Infrared behavior of "tame" two-field hyperbolizable cosmological models

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[M.B. and C.I. Lazaroiu, *Nucl. Phys. B* 983 (2022), 115929]

According to current observational data, ordinary matter and radiation make up only a small fraction of the energy density of the Universe today. The dominant components are dark energy and dark matter, whose nature is not well understood. This motivates the investigation of various cosmological models beyond the standard ΛCDM one. (Lambda Cold Dark Matter is the standard model of big bang cosmology, being the simplest model that is in general agreement with observed phenomena.)

Modern observations have established, to a very good degree of accuracy, that the present day universe is homogeneous and isotropic on large scales. This is naturally explained if one assumes that the early universe underwent a period of accelerated expansion called inflation. This idea can be realized in models where the inflationary expansion is driven by the potential energy of a number of real scalar fields called inflatons. The most studied models of this type contain a single scalar field. However, recent arguments related to quantum gravity suggest that it is more natural, or may even be necessary to have more than one inflaton.

Multifield models are of interest both for understanding cosmological inflation in the early Universe, as well as for describing dark energy in the late Universe.

Multifield cosmological models have richer phenomenology than single field models since they allow for solutions of the equations of motion whose field-space trajectories are not (reparameterized) geodesics. Such trajectories are characterized by a non-zero turn rate.

In the past it was thought that phenomenological viability requires small turn rate. This assumption lead to the well known slow-roll slow-turn (SRST) approximation. However, in recent years it was understood that rapid turn trajectories can also be perturbatively stable and of phenomenological interest.

For instance, a brief rapid turn during slow-roll inflation can induce primordial black hole generation; also, trajectories with large and constant turn rate can correspond to solutions behaving as dark energy.

[L. Anguelova et all., JCAP 03 (2022) 018; L. Anguelova, JCAP 06 (2021) 004]

Definition

A 2-field cosmological model can be described by a 2-dimensional scalar triple, which is an ordered system $(\Sigma, \mathcal{G}, \Phi)$, where:

- $(\Sigma, \mathcal{G})$ is a connected and borderless Riemannian surface (scalar manifold)
- $\Phi \in \mathcal{C}^\infty(\Sigma, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\Sigma, \mathcal{G})$ is complete (to ensure conservation of energy)
2. $\Phi > 0$ on Σ (to avoid certain technical problems)

The action of a cosmological model:

$$\mathcal{S}_{\Sigma,\mathcal{G},\Phi}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|} \left[\frac{M^2}{2} R(g) - \frac{1}{2} g^{\mu\nu} \mathcal{G}_{ij} \partial_\mu \varphi^i \partial_\nu \varphi^j - \Phi \circ \varphi \right]$$

- g is the metric on the space-time $\mathbb{R}^4$
- M is the reduced Planck mass
- $R(g)$ is the Ricci scalar
- $\varphi : \mathbb{R}^4 \longrightarrow \Sigma \quad , \quad \Phi : \Sigma \longrightarrow \mathbb{R}$
- $\mu, \nu \in \{0, .., 3\} \quad , \quad i, j \in \{1, 2\}$

FLRW universe:

$\mathrm{d}s_g^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2$, $x^0 = t$, $\vec{x} = (x^1, x^2, x^3)$, $a(t) > 0$ (*scale factor*)

$\varphi = (\varphi^1, \varphi^2)$ & take $\varphi = \varphi(t)$.

$H(t) \stackrel{\mathrm{def.}}{=} \frac{\dot{a}(t)}{a(t)}$ *(Hubble parameter)*

When $H(t) > 0$, the e.o.m. give the **cosmological equation**:

$$\boxed{\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2\Phi(\varphi(t))\right]^{1/2}\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi(t)) = 0} \qquad (1)$$

plus the condition:

$$\boxed{H(t) = \frac{1}{3M_0}\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2\Phi(\varphi(t))\right]^{1/2}} \qquad (2)$$

$$M_0 = M\sqrt{\frac{2}{3}} \quad (\textit{the rescaled reduced Planck mass})$$

$$\nabla_t \dot{\varphi}^i = \ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k \quad , \quad ||\dot{\varphi}(t)||_{\mathcal{G}}^2 = \mathcal{G}_{ij}\dot{\varphi}^i\dot{\varphi}^j \qquad (3)$$

$$\mathrm{grad}_{\mathcal{G}}\Phi = (\mathrm{grad}_{\mathcal{G}}\Phi)^i\partial_i = \mathcal{G}^{ij}(\partial_j\Phi)\partial_i \quad , \quad \partial_i := \frac{\partial}{\partial\varphi^i} \ .$$

Finding inflationary solutions in multifield models is much harder than in the single-field case. Such models are either studied numerically or solved only approximately.

The cosmological equation is equivalent with a dissipative geometric dynamical system defined on the tangent bundle of that manifold. Little is known in general about this dynamical system, partly because the scalar manifold Σ need not be simply-connected and, also, because it is non-compact in most applications of physical interest and hence cosmological trajectories can "escape to infinity" (i.e. it can approach a Freudenthal end of Σ for early or late cosmological times depending on the behavior of Φ and $\mathcal{G}$ near that end).

A common approach to looking for cosmological trajectories with desirable properties is to first simplify the equations of motion by imposing various approximations (such as slow-roll, slow-turn, rapid-turn etc.).
This leads to an approximate system of equations, obtained by neglecting certain terms in the original nonlinear coupled ODEs. However, there is no guarantee that a solution of the approximate system is a good approximant of the solution of the exact system for a sufficiently long period of time.

It was shown in:

- *C. I. Lazaroiu, Dynamical renormalization and universality in classical multifield cosmological models, Nucl. Phys. B 983 (2022), 115940*

that considering the behavior of the model under scale transformations $t \to t/\epsilon$ of the cosmological time (with parameter $\epsilon > 0$), a renormalization group (RG) action is induced such that:

- the limit $\epsilon \to \infty$ captures the high frequency (or UV) behavior of cosmological curves.

- the limit $\epsilon \to 0$ gives the low frequency (or IR) behavior

- taking ϵ to be large (UV limit) corresponds to replacing the cosmological flow with the geodesic flow,
- taking ϵ to be small (IR limit) corresponds to replacing the cosmological flow with the gradient flow of the classical effective potential $V = M_0\sqrt{2\Phi}$.

Such expansions are natural from a physics perspective. Geodesic flow and gradient flow are well-studied subjects in the theory of dynamical systems (although the generic non-compactness of Σ complicates the analysis).

The use of the scaling limits allows to classify multifield cosmological models into UV and IR universality classes.

The *RG similarity* with parameter ϵ transforms:

$$\varphi \to \varphi_\epsilon \quad , \quad M_0 \to \epsilon M_0 \quad , \quad \mathcal{G} \to \epsilon^2 \mathcal{G} \quad , \quad \Phi \to \Phi_\epsilon \quad (\epsilon > 0) \quad .$$

The ϵ-*scale transform* of a curve $\varphi : I \to \Sigma$ is the curve $\varphi_\epsilon : I_\epsilon \to \Sigma$ s.t.:

$$\varphi_\epsilon(t) \stackrel{\text{def.}}{=} \varphi(t/\epsilon) \ , \ \forall t \in I_\epsilon \quad , \quad I_\epsilon \stackrel{\text{def.}}{=} \epsilon I = \{\epsilon t | t \in I\} \ , \ \epsilon > 0$$

A curve $\varphi : I \to \Sigma$ satisfies the cosmological equation of the model $(\Sigma, \mathcal{G}, \Phi)$ iff φ_ϵ satisfies the ϵ-*rescaled cosmological equation*:

$$\boxed{\nabla_t \dot{\varphi}_\epsilon(t) + \frac{1}{M_0}\left[||\dot{\varphi}_\epsilon(t)||^2_{\mathcal{G}} + 2\Phi_\epsilon(\varphi_\epsilon(t))\right]^{1/2} \dot{\varphi}_\epsilon(t) + (\mathrm{grad}_{\mathcal{G}}\Phi_\epsilon)(\varphi_\epsilon(t)) = 0} \tag{4}$$

The IR limit ($\epsilon \to 0$, i.e. $t \to \infty$) corresponds to low frequency behaviour of cosmological curves $\varphi_\epsilon(t)$.

The IR expansion

When $\epsilon \ll 1$, we expand φ_ϵ in positive powers of ϵ or equivallently expand $\varphi(t)$ in powers of $\frac{1}{\sqrt{2\Phi}}$, such that (4) is Fourier expanded as:

$$\boxed{\frac{1}{\sqrt{2\Phi(\varphi)}}\nabla_t\dot{\varphi} + \frac{1}{M_0}\Big[1 + \Big(\frac{||\dot{\varphi}||_{\mathcal{G}}}{\sqrt{2\Phi(\varphi)}}\Big)^2 - \frac{1}{8}\Big(\frac{||\dot{\varphi}||_{\mathcal{G}}}{\sqrt{2\Phi(\varphi)}}\Big)^4 + \ldots\Big]\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}}\sqrt{2\Phi(\varphi)})(\varphi) = 0}$$

$$\Big(\nabla_t\dot{\varphi}^i = \ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k\ ,\ \ ||\dot{\varphi}(t)||^2_{\mathcal{G}} = \mathcal{G}_{ij}\dot{\varphi}^i\dot{\varphi}^j\ ,\ \ \mathrm{grad}_{\mathcal{G}}\Phi = (\mathrm{grad}_{\mathcal{G}}\Phi)^i\partial_i = \mathcal{G}^{ij}(\partial_j\Phi)\partial_i\Big)$$

First order approximation

To first order in the IR expansion, φ is approximated by the solution φ_{IR} of this gradient flow equation:

$$\boxed{\dot{\varphi}_{\mathrm{IR}}(t) + (\mathrm{grad}_{\mathcal{G}}V)(\varphi_{\mathrm{IR}}(t)) = 0} \tag{5}$$

where $V \overset{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$ is the *classical effective scalar potential* of the model.

Hyperbolizable two-field models

A two-field model $(\Sigma, \mathcal{G}, \Phi)$ is called hyperbolizable if $\mathcal{G}$ is hyperbolizable.

Theorem (Poincaré)

The Weyl equivalence class of any Riemannian metric $\mathcal{G}$ on a borderless connected surface Σ contains a unique complete metric G, called the *uniformizing metric* of $\mathcal{G}$, of constant Gaussian curvature $K = -1$, 0 or $+1$.

A metric G on Σ is called hyperbolic if it is complete and of constant Gaussian curvature $K = -1$.

- The case $K = -1$ is generic: the metric $\mathcal{G}$ defined on Σ is *hyperbolizable* and its uniformizing metric G is called the *hyperbolization* of $\mathcal{G}$.
- The cases $K = +1$ and $K = 0$ occur only for 7 topologies, as follows:
  - When $K = +1$, then Σ must be diffeomorphic with S^2 or $\mathbb{RP}^2 \simeq \mathrm{S}^2/\mathbb{Z}_2$.
  - When $K = 0$, then Σ must be diffeomorphic with T^2, or $\mathrm{K}^2 = \mathbb{RP}^2 \times \mathbb{RP}^2 \simeq \mathrm{T}^2/\mathbb{Z}_2$, or $\mathbb{A}^2$, or $\mathrm{M}^2 \simeq \mathbb{A}^2/\mathbb{Z}_2$, or $\mathbb{R}^2$.

Two-field models whose Σ has constant Gaussian curvature equal to -1, 0 or 1 are IR universal, i.e. they suffice to describe the IR approximants of cosmological orbits for all two-field models with smooth $V > 0$.

There are 4 hyperbolic types of ends: plane, cusp, horn and funnel ends.

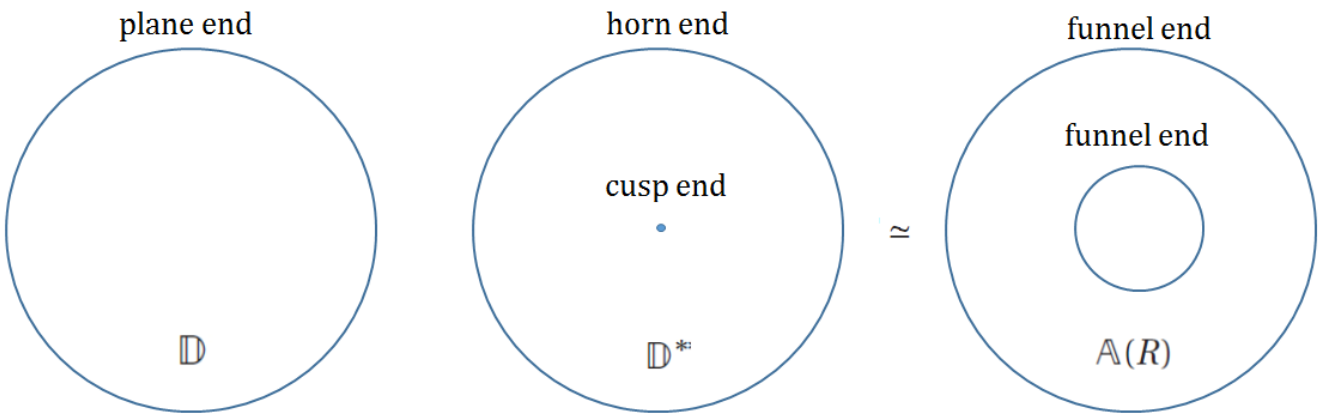
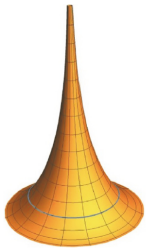
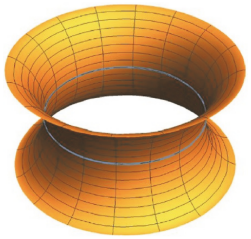


Figure: The elementary hyperbolic surfaces and type of their ends. ($\widehat{\Sigma} = \mathrm{S}^2$)



() $\mathbb{D}^*$ has cusp end & horn end



() $\mathbb{A}$ has funnel ends

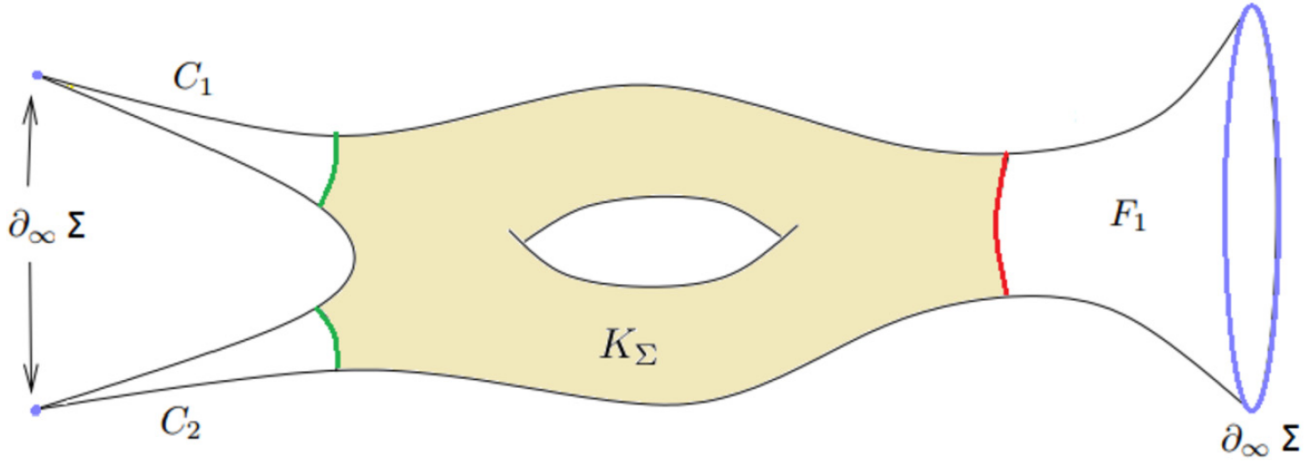


Figure: A non-elementary hyperbolic surface with two cusp ends (the blue dots) and one funnel end (the blue circle), where K_Σ indicates the compact core.

Reminder

Each **end** of a topological space represents a topologically distinct way to move to infinity within the space. Adding a point at each end yields the so-called **end compactification** of the original space, so the set of ends is defined as:

$$\mathrm{Ends}(\Sigma) \stackrel{\text{def.}}{=} \widehat{\Sigma} \setminus \Sigma \quad \Longleftrightarrow \quad \widehat{\Sigma} = \Sigma \sqcup \mathrm{Ends}(\Sigma) \ .$$

Definition

A two-field cosmological model described by a two-dimensional scalar triple $(\Sigma, \mathcal{G}, \Phi)$ is called *"tame"* if the following conditions are satisfied:

1. Σ is oriented and *topologically finite*. Thus, Σ has finite genus and finite number of ends ($\mathrm{Ends}(\Sigma)$ is finite) and its end compactification $\widehat{\Sigma}$ is a compact smooth surface.
2. The scalar potential Φ is *globally well-behaved*, i.e. Φ admits a smooth extension $\hat{\Phi}$ to $\widehat{\Sigma}$. We require that $\hat{\Phi} > 0$ on $\widehat{\Sigma}$.
3. The extended potential $\hat{\Phi}$ is a Morse function on $\widehat{\Sigma}$ (in particular, Φ is a Morse function on Σ). (Thus, $\hat{\Phi}$ and Φ have no degenerate critical points.)

Critical points of V (and Φ) coincide with **interior critical points** of $\widehat{V}$ (and $\hat{\Phi}$)

$$\mathrm{Crit}\, V = \mathrm{Crit}\Phi = \Sigma \cap \mathrm{Crit}\,\widehat{V} = \Sigma \cap \mathrm{Crit}\hat{\Phi} \quad .$$

The set of **critical ends** is defined as:

$$\mathrm{Crit}_\infty V = \mathrm{Crit}_\infty \Phi \stackrel{\mathrm{def.}}{=} \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\,\widehat{V} = \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\hat{\Phi}$$

and we have the disjoint union decomposition:

$$\mathrm{Crit}\,\widehat{V} = \mathrm{Crit}\, V \sqcup \mathrm{Crit}_\infty V \quad .$$

[E. M. Babalic, C. I. Lazaroiu, *The infrared behavior of tame two-field cosmological models*, Nucl. Phys. B 983 (2022), 115929]

Remember the **cosmological equation**:

$$\boxed{\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2\Phi(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi(t)) = 0} \tag{6}$$

and its IR first order approximation, i.e. the **gradient flow equation**, where $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$

$$\boxed{\dot{\varphi}_{\mathrm{IR}}(t) + (\mathrm{grad}_G V)(\varphi_{\mathrm{IR}}(t)) = 0} \tag{7}$$

An **IR optimal cosmological curve** is a solution $\varphi(t)$ of the cosmological equation (6) which satisfies:

$$\dot{\varphi}(0) = -(\mathrm{grad}_G V)(\varphi(0)) \ .$$

We analyze the IR behaviour of hyperbolizable "tame" two-field models by solving the gradient flow equation near critical ends of Σ and near interior critical points of V.

The hyperbolic metric G in the vicinity of an end

The metric in canonical *polar coordinates* (ω, θ) in the vicinity of an end e:

$$\mathrm{d}s^2_G|_{\dot{U}_\mathsf{e}} = \frac{\mathrm{d}\omega^2}{\omega^4} + f_\mathsf{e}(1/\omega)\mathrm{d}\theta^2 \quad ,$$

$$f_\mathsf{e}(1/\omega) = \tilde{c}_\mathsf{e} e^{\frac{2\epsilon_\mathsf{e}}{\omega}} \left[1 + \mathrm{O}\left(e^{-\frac{2}{\omega}}\right)\right] \quad \text{for} \quad \omega \to 0 \quad ,$$

$$\tilde{c}_\mathsf{e} = \begin{cases} \frac{1}{4} & \text{if } \mathsf{e} = \text{plane end} \\ \frac{1}{(2\pi)^2} & \text{if } \mathsf{e} = \text{horn end} \\ \frac{\ell^2}{(4\pi)^2} & \text{if } \mathsf{e} = \text{funnel end of circumference } \ell > 0 \\ \frac{1}{(2\pi)^2} & \text{if } \mathsf{e} = \text{cusp end} \end{cases}$$

$$\epsilon_\mathsf{e} = \begin{cases} +1 & \text{if } \mathsf{e} = \text{flaring (i.e. plane, horn or funnel) end} \\ -1 & \text{if } \mathsf{e} = \text{cusp end} \end{cases}$$

$\mathrm{O}\left(e^{-\frac{2}{\omega}}\right) \equiv 0$ when e is a **cusp** or **horn** end, but for simplicity of calculations we will approximate it to zero for all ends.

In local coordinates on $\widehat{\Sigma}$, we have:

$$\nabla_t \dot{\varphi}^i(t) = \ddot{\varphi}^i(t) + \Gamma^i_{jk}(\varphi(t))\dot{\varphi}^j(t)\dot{\varphi}^k(t) \ ,$$
$$||\dot{\varphi}(t)||^2_G = G_{ij}(\varphi(t))\dot{\varphi}^i(t)\dot{\varphi}^j(t) \ ,$$
$$\mathrm{grad}_G V = G^{ij}\partial_i V \partial_j \ .$$

where $\varphi = (\omega, \theta)$.
The only nontrivial Christoffel symbols are:

$$\Gamma^\omega_{\omega\omega} = -\frac{2}{\omega} \quad , \quad \Gamma^\omega_{\theta\theta} = \tilde{c}_e \epsilon_e \omega^2 e^{\frac{2\epsilon_e}{\omega}} \quad , \quad \Gamma^\theta_{\theta\omega} = \Gamma^\theta_{\omega\theta} = -\frac{\epsilon_e}{\omega^2} \quad .$$

The cosmological equations become (for $M_0 = 1$):

$$\ddot{\omega} - \frac{2}{\omega}\dot{\omega}^2 + \tilde{c}_e \epsilon_e \omega^2 e^{\frac{2\epsilon_e}{\omega}} \dot{\theta}^2 + H\dot{\omega} + \omega^4 \partial_\omega \Phi = 0 \quad ,$$
$$\ddot{\theta} - \frac{2\epsilon_e}{\omega^2}\dot{\omega}\dot{\theta} + H\dot{\theta} + \frac{1}{\tilde{c}_e} e^{-\frac{2\epsilon_e}{\omega}} \partial_\theta \Phi = 0 \quad , \tag{8}$$

where $\Phi = \frac{1}{2}V^2$ and:

$$H = \sqrt{\frac{\dot{\omega}^2}{\omega^4} + \tilde{c}_e e^{\frac{2\epsilon_e}{\omega}} \dot{\theta}^2 + V^2(\omega, \theta)} \quad .$$

The Taylor expansion of V in polar coordinates (ω, θ):

$$V(\omega,\theta) = V(\mathrm{e}) + \frac{1}{2}\omega^2\Big[\lambda_1(\mathrm{e})\cos^2\theta + \lambda_2(\mathrm{e})\sin^2\theta\Big] + \mathrm{O}(\omega^3)\ ,$$

$\lambda_1(\mathrm{e})$ and $\lambda_2(\mathrm{e})$ are the principal real values of the Hessian of V in e.

$$\mathrm{Hess}_{\mathcal{G}}(V) = (\partial_i\partial_j - \Gamma^k_{ij}\partial_k)V\ \mathrm{d}\varphi^i \otimes \mathrm{d}\varphi^j$$

When λ_1 and λ_2 do not both vanish, it is convenient to define:

Definition

- The *critical modulus* of (Σ, G, V) at the critical end e is the ratio:

$$\beta(\mathrm{e}) \stackrel{\mathrm{def.}}{=} \frac{\lambda_1(\mathrm{e})}{\lambda_2(\mathrm{e})} \in [-1,1] \setminus \{0\}\ \ ,$$

- The *characteristic signs* of (Σ, G, V) at e:

$$\varepsilon_i(\mathrm{e}) \stackrel{\mathrm{def.}}{=} \mathrm{sign}(\lambda_i(\mathrm{e})) \in \{-1,1\}\ \ (i = 1,2)$$

In canonical coordinates (ω, θ), for $\theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, the **unoriented gradient flow orbits** around the ends are given in the implicit form:

$$\boxed{\frac{1}{4}[\lambda_1(\mathrm{e}) - \lambda_2(\mathrm{e})]\,\Gamma_2\left(\frac{2\epsilon_{\mathrm{e}}}{\omega}\right) = A + \tilde{c}_{\mathrm{e}}\,[\,\lambda_1(\mathrm{e})\log|\sin\theta| - \lambda_2(\mathrm{e})\log|\cos\theta|\,]} \quad (9)$$

where Γ_2 is the lower incomplete Gamma function of order 2 and A is an integration constant.

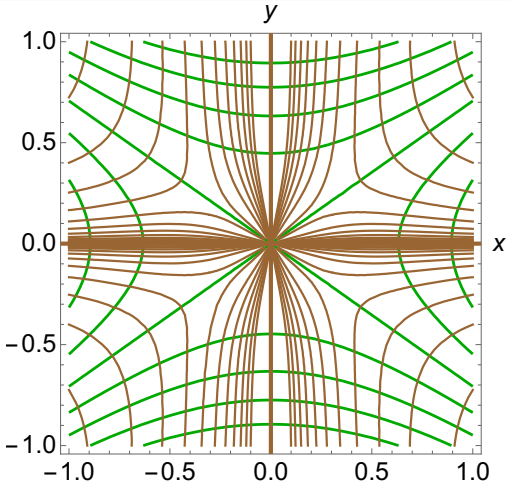
We **graphically compare** the **unoriented gradient flow orbits** to the **IR optimal cosmological curves**, which are the **numerically computed solutions $\varphi(t)$ of the cosmological equation with initial condition**

$$\dot{\varphi}(0) = -(\mathrm{grad}_G V)\varphi(0)$$

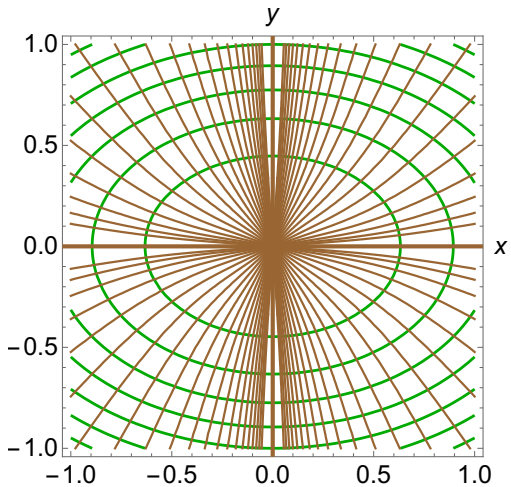
We plot in Cartesian coordinates (x, y), s.t. $\omega = \sqrt{x^2 + y^2}$, $\theta = \arg(x + \mathrm{i}y)$

We take $V(\mathrm{e}) = 1$ and $\lambda_2(\mathrm{e}) = 1$, which gives $\beta(\mathrm{e}) = \lambda_1(\mathrm{e})$.

The IR behaviour near critical plane ends

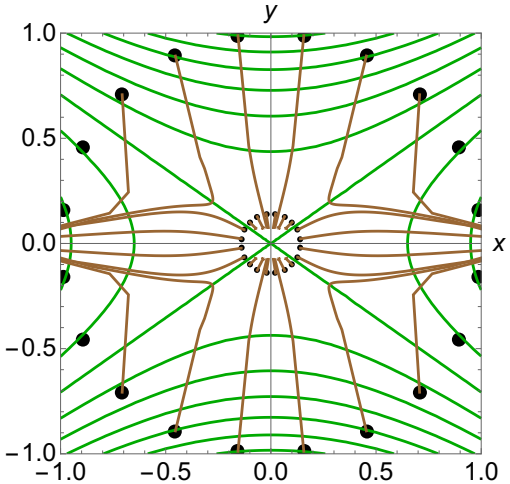


() For $\beta(\mathrm{e}) = -0.5$



() For $\beta(\mathrm{e}) = 0.5$

Figure: Gradient flow orbits (brown) over potential lines (green) for critical plane end.



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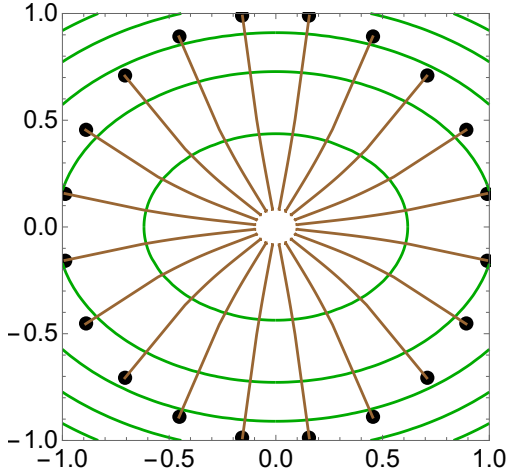
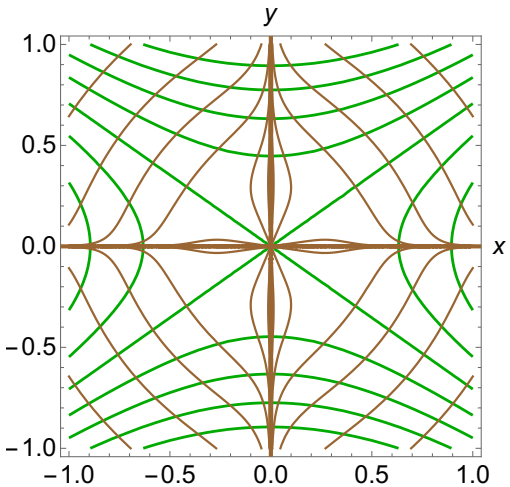


() For $\beta(\mathrm{e}) = 0.5$

Figure: IR optimal cosmological curves (brown) over potential lines (green). The dots are initial points $\varphi(0)$.

The IR behaviour near critical horn ends



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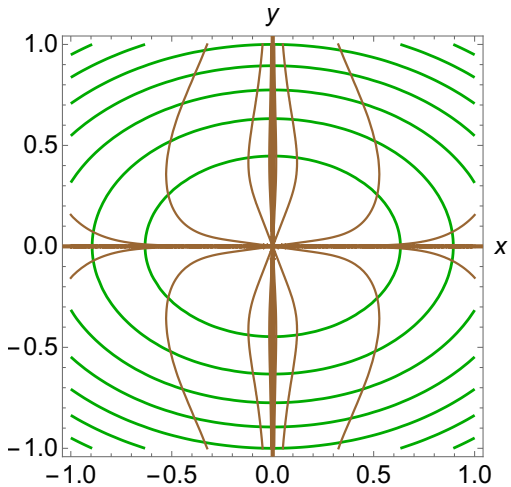
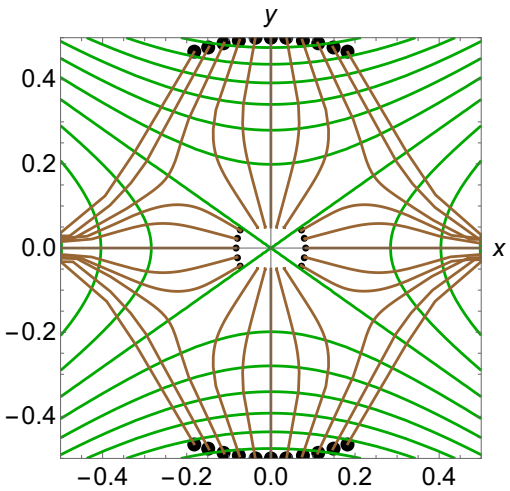


() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical horn end.



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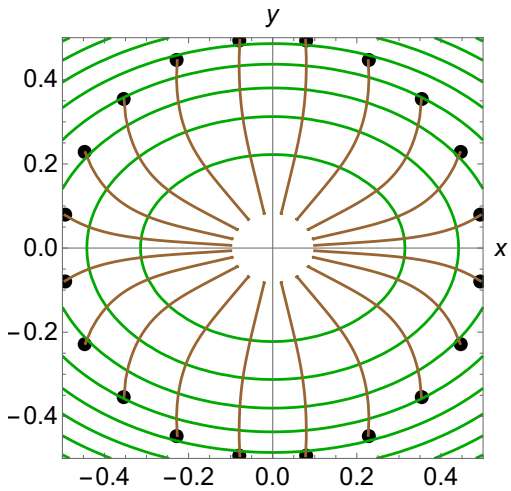
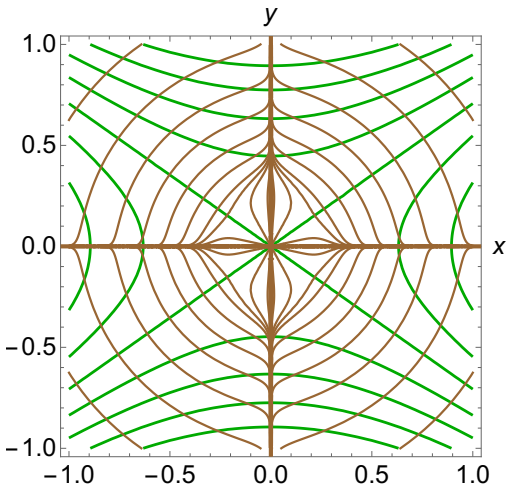


() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical horn end. The dots are initial points $\varphi(0)$.

The IR behaviour near critical funnel ends



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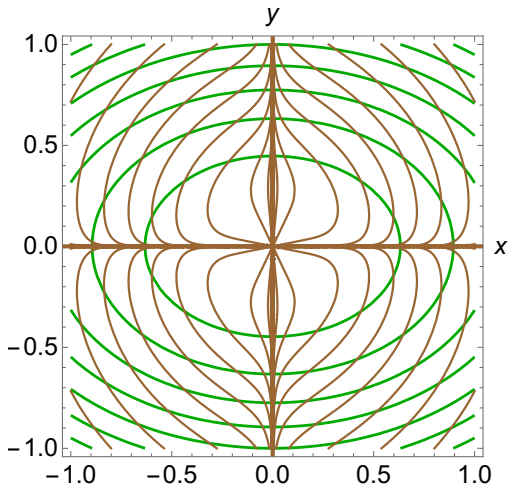
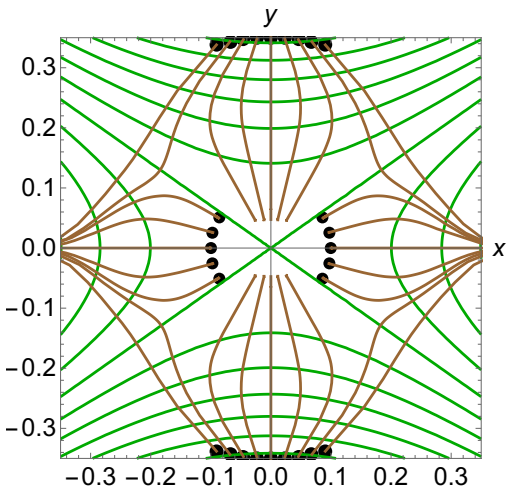


() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical funnel end.



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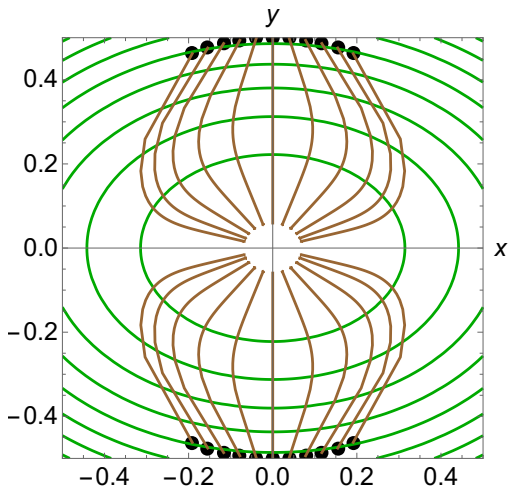
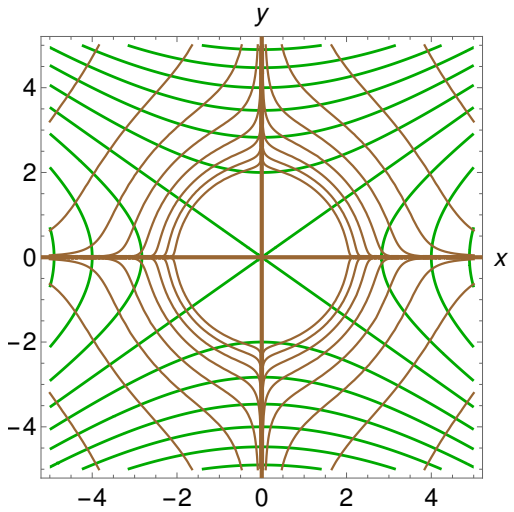


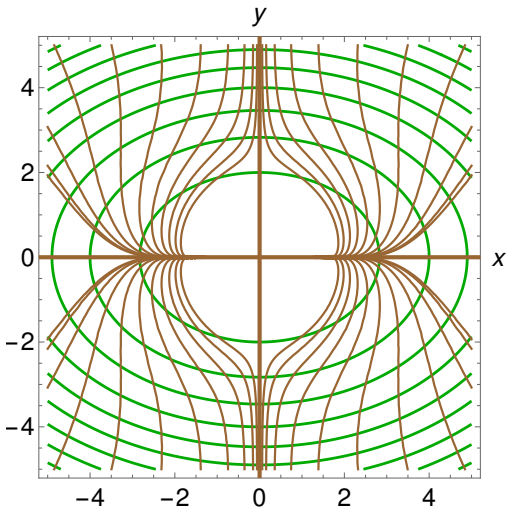
() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical funnel end. The dots are initial points $\varphi(0)$.

The IR behaviour near critical cusp ends

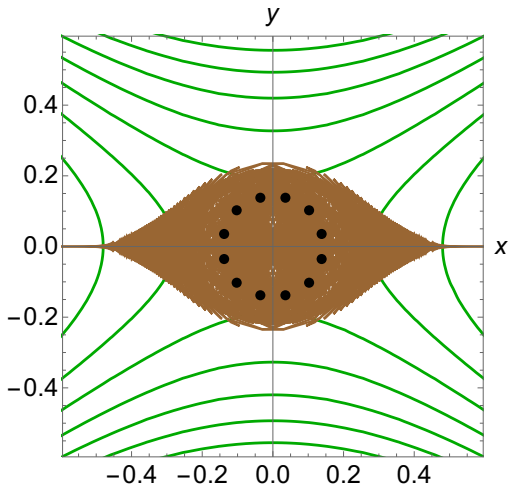


() For $\beta(\mathrm{e}) = -0.5$.

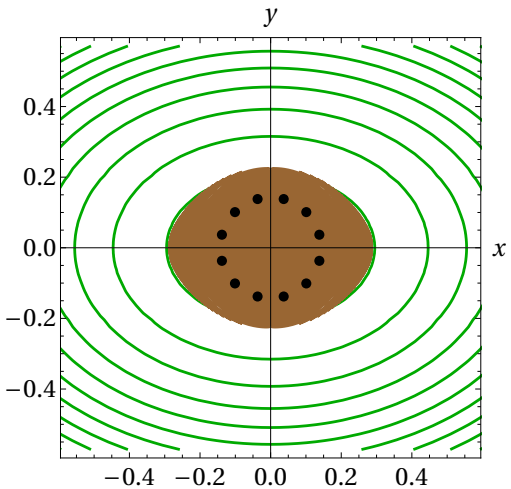


() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical cusp end.



() For $\beta(\mathrm{e}) = -0.5$.



() For $\beta(\mathrm{e}) = 0.5$.

Figure: Critical cusp end. The dots are initial points $\varphi(0)$.

Note: One must consider higher order corrections in the IR expansion to get better aproximation also for the cusp.

The IR behavior near an interior critical point

Let c be **an interior critical point**. In principal polar canonical coordinates (ω, θ) centered at c, we have the metric:

$$\mathrm{d}s_G^2 = \frac{4}{(1-\omega^2)^2}[\mathrm{d}\omega^2 + \omega^2 \mathrm{d}\theta^2]$$

and:

$$V(\omega, \theta) = V(\mathrm{c}) + \frac{1}{2}\omega^2 \left[\lambda_1(\mathrm{c}) \cos^2\theta + \lambda_2(\mathrm{c}) \sin^2\theta\right] + \mathrm{O}(\omega^3) \quad .$$

The *critical modulus* $\beta(\mathrm{c})$ and *characteristic signs* $\epsilon_1(\mathrm{c})$ and $\epsilon_2(\mathrm{c})$ of (Σ, G, V) at c are defined through:

$$\beta(\mathrm{c}) \stackrel{\mathrm{def.}}{=} \frac{\lambda_1(\mathrm{c})}{\lambda_2(\mathrm{c})} \in [-1, 1] \setminus \{0\} \quad , \quad \epsilon_i(\mathrm{c}) \stackrel{\mathrm{def.}}{=} \mathrm{sign}(\lambda_i(\mathrm{c})) \quad (i = 1, 2) \quad .$$

We take $V(c) = 1$ and $\lambda_2(c) = 1$, which gives $\beta(c) = \lambda_1(c)$.

For interior critical points

$$H(\omega,\theta,\dot\omega,\dot\theta)=\sqrt{\frac{4}{(1-\omega^2)^2}\left(\dot\omega^2+\omega^2\dot\theta^2\right)+V^2(\omega,\theta)}$$

$$\Gamma^\omega_{\omega\omega}=\frac{2\omega}{1-\omega^2}\quad,\quad \Gamma^\omega_{\theta\theta}=-\omega^2\Big(\frac{1}{\omega}+\frac{2\omega}{1-\omega^2}\Big)\quad,\quad \Gamma^\theta_{\omega\theta}=\Gamma^\theta_{\theta\omega}=\frac{1}{\omega}+\frac{2\omega}{1-\omega^2}$$

$$(\mathrm{grad}\Phi)^\omega\approx\frac{(1-\omega^2)^2}{4}\partial_\omega\Phi\ ,\quad (\mathrm{grad}\Phi)^\theta\approx\frac{(1-\omega^2)^2}{4\omega^2}\partial_\theta\Phi\ ,\quad \Phi(\omega,\theta)=\frac{1}{2}V^2(\omega,\theta)$$

The cosmological equations become:

$$\ddot\omega+\Gamma^\omega_{\omega\omega}\dot\omega^2+\Gamma^\omega_{\theta\theta}\dot\theta^2+H\dot\omega+(\mathrm{grad}\Phi)^\omega=0\ ,$$
$$\ddot\theta+2\Gamma^\theta_{\omega\theta}\dot\omega\dot\theta+H\dot\theta+(\mathrm{grad}\Phi)^\theta=0\ .$$

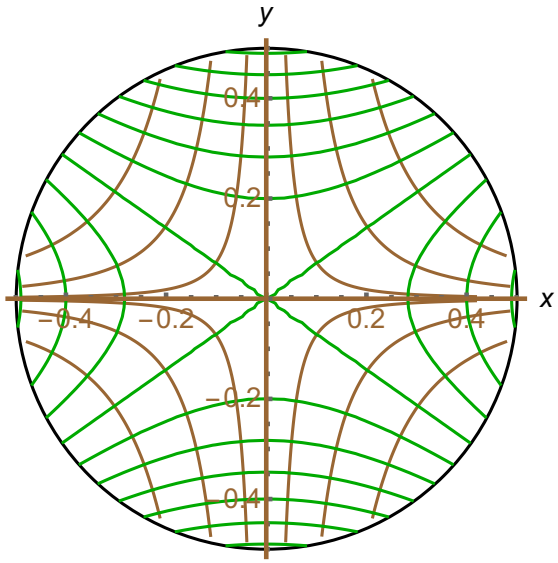
IR optimal initial conditions:

$$\dot\omega(0)=-(\mathrm{grad}_{\mathcal{G}}V)\omega(0)\quad,\quad \dot\theta(0)=-(\mathrm{grad}_{\mathcal{G}}V)\theta(0)$$

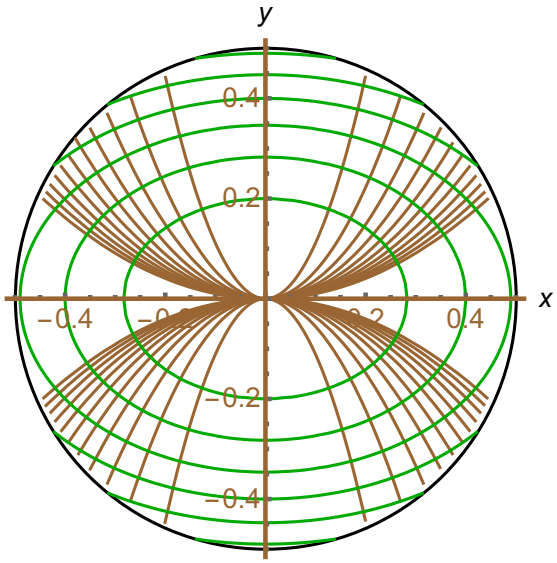
The gradient flow equation gives the general solution:

$$\boxed{\omega=C\frac{|\sin(\theta)|^{\frac{\beta(c)}{1-\beta(c)}}}{|\cos(\theta)|^{\frac{1}{1-\beta(c)}}}\ ,\ \text{for } \theta\notin\{0,\tfrac{\pi}{2},\pi,\tfrac{3\pi}{2}\},}$$ with $C>0$ an integration const.

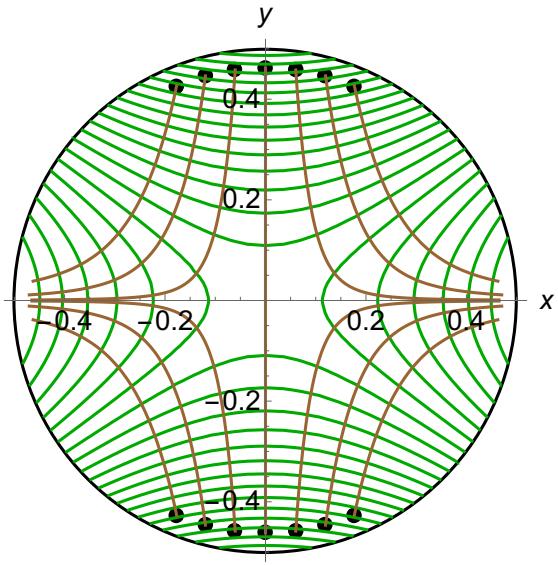
The IR behavior near an interior critical point



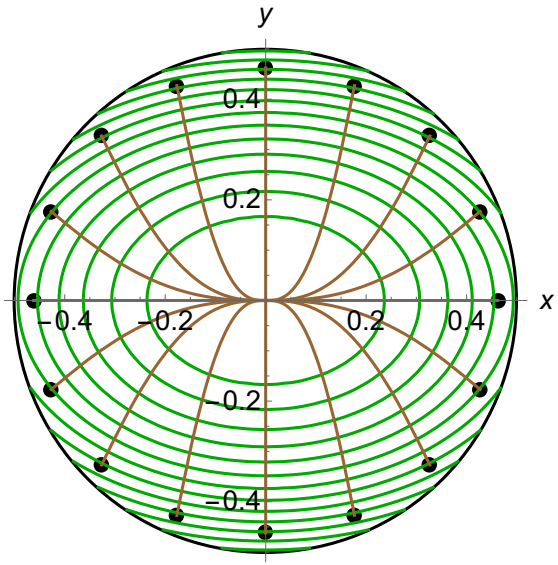
() $\beta(c) = -0.5$, saddle point of V.



() $\beta(c)$=0.5, local extremum of V.



() $\beta(c) = -0.5$, saddle point of V.
The dots are initial points $\varphi(0)$.



() $\beta(c)$=0.5, local extremum of V.
The dots are initial points $\varphi(0)$.

Conclusions

- We studied the first order IR behavior of "tame" hyperbolizable two-field cosmological models by analyzing the asymptotic form of the gradient flow orbits of the classical effective scalar potential V near critical points and ends of Σ and showed that it is characterized by a finite set of parameters associated to these critical points and ends.
- We found particularly interesting behavior near cusp ends, around which generic cosmological trajectories tend to spiral a large number of times before either "falling into the cusp" or being "repelled" back toward the compact core of Σ. In particular, cusp ends lead naturally to "fast turn" behavior of cosmological curves.
- Comparing with numerical computations, we found that the first order IR approximation is already quite good for all interior critical points and all ends except for cusps, for which one must consider higher order corrections in the IR expansion in order to obtain better approximation.
- Our results characterize the IR universality classes of all tame hyperbolizable two-field models in terms of geometric data extracted from the asymptotic behavior of the effective scalar potential and from the uniformizing metric.

Ongoing and further research

- IR expansion in higher ordes (expecially for treating the cusp ends behaviour)
- mean field aproximations (angular approximation, adapted approximation etc) for tame 2-field hyperbolic models

References

In previous work we initiated a geometric study of the classical dynamics of multifield cosmological models.

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