Quasinormal modes of Black Holes

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Introduction

"...we may expect that any intial perturbation will, during its last stages, decay in a manner characteristic of the black hole itself and independent of the cause. In other words, we may expect that during these last stages, the black hole emits gravitational waves with frequencies and rates of damping that are characteristic of the black hole itself, in the manner of a bell sounding its last dying notes."

Chandrasekhar, 1982

Introduction

- Gravitational wave astronomy is now firmly established, with many detections since 2015.
- Massive bodies in motion can excite gravitational waves
- The orbit of binary black hole systems decays over time
- After coalescence the new black hole ‘rings down’.
    - Wave consists of a discrete sum of damped harmonic terms
    - These “quasinormal frequencies” are characteristic of the black hole.

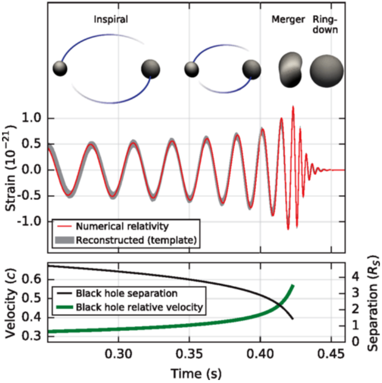


Figure: Waveform from a black hole coalescence [LIGO Scientific Collaboration and Virgo Collaboration Phys. Rev. Lett. **116**, 061102]

Introduction

Event	Redshifted frequency [Hz]			Redshifted damping time [ms]		
	IMR	DS	pSEOB	IMR	DS	pSEOB
GW150914	248^{+8}_{-7}	247^{+14}_{-16}	–	$4.2^{+0.3}_{-0.2}$	$4.8^{+3.7}_{-1.9}$	–
GW170104	287^{+15}_{-25}	228^{+71}_{-102}	–	$3.5^{+0.4}_{-0.3}$	$3.6^{+36.2}_{-2.1}$	–
GW170814	293^{+11}_{-14}	527^{+340}_{-332}	–	$3.7^{+0.3}_{-0.2}$	$25.1^{+22.2}_{-19.0}$	–
GW170823	197^{+17}_{-17}	222^{+664}_{-62}	–	$5.5^{+1.0}_{-0.8}$	$13.4^{+31.8}_{-9.8}$	–
GW190408_181802	319^{+11}_{-20}	504^{+479}_{-459}	–	$3.2^{+0.3}_{-0.3}$	$10.0^{+32.5}_{-8.9}$	–
GW190421_213856	162^{+13}_{-14}	–	171^{+50}_{-16}	$6.3^{+1.2}_{-0.8}$	–	$8.5^{+5.3}_{-4.2}$
GW190503_185404	190^{+17}_{-15}	–	265^{+501}_{-79}	$5.3^{+0.8}_{-0.8}$	–	$3.5^{+3.4}_{-1.8}$
GW190512_180714	382^{+32}_{-42}	220^{+686}_{-42}	–	$2.6^{+0.2}_{-0.2}$	$26.1^{+21.3}_{-22.9}$	–
GW190513_205428	242^{+25}_{-27}	250^{+493}_{-88}	–	$4.3^{+1.2}_{-0.4}$	$5.3^{+19.2}_{-3.8}$	–
GW190519_153544	127^{+10}_{-9}	123^{+11}_{-19}	124^{+12}_{-13}	$9.7^{+1.7}_{-1.6}$	$9.7^{+9.0}_{-3.8}$	$10.3^{+3.6}_{-3.1}$
GW190521	68^{+3}_{-4}	65^{+3}_{-3}	67^{+2}_{-2}	$16.0^{+4.0}_{-2.5}$	$22.1^{+12.4}_{-7.4}$	$30.7^{+7.7}_{-7.4}$
GW190521_074359	198^{+7}_{-8}	197^{+15}_{-15}	205^{+15}_{-12}	$5.4^{+0.4}_{-0.4}$	$7.7^{+6.4}_{-3.3}$	$5.3^{+1.5}_{-1.2}$
GW190602_175927	105^{+10}_{-9}	93^{+13}_{-22}	99^{+15}_{-15}	$10.2^{+2.0}_{-1.5}$	$10.0^{+17.2}_{-4.5}$	$8.8^{+5.4}_{-3.6}$
GW190706_222641	109^{+11}_{-10}	109^{+7}_{-12}	112^{+7}_{-8}	$11.3^{+2.3}_{-2.3}$	$20.4^{+25.2}_{-12.9}$	$19.4^{+7.2}_{-8.9}$
GW190708_232457	497^{+10}_{-46}	642^{+279}_{-596}	–	$2.1^{+0.2}_{-0.1}$	$24.6^{+23.0}_{-22.6}$	–
GW190727_060333	178^{+17}_{-16}	345^{+587}_{-267}	201^{+11}_{-21}	$6.2^{+1.1}_{-0.8}$	$21.1^{+25.6}_{-17.9}$	$15.4^{+5.3}_{-6.1}$
GW190828_063405	239^{+10}_{-11}	247^{+350}_{-15}	–	$4.8^{+0.6}_{-0.5}$	$17.3^{+25.3}_{-10.4}$	–
GW190910_112807	177^{+8}_{-8}	166^{+9}_{-8}	174^{+12}_{-8}	$5.9^{+0.9}_{-0.5}$	$13.2^{+17.1}_{-6.2}$	$9.5^{+3.1}_{-2.7}$
GW190915_235702	232^{+13}_{-18}	534^{+371}_{-493}	–	$4.6^{+0.7}_{-0.6}$	$15.0^{+30.1}_{-13.1}$	–

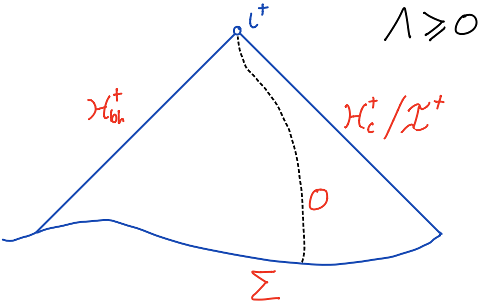
Figure: Estimated ringdown parameters

[The LIGO Scientific Collaboration and the Virgo Collaboration—Tests of General Relativity with Binary Black Holes from the second LIGO–Virgo Gravitational-Wave Transient Catalog, 2010.14529]

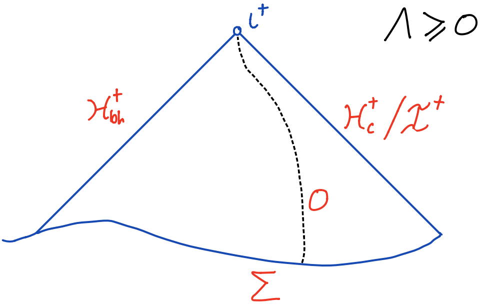
Introduction

- Black holes settle down to equilibrium by producing radiation at fixed (complex) frequencies: *quasinormal ringdown*
- Frequencies are characteristic of the spacetime, carry geometric information
- Recently there has been a lot of work by mathematicians to understand this phenomenon
- Literature on quasinormal modes is vast: I will focus only on a portion
  - See 1910.08479 for a survey
- Restrict attention to $\Lambda \geqslant 0$

The set-up of the problem



- Pose data on Σ and solve to the future. What does the observer O see at late times?
- Could consider:
  - Cauchy data for full vacuum Einstein equations
  - Cauchy data for linearised vacuum Einstein equations about BH
  - Cauchy data for wave/KG equation about BH



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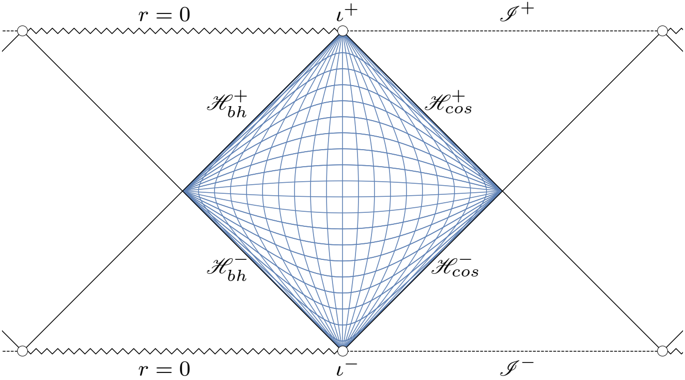
Schwarzschild de Sitter



$$g = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} + r^2 d\Omega_{S^2}$$

Usual definition

- Separate variables:

$$\psi(t, r, \theta, \varphi) = e^{st} Y_{lm}(\theta, \varphi) R_{sl}(r)$$

  satisfies $\Box\psi = 0$ iff:

$$-\frac{d^2}{dr_*^2} R_{sl} + \left[s^2 + V_l\right] R_{sl} = 0, \qquad -\infty < r_* < \infty$$

- The potential V_l decays exponentially as $|r_*| \to \infty$.
- Seek a solution satisfying *outgoing* boundary conditions:

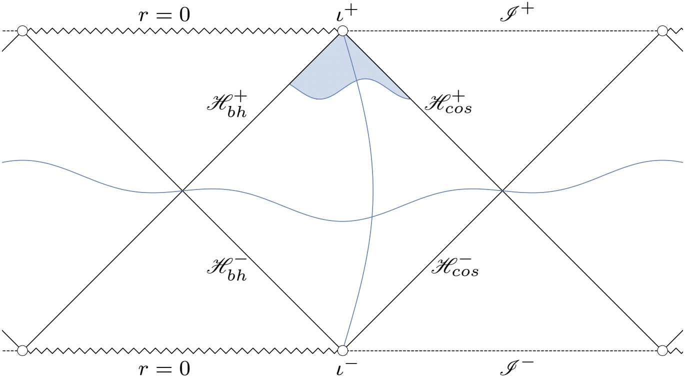
$$R \sim \begin{cases} e^{-sr_*} & r_* \to \infty \\ e^{sr_*} & r_* \to -\infty \end{cases}$$

- Such solutions only occur for a discrete set of $s \in \{\Re z \leq 0\}$, the *quasinormal frequencies*.

Comments

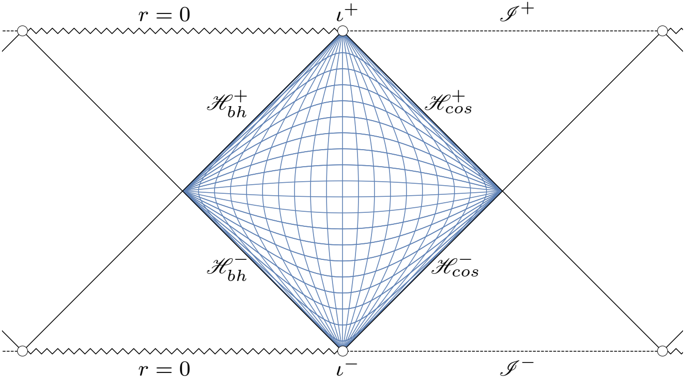
- Expect ψ at late times to decompose as a sum of quasinormal modes
- Difficult in practice to set exponentially decaying branch to zero:
  - "What does $\sim$ mean".
- Looks like an eigenvalue problem, but it isn't!
- Relies on separability of wave equation.
  - Not always possible, particularly for AdS black holes.
- Can make rigorous, but requires an analytic continuation argument!
  - Sá Baretto–Zworski, '97
  - Method of complex scaling / perfectly matched layers

Schwarzschild de Sitter



Region of interest

Schwarzschild de Sitter



$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) dt^2 + \frac{dr^2}{1-\frac{2m}{r}-\frac{\Lambda}{3}r^2} + r^2 d\Omega_{S^2}$$

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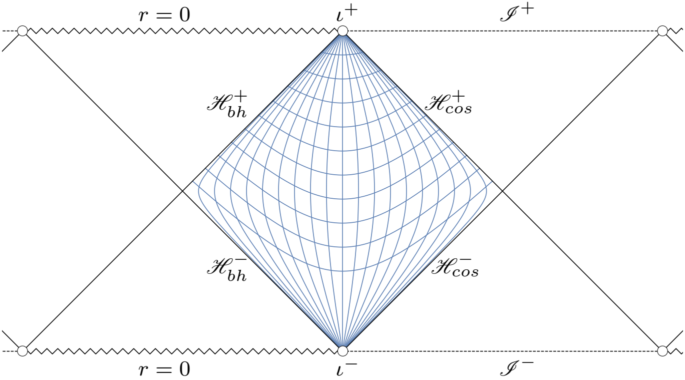
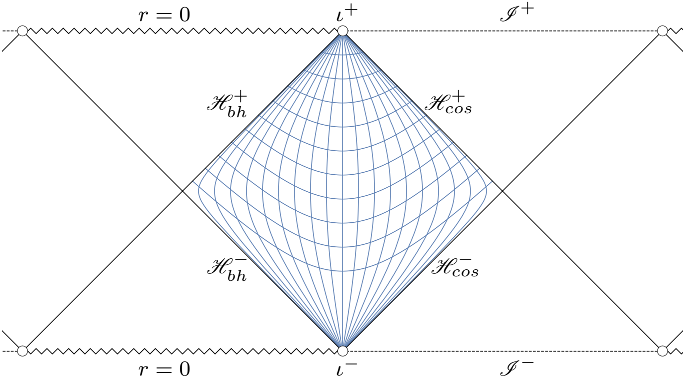


Pick a spacelike surface Σ_0 and push forward by the timelike isometry to give a foliation $\{\Sigma_\tau\}_{\tau \geq 0}$

Schwarzschild de Sitter



[cf B Schmidt, '93]

Semigroups and QNM

- Evolution under the wave equation gives a family of maps $\mathcal{S}[\tau]$ acting on pairs of functions $\psi_i : \Sigma \to \mathbb{C}$:

$$\mathcal{S}[\tau](\psi_0, \psi_1) = (\psi, \partial_\tau \psi)|_{\Sigma_\tau}$$

  where ψ is the solution of:

$$\Box\psi = 0, \qquad \psi|_{\Sigma_0} = \psi_0, \quad \partial_\tau \psi|_{\Sigma_0} = \psi_1.$$

- $\mathcal{S}[\tau]$ has the semigroup property:

$$\mathcal{S}[0] = Id, \qquad \mathcal{S}[\tau_1 + \tau_2] = \mathcal{S}[\tau_1]\mathcal{S}[\tau_2], \quad \tau_1, \tau_2 \geqslant 0$$

- We say that the wave equation *propagates* a Hilbert space H if
  - $\mathcal{S}[\tau] : H \to H$
  - $\lim_{\tau \to 0} \|S[\tau]x - x\| = 0$ for all $x \in H$.

Semigroups and QNM

- Suppose

$$H = \left\{ (\psi_0, \psi_1) : \Sigma \to \mathbb{C}^2 \,\middle|\, \int_\Sigma \left(|\partial \psi_0|^2 + |\psi_0|^2 + |\psi_1|^2 \right) dS < \infty \right\}$$

Then H is propagated by the wave/KG equation.

  - Follows from energy estimates, together with an approximation argument.

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  Then H is propagated by the wave/KG equation.
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- In fact, for $k \in \mathbb{N}$:

$$H = H^k(\Sigma) = \left\{ (\psi_0, \psi_1) : \Sigma \to \mathbb{C}^2 \,\middle|\, \int_\Sigma \left(\sum_{i \leq k} \left| \partial^i \psi_0 \right|^2 + \sum_{i \leq k-1} \left| \partial^i \psi_1 \right|^2 \right) dS < \infty \right\}$$

  is also propagated by the wave/KG equation.

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is also propagated by the wave/KG equation.

- $\mathcal{S}[\tau]$ defines a C^0-*semigroup* on $H^k(\Sigma)$, which can be written:

$$\mathcal{S}[\tau] = e^{\tau \mathcal{A}}$$

for an unbounded operator $\mathcal{A}$ called the generator.

Semigroups and QNM

Theorem (Vasy '10; W '13; Gannot '16)

For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a discrete pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity (minimal if more than one horizon). There are countably many eigenvalues, which do not accumulate at any point.

- Get a representation formula for solutions:

$$\mathbf{\Psi}(t) = \frac{1}{2\pi i}\int_\gamma e^{z\tau}\left(\mathcal{A} - z\right)^{-1}\mathbf{\Psi_0}dz$$

- No need to separate variables or perform analytic continuation
- Same methods immediately apply to Dirac, Maxwell, grav. perturbations etc.
- In the case of AdS black holes, boundary conditions are required at $\mathscr{I}$.
- Proof crucially makes use of *redshift effect* [Dafermos–Rodnianski '05]

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- Every 'outgoing' quasinormal frequency corresponds to an eigenvalue of $\mathcal{A}$ in $\{\Re s > -\varkappa\left(k - \frac{1}{2}\right)\}$ for k large enough
- Not every eigenvalue of $\mathcal{A}$ corresponds to an 'outgoing' quasinormal frequency
  - Bizoń, Chmaj, Mach '20
  - Hintz, Xie '21
- Additional frequencies correspond to modes that are only excited by non-trivial data at horizons
- $\mathcal{A}$ is *not* self adjoint. $\mathcal{A}^\dagger$ has the same eigenvalues, but eigenvectors are 'co-modes'.

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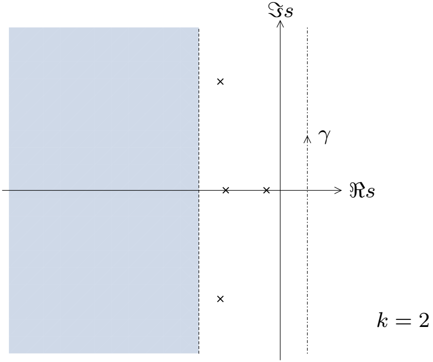

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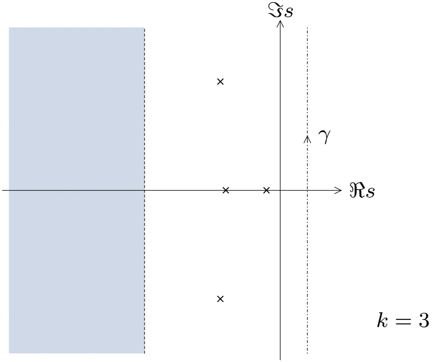

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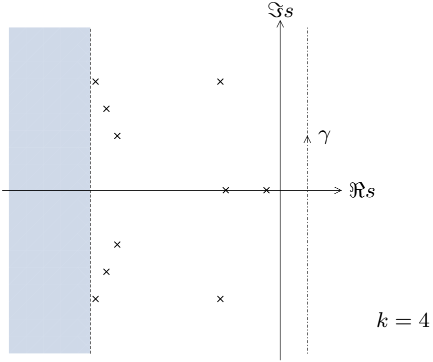

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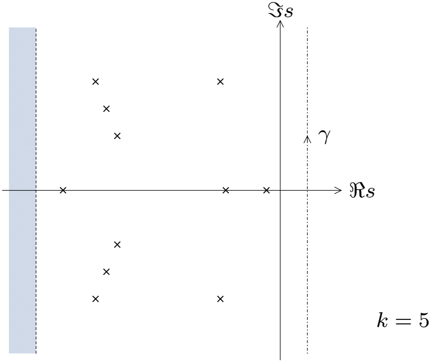


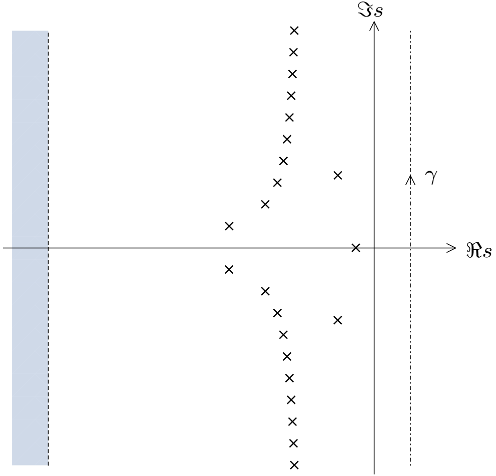
Spectral gaps and expansions

- We can try and shift the contour in the representation formula above.
- If this is possible, get an *asymptotic expansion*: for some $a_i \in \mathbb{C}$

$$\psi(x,\tau) = \sum_{s_i \in \Lambda, \Re s_i > -\nu} a_i u_i(x) e^{s_i \tau} + \mathcal{O}\left(e^{-\nu\tau}\right)$$

- Fundamental obstruction to shifting the contour: *trapped null geodesics*.
- Provided trapping is *normally hyperbolic* (e.g. photon sphere), can shift the contour. [Vasy '10; Dyatlov '14]
- Get a strip near imaginary axis containing only finitely many QNF

Spectral gaps and expansions



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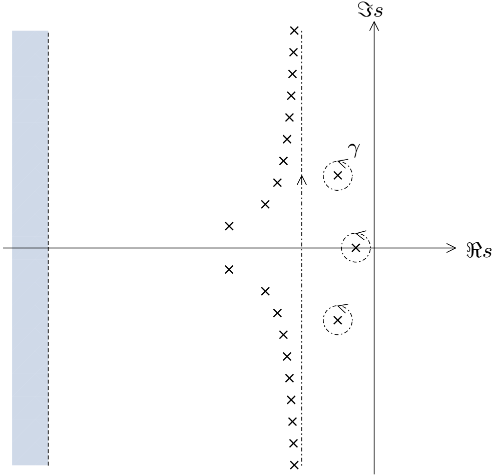


Incompleteness of the quasinormal spectrum

- Since the spectrum is discrete, might hope that any (smooth) solution can be expanded in QNM:

$$\psi(x,\tau) \stackrel{?}{=} \sum_{i=1}^{\infty} a_i u_i(x) e^{s_i \tau}, \quad a_i \in \mathbb{C}$$

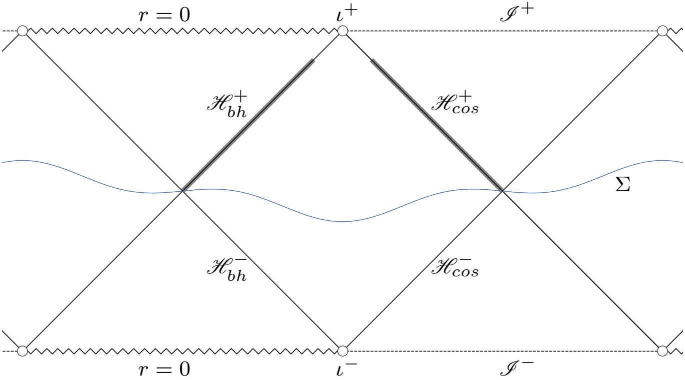
- This is *not* true.

Lemma

On a black hole background, there exist non-trivial solutions of the wave equation which vanish in a neighbourhood of ι^+.

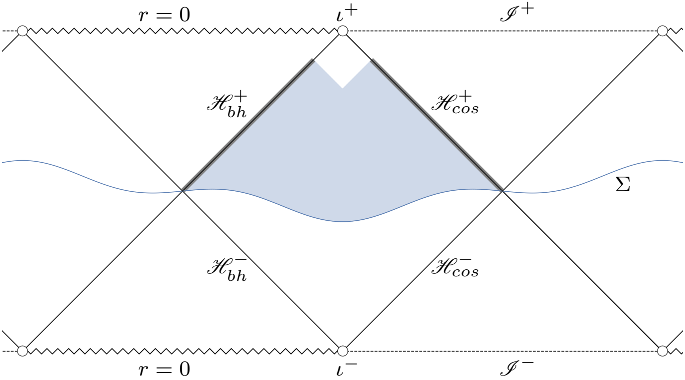
- Such a solution has a quasinormal expansion which is identically zero.
- Construction similar to backwards scattering construction of [Dafermos–Holzegel–Rodnianski '13; Dafermos–Rodnianski–Shlapentokh-Rothman '14]
- Set of such solutions is infinite dimensional.

Incompleteness of the quasinormal spectrum



Specify data on $\mathscr{H}^+_{bh} \cup \mathscr{H}^+_{cos}$, vanishing near ι^+.
Solve wave equation backwards

Incompleteness of the quasinormal spectrum



Solution induces data on Σ which evolve into a solution vanishing near ι^+.

What do we know about distribution of QNFs?

- For many black holes, we can understand the high-frequency distribution ($|\Im s| \gg 0$) of QNFs
  - Key feature: geometry of 'trapped' photon orbits [Vasy '10, Dyatlov '14]
  - Can use WKBJ methods
- For near extremal black holes, or small black holes, a sequence of 'zero-damped modes' emerges [Hintz-Xie '21, Joykutty '21]

Conclusion

- For sub-extremal black holes with $\Lambda \neq 0$, quasinormal modes can be defined in a mathematically satisfactory way as solutions of an eigenvalue problem
- Quasinormal modes do not form a complete basis
- We understand various aspects of the distribution of quasinormal frequencies