Singular limits of hyperbolic PDE and Effective Field Theories

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Introduction

- Often in physical problems, we disregard degrees of freedom whose typical energy scale is very high.
  - When modelling a pendulum we ignore vibrational modes of the rod
  - Can treat a fluid with low compressibility as incompressible
- We are also often presented with situations where we expect our description cannot be valid at all energies
  - Fermi's four-fermion theory of beta decay
  - Gravitational field
- Effective field theory is a systematic way of describing low-energy phenomena while retaining some information about the high-energy degrees of freedom.
  - [Burgess '04, '20; Flanagan-Wald '96]

Introduction: a motivating example

- Consider the Einstein–Hilbert action for a Lorentzian metric g:

$$S[g] = \int_{\mathcal{M}} R_g \, \mathrm{dvol}_g$$

- Requiring that S is stationary with respect to variations in g gives the vacuum Einstein equations:

$$0 = \frac{d}{dt} S[g + \lambda \delta g]\Big|_{\lambda=0} \quad \Longrightarrow \quad Ric_g = 0.$$

- High-energy physics predicts that the action is corrected by a series of terms involving higher derivatives, eg:

$$S[g] = \int_{\mathcal{M}} \left(R_g + \frac{k}{M^2} [R_g^2 - 4 Ric_g^2 + Riem_g^2] + \ldots \right) \mathrm{dvol}_g$$

- Here $M \gg 0$ is a large parameter.
- How should we approach solving such a problem?

A model PDE problem

- Consider the wave equation for a complex scalar $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$

$$\Box_g \phi + \frac{M^2}{2}\phi(|\phi|^2 - 1) = 0$$

  $M > 0$ constant.
- $\Box_g$ is the (hyperbolic) Laplace-Beltrami operator associated to the Lorentzian metric

$$g = -dt^2 + \delta_{ij} dx^i dx^j$$

- Admits a conserved energy

$$E = \frac{1}{2}\int_{\{t=const.\}} \left((\partial_t \phi)^2 + \delta^{ij}\partial_j \phi \partial_j \phi + \frac{M^2}{4}(|\phi|^2 - 1)^2 \right)$$

- Global smooth solutions for $n \leqslant 3$.
- Want to understand the limit $M \to \infty$ while E remains finite and energy does not concentrate.
- Expect $|\phi| \to 1$, so the *phase* of ϕ is the only remaining degree of freedom - radial degree of freedom becomes infinitely 'stiff'.

The EFT expansion

- Write $\phi = e^{\rho+i\theta}$. Obtain coupled equations (the UV equations)

$$-\Box\rho + M^2\rho = \partial_\mu\rho\partial^\mu\rho - \partial_\mu\theta\partial^\mu\theta - M^2\rho^2 W(\rho) \tag{1}$$

$$-\Box\theta = 2\partial^\mu\rho\partial_\mu\theta. \tag{2}$$

where $W(\rho) := (e^{2\rho} - 1 - 2\rho)/(2\rho^2)$
$\partial_\mu\theta\partial^\mu\rho = -\partial_t\theta\partial_t\rho + \nabla\theta\cdot\nabla\rho$, etc.

- Crudely, from finite energy expect $\rho \sim M^{-1}$

- At top order, θ solves the wave equation.

- Idea:
    1. 'solve' (1) for ρ in terms of θ, perturbatively in M^{-1}
    2. Insert $\rho = \rho[\theta]$ into (2) to get an equation involving only θ
    3. Treat this equation as a corrected equation for θ at higher order in M^{-1}.

The EFT expansion

- We can rewrite (1) as

$$\rho = -\frac{\partial_\mu \theta \partial^\mu \theta}{M^2} + \frac{\Box \rho}{M^2} + \frac{\partial_\mu \rho \partial^\mu \rho}{M^2} - \rho^2 W(\rho)$$

and formally solve iteratively to find

$$\rho = -\frac{\partial_\mu \theta \partial^\mu \theta}{M^2} - \frac{\Box(\partial_\mu \theta \partial^\mu \theta) + (\partial_\mu \theta \partial^\mu \theta)^2}{M^4} + O(M^{-6})$$

- We obtain our EFT equation for θ:

$$-\Box \theta = 2\partial_\nu \theta \partial^\nu \left(-\frac{\partial_\mu \theta \partial^\mu \theta}{M^2} - \frac{\Box(\partial_\mu \theta \partial^\mu \theta) + (\partial_\mu \theta \partial^\mu \theta)^2}{M^4} \right) + O(M^{-6})$$

EFT: the bottom-up approach

$$-\Box\theta = 2\partial_\nu\theta\partial^\nu\left(-\frac{\partial_\mu\theta\partial^\mu\theta}{M^2} - \frac{\Box(\partial_\mu\theta\partial^\mu\theta) + (\partial_\mu\theta\partial^\mu\theta)^2}{M^4}\right) + O(M^{-6})$$

- Suppose we wished to guess the form of this EFT equation. Our original problem had the following symmetries:
  - Poincaré invariance (translations, rotations, boosts of spacetime coordinates)
  - $\theta \to \theta + c$
  - $\theta \to -\theta$
  - $x^\mu \to \lambda x^\mu$ and $M \to \lambda^{-1} M$
- Any EFT equation for θ consistent with these symmetries may be written in the form

$$-\Box\theta = 2\partial_\nu\theta\partial^\nu\left(\frac{a\partial_\mu\theta\partial^\mu\theta}{M^2} + \frac{b\Box(\partial_\mu\theta\partial^\mu\theta) + c(\partial_\mu\theta\partial^\mu\theta)^2}{M^4}\right) + O(M^{-6})$$

  for constants a, b, c.
- Can also work at the level of a Lagrangian.

Questions

A) Does a class of solutions to (1), (2) exist for which the derivation of the EFT can be justified?

B) Can we meaningfully solve the EFT for θ directly?
- Truncating at finite order gives a PDE
- Typically higher than second order - need additional initial data?

C) Can we approximate a solution of the original problem by a solution of the EFT?

D) Does every solution to the EFT necessarily arise as a limit of the original problem?

[From now on, assume we solve on the domain $[0, T] \times \mathbb{T}^n$]

Question A

A) Does a class of solutions to (1), (2) exist for which the derivation of the EFT can be justified?

Short answer: Yes

- Provided the expansion for ρ holds at time 0, it will continue to hold for a time T independent of M

Question A

A) Does a class of solutions to (1), (2) exist for which the derivation of the EFT can be justified?

Theorem (Long answer A)

Suppose $\theta|_{\{t=0\}}$, $\partial_t\theta|_{\{t=0\}}$ *are given and smooth, and suppose that the expansion*

$$\partial_t^j \rho\Big|_{\{t=0\}} = \partial_t^j \left(-\frac{\partial_\mu\theta\partial^\mu\theta}{M^2} - \frac{\Box(\partial_\mu\theta\partial^\mu\theta) + (\partial_\mu\theta\partial^\mu\theta)^2}{M^4} + \dots \right)\Bigg|_{\{t=0\}}$$

holds to order $l - j$ *for* $j \leqslant 1$. *Then there exists* $T > 0$, *independent of* M, *such that a solution to (1), (2) exists for* $t \in [0, T]$, *and in that interval the expansion*

$$-\Box\theta = 2\partial_\nu\theta\partial^\nu \left(-\frac{\partial_\mu\theta\partial^\mu\theta}{M^2} - \frac{\Box(\partial_\mu\theta\partial^\mu\theta) + (\partial_\mu\theta\partial^\mu\theta)^2}{M^4} + \dots \right)$$

holds to order $l - 1$.

Question A

- Use (2) to relate $\partial_t^j \theta|_{\{t=0\}}$ to $\theta|_{\{t=0\}}$, $\partial_t \theta|_{\{t=0\}}$.
- We say $f = f_0 + M^{-1} f_1 + \ldots$ holds to order l if for each k there exists C_k such that

$$\left|\left|f - f_0 + M^{-1} f_1 - \ldots - M^{-l} f_l\right|\right|_{H^k} \leq M^{-l-1} C_k.$$

  (Spatial derivatives only)
- Key observation is that the derivation of the EFT relied on $M\rho$ *and its derivatives* being bounded
- Closely related to theory of singular limits of symmetric hyperbolic systems developed in '80s
  - [Kreiss '80; Klainerman–Majda '81; Browning–Kreiss '82]

Question B

B) Can we meaningfully solve the EFT for θ directly?

- Truncating at finite order gives a PDE
- Typically higher than second order - need additional initial data?

Short answer: Yes

- We can assign a meaning to solving the EFT to order l such that the problem is 'well posed'.

Question B

B) Can we meaningfully solve the EFT for θ directly?

Definition

Given an EFT in the form

$$-\Box\theta = M^{-1}\mathcal{F}_1[\theta] + M^{-2}\mathcal{F}_2[\theta] + \ldots, \tag{3}$$

where $\mathcal{F}_k[\theta]$ is a polynomial in θ and finitely many derivatives, independent of M.

We say ϑ is an EFT solution of (3) to order l on $[0, T]$ if ϑ and all derivatives are bounded uniformly in M, and

$$-\Box\vartheta = M^{-1}\mathcal{F}_1[\vartheta] + M^{-2}\mathcal{F}_2[\vartheta] + \ldots + M^{-l}\mathcal{F}_l[\vartheta] + M^{-l-1}R_l$$

where R_l and all its derivatives are bounded uniformly in M.

Question B

B) Can we meaningfully solve the EFT for θ directly?

Theorem (Long answer B)

Given Cauchy data for at $t = 0$, independent of M, there exists an EFT solution to (3) to order l.

EFT Solutions are unique in the sense that if ϑ_1, ϑ_2 are two solutions with the same Cauchy data, then

$$\vartheta_1 = \vartheta_2 + \frac{\delta\vartheta}{M^{l+1}}$$

where $\delta\vartheta$ and all of its derivatives are bounded uniformly in M.

The EFT solution depends continuously on the initial data in an appropriate topology.

Question B

- Existence: make an ansatz $\vartheta = \vartheta_0 + M^{-1}\vartheta_1 + \ldots$ and solve a sequence of linear wave equations.
- Uniqueness: look at equation satisfied by $\delta\vartheta$ and iteratively improve bound in powers of M.
- Uniqueness statement is the strongest possible given the expansion.
- Uniform boundedness assumption in definition crucial to uniqueness result.

Question C

C) Can we approximate a solution of the original problem by a solution of the EFT?

Short answer: Yes

- Provided the solution to the original problem obeys the EFT expansion to a certain order.

Question C

C) Can we approximate a solution of the original problem by a solution of the EFT?

Theorem (Long answer C)

Suppose we are given a solution to (1), (2) for $t \in [0, T]$, and in that interval the expansion

$$-\Box\theta = 2\partial_\nu\theta\partial^\nu\left(-\frac{\partial_\mu\theta\partial^\mu\theta}{M^2} - \frac{\Box(\partial_\mu\theta\partial^\mu\theta) + (\partial_\mu\theta\partial^\mu\theta)^2}{M^4} + \dots\right) \qquad (4)$$

holds to order $l - 1$.
Suppose ϑ is an EFT solution of (4) to order l, and that ϑ has the same initial Cauchy data as θ. Then there exists C, independent of M, such that

$$||\theta - \vartheta||_{L^\infty} \leqslant \frac{C}{M^{l+1}}$$

Question D

D) Does every solution to the EFT necessarily arise as a limit of the original problem?

Short (and long) answer: Yes

- This follows from answers A) B) and C) above.

Two further questions

E) Suppose we solve the UV equations (1), (2) with initial data which does *not* respect the EFT expansion. Can anything be said?

F) What about long timescales?

Two further questions

E) Suppose we solve the UV equations (1), (2) with initial data which does *not* respect the EFT expansion. Can anything be said?

Theorem

Suppose θ, ρ *solve the UV equations (1), (2) with* $\theta|_{t=0} = \theta_0, \partial_t\theta|_{t=0} = \theta_1$*,* $\rho_{t=0} = \rho_0, \partial_t\rho|_{t=0} = \rho_0$*. Let* $\hat{\theta}$ *be the solution to the initial value problem*

$$-\Box\hat{\theta} = -\frac{6}{M^2}\partial_\mu\left(\varepsilon\partial^\mu\hat{\theta}\right) - \frac{2}{M^2}\partial^\mu\left(\partial_\nu\hat{\theta}\partial^\nu\hat{\theta}\partial_\mu\hat{\theta}\right),$$

$$\hat{\theta}|_{t=0} = \theta_0 - \frac{2}{M^2}\rho_1\theta_1,$$

$$\partial_0\hat{\theta}|_{t=0} = \theta_1\left(1 + 2\rho_0 + 2\rho_0^2 + \frac{2}{M^2}\left(\partial_i\theta_0\partial^i\theta_0 - \theta_1^2\right) + \frac{6}{M^2}\varepsilon\right) + \frac{2}{M^2}\partial_i(\rho_1\partial_i\theta_0),$$

where $\varepsilon = \frac{1}{2}(\rho_1^2 + M^2\rho_0^2)$*.*

Then $\theta - \hat{\theta} = O(M^{-3})$ *in the sense of distributions.*

What about long timescales?

- Our methods can be extended to permit $T \propto M^\mu$ for $\mu < 2$.
- In 2208.09194, Kadar considers a similar but different theory posed on $\mathbb{R} \times \mathbb{R}^3$:
$$(\Box - 1)U = UV, \qquad (\Box - M^2)V = U^2/2$$
- Establishes
  - Global existence for initial data which is $O(1)$ in an *EFT* norm (in particular, U can be $O(1)$).
  - Scattered states are well approximated by an EFT approach.

Conclusions

- Effective field theory allows us to include the effects of high energy physics when modelling low energy phenomena
- Applied to an example geometric PDE problem this gives rise to an expansion involving higher derivatives
- We can meaningfully solve this expansion
- Certain true solutions can be well approximated by this process

Thank-you!