "\tilde{\Lambda}"'] &
\end{tikzcd}
$$

$$q^i \circ \varepsilon = q^i$$
$$\xi_a \circ \varepsilon = \eta_b$$
$$\dot{q}^j \circ \varepsilon = \rho^j_a(q) y^a$$
$$\dot{\xi}_b \circ \varepsilon = C^a_{db}(q)\, y^d \eta_a + \rho^i_a(q) p_i$$

COORDINATES

$M : (q^i)$ $\qquad TE^* \; (q^i, \xi_a, \dot{q}^j, \dot{\xi}_b)$

$E : (q^i, y^a)$ $\qquad T^*E \; (q^i, y^a, p_j \; \eta_b)$

$E^* : (q^i, \xi_a)$

$$\varepsilon(q^i, y^a, p_j, \eta_b) = (q^i, \eta_b, \rho^i_a(q) y^a, C^a_{db}(q)\, y^d \eta_a + \rho^i_a(q) p_i)$$

1. BRACKET OF $Sec(\tau)$ & ANCHOR
2. LINEAR POISSON STRUCTURE ON E^*
3. A HOMOLOGICAL DERIVATION d_E OF DEGREE 1
4. A DVB MORPHISM $\varepsilon : T^*E \longrightarrow TE^*$
5. IN SUPERGEOMETRIC LANGUAGE :

A HOMOLOGICAL VECTOR FIELD ON THE SUPERMANIFOLD ΠE ASSOCIATED TO THE V.B. $E \longrightarrow M$

$\mathcal{M} = \Pi E$

$C^\infty(\mathcal{M}) = \mathcal{A}(E^*)$

VECTOR FIELD d_E

TULCZYJEW TRIPLE IN LIE ALGEBROID SETTING

$$
\begin{tikzcd}[column sep=small]
 & T^*E^* \arrow[dl, dash] \arrow[dr, dash] \arrow[rrr, "\tilde{\Lambda}"] & & & TE^* \arrow[dl, dash] \arrow[dr, dash] & & & T^*E \arrow[lll, "\varepsilon"'] \arrow[dl, dash] \arrow[dr, dash] & \\
E^* \arrow[ur, bend left, "d\mathcal{H}"] \arrow[rrr, "="] \arrow[dr, dash] & & & E^* \arrow[dr, dash] & & & E^* \arrow[lll, "="'] \arrow[dr, dash] & & \\
 & & E \arrow[rrr, "\rho"] \arrow[dl, dash] & & & TM \arrow[dl, dash] & & & E \arrow[lll, "\rho"'] \arrow[dl, dash] \arrow[ull, "\lambda"'] \arrow[uul, bend right, "d\mathcal{L}"'] \\
 & M \arrow[rrr, dash, "="] & & & M \arrow[rrr, dash, "="] & & & M &
\end{tikzcd}
$$

2, 4

THE TWO WAYS OF ENCODING A LIE ALGEBROID STRUCTURE PRODUCE THE TULCZYJEW TRIPLE. THE PHASE SPACE IS NOW E^*, WITH $\mathcal{D} \subset TE^*$.

$\mathcal{D}$ CAN BE GENERATED BOTH FROM LAGRANGIAN AND HAMILTONIAN GENERATING OBJECT.

EULER-LAGRANGE EQUATIONS IN LIE ALGEBROID SETTING

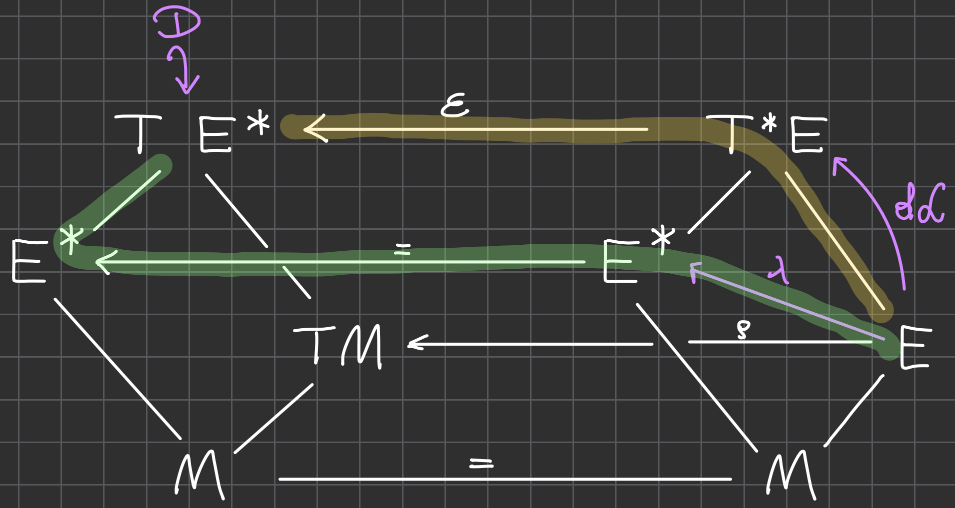
$$
\begin{tikzcd}
 & D \arrow[d] & & & & \\
 & TE^* \arrow[dl] \arrow[ddr] & & & T^*E \arrow[lll, "\varepsilon"'] \arrow[dl] & \\
E^* \arrow[ddr] & & & E^* \arrow[lll, "="'] \arrow[ddr] & & \\
 & & TM \arrow[dl] & & & E \arrow[lll, "\rho"'] \arrow[ull, "\lambda"'] \arrow[uul, bend right, "d\mathcal{L}"'] \arrow[dl] \\
 & M \arrow[rrrr, "="] & & & M &
\end{tikzcd}
$$

$$\varepsilon(q^i, y^a, p_j, \eta_b) = (q^i, \eta_b, \rho^i_a(q) y^a, C^a_{db}(q) y^d \eta_a + \rho^i_a(q) p_i)$$

$$\gamma : I \longrightarrow E$$

$$\varepsilon(d\mathcal{L}(\gamma(t))) = \frac{d}{dt} \lambda(\gamma(t))$$

$$\mathcal{E} \circ dL = \Lambda_T$$



EULER-LAGRANGE EQUATIONS IN LIE ALGEBROID SETTING

$$\gamma : I \longrightarrow E$$

$$\varepsilon\left(dL(\gamma(t))\right) = \frac{d}{dt}\lambda(\gamma(t))$$

$$\varepsilon(q^i, y^a, p_j, \eta_b) = \left(q^i, \eta_b, \rho^i_a(q) y^a, C^a_{db}(q) y^d \eta_a + \rho^i_a(q) p_i\right)$$

$$\frac{d}{dt} q^i(t) = \rho^i_a(q(t))\, y^a(t) \longleftarrow \text{ADMISSIBILITY CONDITION}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}(q(t), y(t))\right) = C^a_{db}(q(t))\, y^d(t) \frac{\partial L}{\partial y^a}(q(t), y(t)) + \rho^i_b(q(t)) \frac{\partial L}{\partial q^i}(q(t), y(t))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) - C^a_{db}\, y^d \frac{\partial L}{\partial y^a} - \rho^i_b \frac{\partial L}{\partial q^i} = 0$$

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GEOMETRICAL MECHANICS ON ALGEBROIDS

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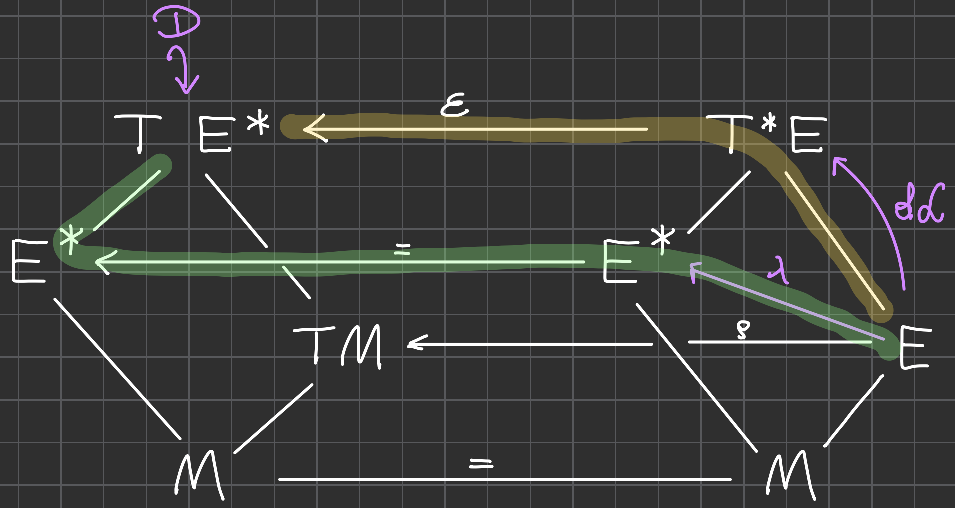

EULER-LAGRANGE EQUATIONS IN LIE ALGEBROID SETTING

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$$\frac{d}{dt} q^i(t) = \rho^i_a(q(t))\, y^a(t)$$ ADMISSIBILITY CONDITION

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^a}(q(t), y(t))\right) = C^a_{db}(q(t))\, y^d(t) \frac{\partial \mathcal{L}}{\partial y^a}(q(t), y(t)) + \rho^i_b(q(t)) \frac{\partial \mathcal{L}}{\partial q^i}(q(t), y(t))$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^e}\right) - C^a_{db}\, y^d \frac{\partial \mathcal{L}}{\partial y^e} - \rho^i_b \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

EXERCISE FOR STUDENTS $\longrightarrow$ WRITE EULER-LAGRANGE EQUATION $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0$ IN COORDINATES (q^i, y^j) ASSOCIATED TO A BASIS OF SECTIONS OF $TM \to M$ DIFFERENT THAN $(\partial_{q^1}, \dots, \partial_{q^n})$. YO WILL SEE SOMETHING LIKE THE E-L EQUATION ON LIE ALGEBROID WITH APPROPRIATE ρ AND C

DO WE SEE SITUATIONS LIKE THIS IN MECHANICS?

CHAIR OF MATHEMATICAL METHODS IN PHYSICS
KATEDRA METOD MATEMATYCZNYCH FIZYKI

MECHANICS ON LIE ALGEBROIDS - FIRST EXAMPLE

ALGEBROIDS IN MECHANICS APPEAR E.G. AS A RESULT OF REDUCTION WITH RESPECT TO SYMMETRIES. THE SIMPLEST EXAMPLE IS THE SYSTEM ON A LIE GROUP WITH INVARIANT LAGRANGIAN

G - A LIE GROUP

l_g - LEFT MULTIPLICATION BY g

$\mathcal{L}: TG \longrightarrow \mathbb{R}$ $\quad \mathcal{L}(Tl_g(v)) = \mathcal{L}(v)$ $\quad TG \simeq G \times \mathfrak{g}$ $\quad \mathcal{L}(g, x) = L(x)$

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$\mathcal{L}: TG \longrightarrow \mathbb{R}$ $\quad \mathcal{L}(Tl_g(v)) = \mathcal{L}(v)$ $\quad TG \simeq G\times\mathfrak{g}$ $\quad \mathcal{L}(g,x) = L(x)$

FULL TULCZYJEW TRIPLE

$$T^*T^*G \xleftarrow{\beta_G} TT^*G \xrightarrow{\alpha_G} T^*TG$$

IN TRIVIALIZATION BY LEFT MULTIPLICATION $TG \simeq G\times\mathfrak{g}$, $T^*G \simeq G\times\mathfrak{g}^*$

$$G\times\mathfrak{g}^*\times\mathfrak{g}^*\times\mathfrak{g} \xleftarrow{\beta_G} G\times\mathfrak{g}^*\times\mathfrak{g}\times\mathfrak{g}^* \xrightarrow{\alpha_G} G\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^*$$

MECHANICS ON LIE ALGEBROIDS - FIRST EXAMPLE

ALGEBROIDS IN MECHANICS APPEAR E.G. AS A RESULT OF REDUCTION WITH RESPECT TO SYMMETRIES. THE SIMPLEST EXAMPLE IS THE SYSTEM ON A LIE GROUP WITH INVARIANT LAGRANGIAN

G - A LIE GROUP

l_g - LEFT MULTIPLICATION BY g

$$\mathcal{L} : TG \longrightarrow \mathbb{R} \qquad \mathcal{L}(Tl_g(v)) = \mathcal{L}(v)$$

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$$\begin{array}{ccccc} G\times\mathfrak{g}^*\times\mathfrak{g}^*\times\mathfrak{g} & \xleftarrow{\ \beta_G\ } & G\times\mathfrak{g}^*\times\mathfrak{g}\times\mathfrak{g}^* & \xrightarrow{\ \alpha_G\ } & G\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^* \\ (g,\xi,\mathrm{ad}^*_X\xi-\eta,X) & \longleftarrow\!\shortmid & (g,\xi,X,\eta) & \longmapsto & (g,X,\eta-\mathrm{ad}^*_X\xi,\xi) \end{array}$$

CALCULATED BY MARCIN ZAJĄC IN HIS MASTER THESIS

IN TRIVIALIZATION $Tl_h(g,X) = (hg, X)$

INVARIANT LAGRANGIAN $\mathcal{L}(g,X) = \mathcal{L}(hg,X) = L(X)$

$$d\mathcal{L}(g,X) = (g, X, 0, dL(X))$$

$$
\begin{tikzcd}
G\times\mathfrak{g}^*\times\mathfrak{g}^*\times\mathfrak{g} & \cancel{G}\times\mathfrak{g}^*\times\cancel{\mathfrak{g}}\times\mathfrak{g}^* \arrow[l, "\beta_G"'] \arrow[r, "\alpha_G"] \arrow[dd] & G\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^* \\
& & K = G\times\mathfrak{g}\times\{0\}\times\mathfrak{g}^* \arrow[u, hook] \arrow[d] \\
& \overset{T\mathfrak{g}^*}{\mathfrak{g}^*\times\mathfrak{g}^*} & \mathfrak{g}\times\mathfrak{g}^* = T^*\mathfrak{g} \arrow[l, "\varepsilon"']
\end{tikzcd}
$$

COISOTROPIC SUBMANIFOLD

TANGENT PROJECTION FROM GROUP ACTION

SYMPLECTIC REDUCTION

AFTER REDUCTION I HAVE A MAP IN THE OTHER DIRECTION

$$
\begin{tikzcd}
G\times\mathfrak{g}^*\times\mathfrak{g}^*\times\mathfrak{g} & \cancel{G}\times\mathfrak{g}^*\times\cancel{\mathfrak{g}}\times\mathfrak{g}^* \arrow[l, "\beta_G"'] \arrow[r, "\alpha_G"] \arrow[dd] & G\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^* \\
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 & \mathfrak{g}^*\times\mathfrak{g}^* & \mathfrak{g}\times\mathfrak{g}^* \arrow[l, "\varepsilon"']
\end{tikzcd}
$$

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$$
\begin{array}{cccc}
T\mathfrak{g}^* & \mathfrak{g}^*\times\mathfrak{g}^* & \longleftarrow & \mathfrak{g}\times\mathfrak{g}^* \simeq T^*\mathfrak{g} \\
 & (\xi, \mathrm{ad}^*_x\xi) & \longmapsfrom & (x,\xi)
\end{array}
$$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[d] \\
\{\bullet\}
\end{tikzcd}
$$

TG IS REDUCED TO LIE ALGEBRA OVER A POINT

$$
\begin{tikzcd}
G\times\mathfrak{g}^*\times\mathfrak{g}^*\times\mathfrak{g} & \cancel{G}\times\mathfrak{g}^*\times\cancel{\mathfrak{g}}\times\mathfrak{g}^* \arrow[l, "\beta_G"'] \arrow[r, "\alpha_G"] \arrow[dd] & G\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^* \\
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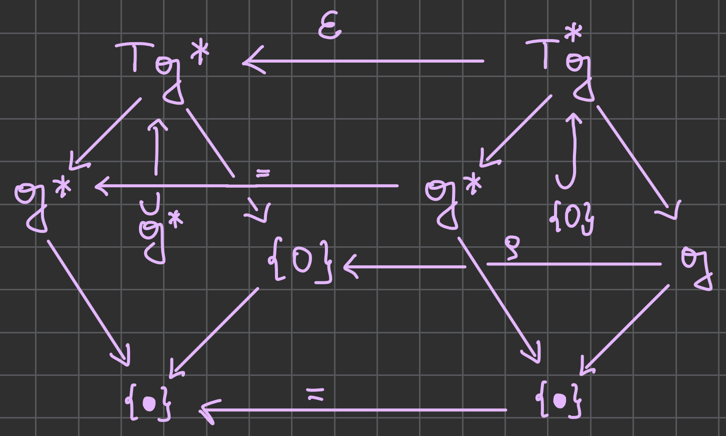
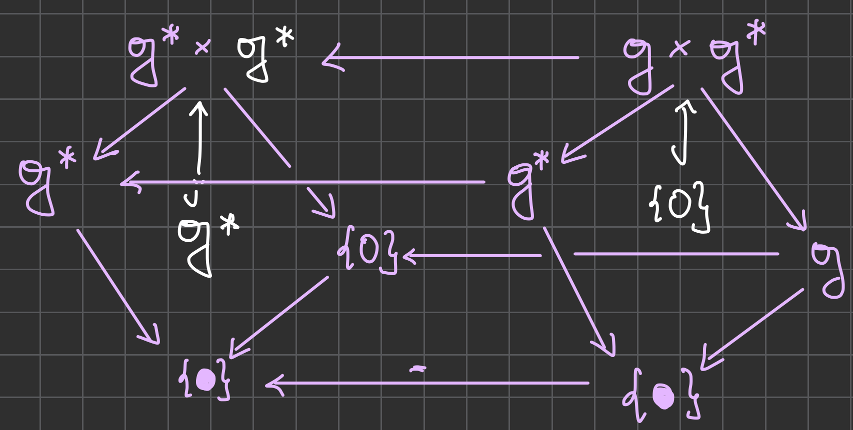
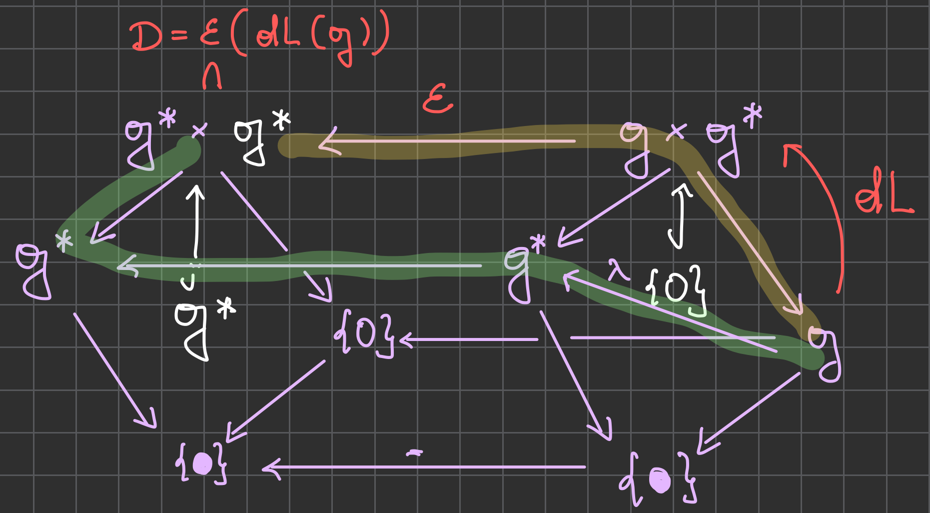
$$
\begin{tikzcd}
\mathfrak{g} \arrow[d] \\
\{\bullet\}
\end{tikzcd}
$$

TG IS REDUCED TO LIE ALGEBRA OVER A POINT

FULL DVB MORPHISM FOR $\mathfrak{g} \to \{\bullet\}$







LEGENDRE MAP: $\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ $\quad X \longmapsto dL(X)$

DYNAMICS $\quad D = \{ (\xi, \dot{\xi}) : \quad \xi = dL(X) \quad \dot{\xi} = ad^*_X (dL(X)) \}$

EULER-POINCARÉ EQUATION ON $\mathfrak{g}$: $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial X}\right) = ad^*_X \frac{\partial L}{\partial X}$

EULER LAGRANGE ON $\mathfrak{g} \to \{0\}$

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The Tulczyjew triple in mechanics on a Lie group

Katarzyna Grabowska[1] and Marcin Zając[1]

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FURTHER GENERALISATIONS : SKEW ALGEBROIDS ...

WE HAVE SEEN ON PREVIOUS SLIDES THAT WHAT WE REALLY USE IN MECHANICS IS A LIE ALGEBROID IN A FORM OF DOUBLE VECTOR BUNDLE MORPHISM. WE CAN THEN GENERALIZE IT DROPPING ASSUMPTIONS : JACOBI IDENTITY, ANTISYMMETRY ... UP TO JUST DVB MORPHISM $T^*E \longrightarrow TE^*$ OVER THE IDENTITY ON E^* DO WE NEED IT? SOME OF IT — MAYBE

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EXAMPLE : NONHOLONOMIC CONSTRAINTS + MECHANICAL LAGRANGIAN

WE START FROM LIE ALGEBROID (E, ε) AND SUBBUNDLE $K \subset E$ OVER M (FOR SIMPLICITY)

LAGRANGIAN IS ANY SMOOTH FUNCTION $\mathcal{L} : E \longrightarrow \mathbb{R}$

FROM VARIATIONAL APPROACH —
— d'ALEMBERT PRINCIPLE

CONSTRAINTS

$$(q^i, y^A, y^\alpha) \qquad K = \{ y^\alpha = 0 \}$$

$$y^\alpha = 0$$

$$\dot{q}^i = \rho^i_A \, y^A$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^A}\right) - C^B_{DA}\, y^D \frac{\partial \mathcal{L}}{\partial y^B} - C^\beta_{DA}\, y^D \frac{\partial \mathcal{L}}{\partial y^\beta} - \rho^i_A \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

THIS PART, PRESENT FOR ARBITRARY LAGRANGIAN, MEANS THAT WE CANNOT LOOK AT K ONLY

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GEOMETRICALLY

$$\imath : K \hookrightarrow E$$
$$\imath^* : E^* \longrightarrow K^*$$

$$\begin{tikzcd}
T^*E \arrow[r, "\varepsilon"] & TE^* \arrow[r, "T\imath^*"] & TK^* \arrow[d, "\tau_K"] \\
K \arrow[u, "d\mathcal{L}"] \arrow[r, "\lambda_{|K}"] & E^* \arrow[r, "\imath^*"] & K^*
\end{tikzcd}$$

FOR A MECHANICAL LAGRANGIAN:

g - BUNDLE METRIC ON E

$g : E \times_M E \longrightarrow \mathbb{R}$

BILINEAR
POSITIVE DEFINITE
SYMMETRIC

$$E = \underset{y^A}{K} \oplus \underset{y^\alpha}{K^\perp}$$

$$g_{A\alpha} = 0$$

$$\mathcal{L}(q^i, y^A, y^\alpha) = \frac{1}{2} g_{AB} y^A y^B + \frac{1}{2} g_{\alpha\beta} y^\alpha y^\beta - V(q)$$

FOR A MECHANICAL LAGRANGIAN:

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THIS PART DOES NOT MATTER
i.E. WE CAN WORK WITH
LAGRANGIAN DEFINED
ON K ONLY

$$\frac{\partial \mathcal{L}}{\partial y^\beta} = g_{\alpha\beta} y^\alpha \Big|_{y^\alpha = 0} = 0$$

$$y^\alpha = 0$$

$$\dot{q}^i = \rho^i_A \, y^A$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^A}\right) - C^B_{DA} \, y^D \frac{\partial \mathcal{L}}{\partial y^B} - C^\beta_{DA} \, y^D \frac{\partial \mathcal{L}}{\partial y^\beta} - \rho^i_A \frac{\partial \mathcal{L}}{\partial q^i}$$

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$E = K \oplus_M K^{\perp}$ $\quad$ $E^* = K^* \oplus_M K^\circ$ $\quad$ $i : K \hookrightarrow E$ $\quad$ $P : E \longrightarrow K$ $\quad$ $a = (A, \alpha)$

$$\begin{tikzcd}
T^*_D E \arrow[r, "\varepsilon"] & TE^* \arrow[d, "Ti^*"] \\
T^*K \arrow[u, "T^*P"] \arrow[r, "\varepsilon_K"] & TK^*
\end{tikzcd}$$

$$\begin{tikzcd}
(q^i, y^A, 0, p_j, \xi_A, 0) \arrow[r, "\varepsilon"] & (q^i, \xi_A, 0, \rho^i_e y^a, C^a_{bD} y^b \xi_a + \rho^i_D p_i, \cancel{C^\alpha_{b\beta} y^b \xi_\alpha + \rho^i_\beta p_i}) \arrow[d] \\
(q^i, y^A, p_j, \xi_A) \arrow[u] \arrow[r, "\varepsilon_K"] & (q^i, \xi_A, \rho^i_A y^A, C^A_{BD} y^B \xi_A + \rho^i_D p_i)
\end{tikzcd}$$

$y^\alpha = 0$

$\rho^i_A y^A + \cancel{\rho^i_\alpha y^\alpha}$

$y^\alpha = 0$

$\xi_\alpha = 0$

$E = K \oplus_M K^\perp \qquad E^* = K^* \oplus_M K^\circ \qquad i : K \hookrightarrow E \qquad P : E \longrightarrow K$

$$
\begin{tikzcd}
T^*_K E \arrow[r, "\varepsilon"] & TE^* \arrow[d] \\
T^*K \arrow[u, "T^*P"] \arrow[r, "\varepsilon_K"] & TK^*
\end{tikzcd}
$$

$y^\alpha = 0$ $\qquad\qquad$ $\rho^i_A y^A + \cancel{\rho^i_\alpha y^\alpha}$ $\qquad$ $y^\alpha = 0$

$\|$

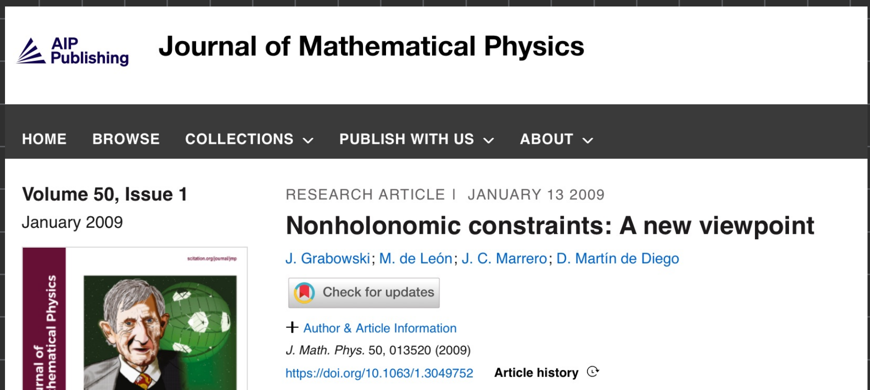
$$
\begin{tikzcd}
(q^i, y^A, 0, p_j, \xi_A, 0) \arrow[r, "\varepsilon"] & (q^i, \xi_A, 0, \rho^i_a y^a, C^a_{bD} y^b \xi_a + \rho^i_D p_i, C^\alpha_{b\beta} y^b \xi_\alpha + \rho^i_\beta p_i) \arrow[d] \\
(q^i, y^A, p_j, \xi_A) \arrow[u] \arrow[r, "\varepsilon_P"] & (q^i, \xi_A, \rho^i_A y^A, C^A_{BD} y^B \xi_A + \rho^i_D p_i)
\end{tikzcd}
$$

$\xi_\alpha = 0$

WE HAVE A DOUBLE VECTOR BUNDLE MORPHISM $T^*K \longrightarrow TK^*$ OVER THE IDENTITY ON K^* CORRESPONDING TO A BIVECTOR ON K^*, BUT NO JACOBI IDENTITY

THE BRACKET ON K^* IS CALLED A NONHOLONOMIC BRACKET

THE THEORY WORKS FOR PARTICULAR LAGRANGIANS / HAMILTONIANS ONLY



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Oscar Cosserat, Alexei Kotov, Camille Laurent-Gengoux, Leonid Ryvkin, Vladimir Salnikov

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