Hamiltonian systems on almost cosymplectic manifolds

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DFT, Magurele-Bucharest, Romania, until 02.02.2022

IFIN-HH, March 30, 2023
Magurele

Outline

$G_n^J(\mathbb{R})$ - real Jacobi group of degree n; G_n^J, Mathematics, Math Phys, Theoretical Phys, applications in Quantum Optics, squeezed states, teleportation...
$\mathcal{X}_n^J$– Siegel-Jacobi upper half space; $\mathcal{D}_n^J$– S-J ball; $\tilde{\mathcal{X}}_n^J$ extended S-J u h s.
S. B, *The real Jacobi group revisited*, SIGMA **15** (2019) 096, 50 pages; arXiv:1903.1072 [math.DG], v1, 93 pages; v2, 54 pages:
Technique to calculate equations of classical and quantum motion on a homogenous manifold $M = G/H$ attached to a linear Hamiltonian in the generators of the group G, $+$ determination of the Berry phase (Gheorghe Cezar). CS manifolds, i.e. *Kähler manifolds* for which the generators of group G admits a realisation as first order holomorphic differential operators with polynomial coefficients
Question: *how can we distinguish between the different invariant metrics obtained on* $\mathcal{X}_1^J$ and $\tilde{\mathcal{X}}_1^J$?
Answer: *we underline the differences between equations of motion on* $\mathcal{X}_1^J$ *and* $\tilde{\mathcal{X}}_1^J$. Technique: *find the equations of motion on odd dimensional manifolds* M_{2n+1} with *almost cosymplectic structure* (ACOS) in the meaning of Paulette Libermann. This talk is based on JGP 2023, S. B.

Balanced metric

• Donaldson 2001; Arezzo & Loi 2004

ω: G-invariant Kähler two-form on homogeneous manifold $M = G/H$

$\omega_M(z) = \mathrm{i}\sum_{\alpha,\beta=1}^n h_{\alpha\bar{\beta}}(z)\,\mathrm{d}\,z_\alpha \wedge \mathrm{d}\,\bar{z}_\beta$, $h_{\alpha\bar{\beta}} = \bar{h}_{\beta\bar{\alpha}} = h_{\bar{\beta}\alpha}$, $h_{\alpha\bar{\beta}} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta}$.

A Kählerian metric: $\mathrm{d}\,\omega = 0 \leftrightarrow \frac{\partial h_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial h_{\gamma\bar{\beta}}}{\partial z_\alpha}$. the balanced hermitian metric corresponds to the Ka potential $f(z,\bar{z}) = \ln K(z,\bar{z}) = (e_{\bar{z}}, e_{\bar{z}})$, $e_z \in \mathfrak{H}$ - Perelomov's CS vectors, $z \in M$.

• Berezin quantization of Kähler manifolds 1973-1975: $\mathbb{C}^n$, classical bounded domains (=Hermitian symmetric spaces);

Loi & Mossa: Berezin quantization on homogeneous bounded domains , ...

homogeneous Kähler manifolds 2015

• ϵ-function: Rawnsley, Cahen, Gutt; globalization of Berezin's construction to non-homogeneous Kähler manifolds;

J.-H. Yang

S. Berceanu, SIGMA, 2016

Geometric structures on odd dimensional manifolds

$$
\begin{array}{ccccccccc}
\mathrm{COS} & \subset & \color{red}{\mathrm{GTACOS}} & \supset & \mathrm{TACS} & & & & \\
 & & \cap & & & & & & \\
 & & \mathrm{ACOS} & & & & & & \\
 & & \cup & & & & & & \\
\mathrm{CH} & \subset & \mathrm{C} & \subset & \mathrm{ACM} & \subset & \mathrm{SAC} & \supset & \mathrm{N{=}SAS} \\
 & & & & \cap & & & & \\
 & & & & \mathrm{ACOK} & & & &
\end{array}
$$

COS: cosymplectic; ACOS: almost cosymplectic, GTACOS: generalized transitive almost cosymplectic, TACS: transitive almost contact structure, CH: contact Hamiltonian, C: contact, ACM: almost contact metric, SAC: (strict) almost contact, N: normal, SAS: Sasakian, ACOK: almost coKähler.

SH: symplectic Hamiltonian

(M, Ω), $\dim M = 2n$, $n \in \mathbb{N}$, Ω - closed non-degenerate two-form, $\Omega^n \neq 0$.

If $H : M \to \mathbb{R}$ is a Hamiltonian function, then the *Hamiltonian vector field* X_H, grad H, is the solution of the equation:
$\flat(X_H) = \mathrm{d}\, H$, where $\flat : TM \to T^*M$, $\flat(X) = X \lrcorner \Omega$.

$X \in \mathfrak{D}^1$- vector field, ω-differential form, the interior product $X \lrcorner \omega := i_X \omega$ (interior multiplication, contraction)

Darboux coordinates (q^i, p_i), $i = 1, \ldots, n$, $\Omega = \mathrm{d}\, q^i \wedge \mathrm{d}\, p_i$,
$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$,

Hamilton equations of motion:
$\dot{q}^i = \frac{\partial H}{\partial p_i} = \{q^i, H\}_P$, $\dot{p}_i = -\frac{\partial H}{\partial q^i} = \{p_i, H\}_P$,
Poisson bracket $\{f, g\}_P = \Omega(X_f, X_g) = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i}$, $f, g \in C^\infty(M)$.

ACOS, COS: almost cosymplectic, cosymplectic

Paulette : **ACOS**: – (M, θ, Ω), $\theta \in \mathfrak{D}^1$, Ω – 2-form, $\text{rank}(\Omega) = 2n$,
$\theta \wedge \Omega^n \neq 0$.
Reeb vector $R \in \mathfrak{D}^1$– solution of the equation: $R \lrcorner \Omega = 0, \quad R \lrcorner \theta = 1$.
Albert terminology : Paulette's ACOS is called *almost contact manifold*.
Albert proved: *the application* $\flat : TM \to TM^* \; X \to X^\flat = X \lrcorner \Omega + (X \lrcorner \theta)\theta$ *is a vector bundle isomorphism*.

COS : Paulette **ACOS** : – (M, θ, Ω), $+ \, \mathrm{d}\,\theta = 0, \quad \mathrm{d}\,\Omega = 0$.
the isomorphism $\flat$ for the manifold (M, η, Ω), $(z, q^i, p_i,)$, $i = 1, \dots, n$, Ω, $\eta = \mathrm{d}\,z$. $\flat(\text{grad}\, H) = \mathrm{d}\, H, X_H = \text{grad}\, H - R(H)R$, *gradient vector field* $\text{grad}\, H$, the *Hamiltonian vector field* X_H,
$\mathcal{E}_H = X_H + R$, *evolution vector field*

TACS transitive almost contact structure

Albert; Cantrijn ...: *TACS* - an

1. ACOS (M, θ, Ω)
2. $\mathrm{d}\,\Omega = 0$,
3. around every point of M there is a neighbourhood where there are local Darboux coordinates $(\kappa, q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that $\theta = \mathrm{d}\,\kappa + \epsilon p_i \,\mathrm{d}\, q^i, \quad i = 1, \ldots, n, \quad \epsilon \in \mathbb{R}$.

Albert: To a function $f \in C^\infty(M) \rightarrow X_f$ Hamiltonian vector field

$$X_f \lrcorner \theta = \epsilon f, \quad X_f \lrcorner \Omega = \mathrm{d}\, f - (Rf)\theta.$$

GTACS- generalized transitive almost contact structure

GTACOS = ACOS (M, θ, Ω) s.t. $\mathrm{d}\,\Omega = 0$
S.B. 2023

CH – Contact Hamiltonian system

Leo, Lainz; Leon, Valczar : (M,η) – ACOS, $(M,\eta,\mathrm{d}\,\eta)$, $\eta\wedge\mathrm{d}\,\eta^n\neq 0$.
Gray: *contact structure* for a manifold (M,η)
$\eta=\mathrm{d}\,\kappa-p_i\,\mathrm{d}\,q^i,\quad \mathrm{d}\,\eta=\mathrm{d}\,q^i\wedge\mathrm{d}\,p_i.$
Reeb vector: $R\lrcorner\,\mathrm{d}\,\eta=0,\quad R\lrcorner\eta=1,\quad R=\frac{\partial}{\partial\kappa}.$
Hamiltonian vector field : $\quad\flat(X_H)=\mathrm{d}\,H-(R\lrcorner H+H)\eta$
The vector field X_H in Darboux coordinates
$X_H=\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}-(\frac{\partial H}{\partial q^i}+p_i\frac{\partial H}{\partial\kappa})\frac{\partial}{\partial p_i}+(p_i\frac{\partial H}{\partial p_i}-H)\frac{\partial}{\partial\kappa}.$
$\mathrm{grad}\,H=\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}-(\frac{\partial H}{\partial q^i}+p_i\frac{\partial H}{\partial\kappa})\frac{\partial}{\partial p_i}+(\frac{\partial H}{\partial\kappa}+p_i\frac{\partial H}{\partial p_i})R$
$X_H=\mathrm{grad}\,H-(H+R(H))R.$
Ignore the "green parts" $\to X_H$ eqs on the symplectic manifold (M,Ω).
$X\to X^\flat=X\lrcorner\,\mathrm{d}\,\eta+(X\lrcorner\eta)\eta.$
$X=A_i\frac{\partial}{\partial q^i}+B_i\frac{\partial}{\partial p^i}+C\frac{\partial}{\partial\kappa},\to X^\flat=\alpha_i\,\mathrm{d}\,q^i+\beta_i\,\mathrm{d}\,p_i+\gamma\,\mathrm{d}\,\kappa,\quad i=1,\dots,n,$
$\flat:\quad\alpha_i=-B_i+p_i(p_jA_j-C),\quad\beta_i=A_i,\quad\gamma=C-p_jA_j,$
$\sharp=\flat^{-1}:\quad A_i=\beta_i,\quad B_i=-\alpha_i-\gamma p_i,\quad C=p_i\beta_i+\gamma.$
Jacobi bracket : $\{f,g\}_J=\{g,f\}_P+\frac{\partial f}{\partial\kappa}g_e-\frac{\partial g}{\partial\kappa}f_e,$
Euler's operator: $f_e:=f-p_i\frac{\partial f}{\partial p_i}.$

SAC - strict almost contact; C -contact struct; ACM - almost contact metric

Sasaki, Boyer, Galicki: The manifold M_{2n+1} has a SAC structure (Φ,ξ,η) if $\exists$ Φ– $(1,1)$-tensor field, $\xi\in\mathcal{D}^1$ contravariant vector field (*Reeb vector field*, or *characteristic vector field*), $\eta\in\mathfrak{D}_1$ – covariant vector field s. t.
$\eta\lrcorner\xi=1,\quad \Phi^2X=-X+\eta(X)\xi.$
Boyer: (M_{2n+1},η) C when $\eta\in\mathcal{D}_1$, $\eta\wedge \mathrm{d}\,\eta^n\neq 0$.
Sasaki: (Φ,ξ,η) - SAC $\Rightarrow$
$\Phi\xi=0,\quad \eta\Phi=0,\quad Rank\ (\Phi^i_j)=2n,\quad \xi,\eta^t\in M(n,1,\mathbb{R}).$
$\exists$ g, positive Riemannian metric s. t. $\eta=g\xi,\quad \Phi^tg\Phi=g-\eta^t\otimes\eta.$
If $(\star)$ $\hat{\Phi}:=g\Phi\rightarrow$ the two-form $\hat{\Phi}$ is antisymmetric.

Sasaki: (M,η) – C $\Rightarrow$ ACM (M,Φ,ξ,η,g) s. t. $(\star)$,
$\mathrm{d}\,\eta(X,Y)=g(X,\Phi(Y))$,
and
$\mathrm{d}\,\eta=\frac{1}{2}\sum_{i,j=1}^{2n+1}\hat{\Phi}_{ij}\,\mathrm{d}\,x^i\wedge\mathrm{d}\,x^j=\sum_{1\leq i<j\leq 2n+1}\hat{\Phi}_{ij}\,\mathrm{d}\,x^i\wedge\mathrm{d}\,x^j,\quad \hat{\Phi}_{ij}=\partial_i\eta_j-\partial_j\eta_i.$

N=SAS — normal, Sasakian

Blair: $h=(1,1)$-tensor field. *Nijenhuis torsion* $[h,h]$ of h is the tensor field of type $(1,2)$
$[h,h](X,Y)=h^2[X,Y]+[hX,hY]-h[hX,Y]-h[X,hY], \quad X,Y\in\mathfrak{D}^1.$

SAS structure is Normal if the Φ is integrable on the cone $C(M)$.
Define $(1,2)$-tensor $N^1 := [\Phi,\Phi]+2\,\mathrm{d}\,\eta\otimes\xi$.
Boyer, Galicki: An almost contact structure (ξ,η,Φ) on M is normal if and only if $N^1=0$.
A normal contact metric structure (M,ξ,η,Φ,g) is called a *Sasakian* structure.

Jacobi group G_n^J

• Jacobi group: $G_n^J(\mathbb{R}) = \mathrm{H}_n(\mathbb{R}) \rtimes \mathrm{Sp}(\mathrm{n},\mathbb{R})$, $\mathrm{H}_n(\mathbb{R})$- Heisenberg
Siegel-Jacobi upper half plane: $\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n$, $\mathcal{X}_n = \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$
Siegel upper half plane:
$\mathcal{X}_n = \{V \in M_n(\mathbb{C}) | V = S + \mathrm{i}R, V = V^t, S, R \in M_n(\mathbb{R}), R > 0\}$

• complex version:
$G_n^J = \mathrm{H}_n \rtimes \mathrm{Sp}(\mathrm{n},\mathbb{R})_{\mathbb{C}}$, $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}} = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$
Siegel-Jacobi ball: $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$, $\mathcal{D}_n = \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\mathrm{U}(n)$
Siegel ball: $\mathcal{D}_n = \{W \in M_n(\mathbb{C}) | W = W^t, \mathbb{1}_n - W\bar{W} > 0\}$

• Partial Cayley transform $\Phi : \mathcal{X}_n^J \to \mathcal{D}_n^J$, $\Phi(V,u) = (W,z)$

$$\begin{aligned} W &= (V - \mathrm{i}\,\mathbb{1}_n)(V + \mathrm{i}\,\mathbb{1}_n)^{-1}, \\ z &= 2\,\mathrm{i}(V + \mathrm{i}\,\mathbb{1}_n)^{-1}u, \end{aligned}$$

Jacobi group $G_1^J(\mathbb{R})$ à la Berndt & Schimdt

$G_1^J(\mathbb{R})$ - subgroup of $Sp(2,\mathbb{R})$: 4×4 real matrices $g=((\lambda,\mu,\kappa),M)$, $(\lambda,\mu,\kappa)\in H_1(\mathbb{R})$, $M\in SL(2,\mathbb{R})$

$$\begin{pmatrix} a & 0 & b & q \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -p \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M = 1,$$

$$Y := (p,q) = XM^{-1} = (\lambda,\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (\lambda d - \mu c, -\lambda b + \mu a)$$

is related to the Heisenberg group H_1.

$$H_1 \ni g = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ -\lambda & 1 & -\mu & -\kappa \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{g}_1^J(\mathbb{R}) := \langle \mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathbb{R}}$, $\mathfrak{h} = \langle \mathrm{P},\mathrm{Q},\mathrm{R}\rangle_{\mathbb{R}}$, $\mathfrak{sl}(2,\mathbb{R}) = \langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathbb{R}}$.

The Heisenberg subgroup of Sp$(2, \mathbb{R})$

The composition law of the 3-dimensional Heisenberg group $H_1(\mathbb{R})$

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \lambda'\mu).$$

The Lie algebra of H_1 in the space $M(4, \mathbb{R})$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Proposition

a) Kähler two-form on $\mathcal{D}_1^J$, invariant to the action of $G_0^J = \mathrm{SU}(1,1) \ltimes \mathbb{C}$

$-\mathrm{i}\,\omega_{\mathcal{D}_1^J}(w,z) = \frac{2k}{P^2}\,\mathrm{d}\,w \wedge \mathrm{d}\,\bar{w} + \nu\frac{\mathcal{A}\wedge\bar{\mathcal{A}}}{P}, P := 1 - |w|^2, \mathcal{A}(w,z) := \mathrm{d}\,z + \bar{\eta}\,\mathrm{d}\,w.$

$FC : (w,z) \to (w,\eta), \mathrm{FC} : z = \eta - w\bar{\eta},\ \mathrm{FC}^{-1} : \eta = \frac{z+\bar{z}w}{P},$

$\mathrm{FC} : \mathcal{A}(w,z) \to \mathrm{d}\,\eta - w\,\mathrm{d}\,\bar{\eta}.$

b) Partial Cayley transform

$\Phi : (w,z) \to (v,u)\ \Phi : w = \frac{v-\mathrm{i}}{v+\mathrm{i}}, \quad z = 2\,\mathrm{i}\,\frac{u}{v+\mathrm{i}}, \quad v,u \in \mathbb{C},\ \Im v > 0,$

$\Phi^{-1} : v = \mathrm{i}\,\frac{1+w}{1-w}, \quad u = \frac{z}{1-w}, \quad w,z \in \mathbb{C},\ |w| < 1$

$\mathcal{A}\left(\frac{v-\mathrm{i}}{v+\mathrm{i}}, \frac{2\,\mathrm{i}\,u}{v+\mathrm{i}}\right) = \frac{2\,\mathrm{i}}{v+\mathrm{i}}\mathcal{B}(v,u), \mathcal{B}(v,u) := \mathrm{d}u - r\mathrm{d}v,\ r := \frac{u-\bar{u}}{v-\bar{v}}.$

Berndt–Kähler's two-form, inv. action $G^J(\mathbb{R})_0 = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{C}$:

$-\mathrm{i}\,\omega_{\mathcal{X}_1^J}(v,u) = -\frac{2k}{(\bar{v}-v)^2}\,\mathrm{d}\,v \wedge \mathrm{d}\,\bar{v} + \frac{2\nu}{\mathrm{i}(\bar{v}-v)}\mathcal{B}\wedge\bar{\mathcal{B}}.(\star)$

$\mathrm{FC}_1 : (v,u) \to (v,\eta), \mathrm{FC}_1 :\ 2\,\mathrm{i}\,u = (v+\mathrm{i})\eta - (v-\mathrm{i})\bar{\eta}, \mathrm{FC}_1^{-1} : \eta = \frac{u\bar{v}-\bar{u}v}{\bar{v}-v} + \mathrm{i}\,r.$

c) $\mathcal{D}_1^J \ni (v,u) \to (x,y,p,q) \in \mathcal{X}_1^J,\ \eta = q + \mathrm{i}\,p,$

$\mathbb{C} \ni u := pv + q, \quad p,q \in \mathbb{R}, \quad \mathbb{C} \ni v := x + \mathrm{i}\,y,\ x,y \in \mathbb{R},\ y > 0,$

$r = p, \quad \mathcal{B}(v,u) = \mathrm{d}\,u - p\,\mathrm{d}\,v,\ \mathcal{B}(v,u) = \mathcal{B}(x,y,p,q) := F\,\mathrm{d}\,t =$

$v\,\mathrm{d}\,p + \mathrm{d}\,q = (x + \mathrm{i}\,y)\,\mathrm{d}\,p + \mathrm{d}\,q, \quad F := \dot{p}v + \dot{q}.$

Proposition- continuation

d) The two-parameter balanced metric on the $\mathcal{X}^J_1$, associated to the Kähler two-form ($\star$),

$$\mathrm{d}\, s^2_{\mathcal{X}^J_1}(x,y,p,q) = \alpha\frac{\mathrm{d}\, x^2 + \mathrm{d}\, y^2}{y^2} + \frac{\gamma}{y}(S\,\mathrm{d}\, p^2 + \mathrm{d}\, q^2 + 2x\,\mathrm{d}\, p\,\mathrm{d}\, q),$$

$S = x^2 + y^2,\ \alpha := k/2, \quad \gamma := \nu)$

$$g_{\mathcal{X}^J_1} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 \\ 0 & 0 & g_{pp} & g_{pq} \\ 0 & 0 & g_{qp} & g_{qq} \end{pmatrix}, \quad \begin{array}{ccc} g_{xx} = \frac{\alpha}{y^2}, & g_{yy} = \frac{\alpha}{y^2}; & \\ g_{pq} = \gamma\frac{x}{y}, & g_{pp} = \gamma\frac{S}{y}, & g_{qq} = \frac{\gamma}{y}. \end{array}$$

Proposition -end

Three parameter left invariant metric to the action of $G_1^J(\mathbb{R})$ on $\tilde{\mathfrak{X}}_1^J$ in S-coordinates (x, y, q, p, κ)

$$\mathrm{d}s^2_{\tilde{\mathfrak{X}}_1^J}(x,y,p,q,\kappa) = \frac{\alpha}{y^2}(\mathrm{d}x^2+\mathrm{d}y^2) + \left(\frac{\gamma}{y}S+\delta q^2\right)\mathrm{d}p^2 + \left(\frac{\gamma}{y}+\delta p^2\right)\mathrm{d}q^2+\delta\mathrm{d}\kappa^2$$
$$+2\left(\gamma\frac{x}{y}-\delta pq\right)\mathrm{d}p\mathrm{d}q + 2\delta(q\mathrm{d}p\mathrm{d}\kappa - p\mathrm{d}q\mathrm{d}\kappa).$$

$$\begin{pmatrix} g_{xx} & 0 & 0 & 0 & 0\\ 0 & g_{yy} & 0 & 0 & 0\\ 0 & 0 & g_{pp} & g_{pq} & g_{p\kappa}\\ 0 & 0 & g_{qp} & g_{qq} & g_{q\kappa}\\ 0 & 0 & g_{\kappa p} & g_{\kappa q} & g_{\kappa\kappa}\end{pmatrix}, \quad \begin{matrix} g_{xx} = \frac{\alpha}{y^2}, & g_{yy} = \frac{\alpha}{y^2},\\ g_{pq} = \gamma\frac{x}{y} - \delta pq, & g_{p\kappa} = \delta q, g_{q\kappa} = -\delta p,\\ g_{pp} = \gamma\frac{S}{y} + \delta q^2, & g_{qq} = \frac{\gamma}{y} + \delta p^2, g_{\kappa\kappa} = \delta.\end{matrix}$$

$\tilde{\mathfrak{X}}_1^J$ does not admit an ACOS (Φ, ξ, η),
$\eta = \lambda_6 = \sqrt{\delta}(\mathrm{d}\,\kappa - p\,\mathrm{d}\,q + q\,\mathrm{d}\,p), \quad \delta > 0,, \xi = \mathrm{Ker}(\eta).$

Equations of motion on ACOS manifolds

Theorem

$(M_{2n+1}, \theta, \Omega)$ - *ACOS manifold,*
$\theta = a_i \, \mathrm{d}\, q^i + b_i \, \mathrm{d}\, p_i + c\kappa, \quad a_i, b_i \in \mathbb{R}, \ i = 1, \dots, n, \quad c \neq 0,$
- *H smooth Hamiltonian.* $X_H = A_i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{p_i} + C \frac{\partial}{\partial \kappa},$
$\flat(X_H) = \mathrm{d}\, H - (R \lrcorner H + H)\theta, \quad R = \frac{1}{c}\frac{\partial}{\partial \kappa},$
$A_i = \frac{\partial H}{\partial p^i} - b_i R(H), \ B_i = -\frac{\partial H}{\partial q_i} + a_i R(H), \ C = \frac{1}{c}(-a_i \frac{\partial H}{\partial p^i} + b_i \frac{\partial H}{\partial q^i} - H),$
grad $H =$
$(\frac{\partial H}{\partial p_i} - b_i R(H))\frac{\partial}{\partial q^i} + (-\frac{\partial H}{\partial p_i} + a_i R(H))\frac{\partial}{\partial p_i} + (-a_i \frac{\partial H}{\partial p_j} + b_i \frac{\partial H}{\partial q^j} + R(H))R.$

$$\text{grad}\, q^i = -\frac{\partial}{\partial p_i} + b_i R, \text{grad}\, p_i = \frac{\partial}{\partial q^i} - a_i R, \ \text{grad}\, \kappa = \frac{1}{c}(-b_i \frac{\partial}{\partial q^i} + a_i \frac{\partial}{\partial p_i} + R),$$

$$X_{q^i} = -\frac{\partial}{\partial p_i} + (b_i - q^i)R, X_{p_i} = \frac{\partial}{\partial q^i} - (a_i + p_i)R, X_\kappa = \frac{1}{c}(-b_i \frac{\partial}{\partial q^i} + a_i \frac{\partial}{\partial p_i} - \kappa \frac{\partial}{\partial \kappa}).$$

GTACOS on $\tilde{\mathfrak{X}}_1^J$

Endow $\tilde{\mathfrak{X}}_1^J$ with a GTACOS struct (M, θ, Ω), i.e. an ACOS structure s. t. $\mathrm{d}\,\Omega = 0$.

Lemma

$(w, z) \rightarrow (x, y, q, p), y > 0$ in $\omega_{\mathcal{D}_1^J}(w, z) \Rightarrow$
$\omega_{\mathfrak{X}_1^J}(x, y, p, q) = \frac{k}{y^2}\,\mathrm{d}\,x \wedge \mathrm{d}\,y + 2\nu\,\mathrm{d}\,q \wedge \mathrm{d}\,p$
$(\tilde{\mathfrak{X}}_1^J, \theta, \omega)$, parametrized in (x, y, p, q, κ) is ACOS
$\theta = \lambda_6 = \sqrt{\delta}(\mathrm{d}\,\kappa - p\,\mathrm{d}\,q + q\,\mathrm{d}\,p), \quad \delta > 0, \quad \omega = \omega_{\tilde{\mathfrak{X}}_1^J}.$
$\mathrm{d}\,\omega = 0, \quad \theta \wedge \omega^2 = 2\frac{k\nu\sqrt{\delta}}{y^2}\,\mathrm{d}\,x \wedge \mathrm{d}\,y \wedge \mathrm{d}\,q \wedge \mathrm{d}\,p \wedge \mathrm{d}\,\kappa,$
$a_1 = b_1 = 0, \quad a_2 = -\sqrt{\delta}p, \quad b_2 = \sqrt{\delta}q, \quad c = \sqrt{\delta}, \quad n = 2,$
$q^1 = kx, \quad p_1 = -\frac{1}{y}, \quad q^2 = 2\nu q, \quad p_2 = p, \quad n = 2.$
In Darboux coordinates ACOS $(\tilde{\mathfrak{X}}_1^J, \theta, \omega)$. $+\,\mathrm{d}\,\omega = 0$, $(\tilde{\mathfrak{X}}_1^J, \theta, \omega)$ is a GTACOS .

Equations of motion on $\tilde{\mathfrak{X}}^J_1$ as GTACOS

Proposition

Eqs of motion on the 5-dimensional $\tilde{\mathfrak{X}}^J_1$:

$$\dot{x} = \frac{y^2}{k}\frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{y^2}{k}\frac{\partial H}{\partial x},$$

$$\dot{q} = \frac{1}{2\nu}\left(\frac{\partial H}{\partial p} - q\frac{\partial H}{\partial \kappa}\right), \qquad \dot{p} = -\frac{1}{2\nu}\left(\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial \kappa}\right),$$

$$\dot{\kappa} = \frac{1}{2\nu}\left(p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q}\right) - H.$$

$\tilde{X}_1^J$ as CH ans ACM

Proposition

Let $\eta_o = \sqrt{\delta}\,\mathrm{d}\,\kappa + \frac{k}{y}\,\mathrm{d}\,x + \nu(-p\,\mathrm{d}\,q + q\,\mathrm{d}\,p)$ *s. th.*
$\mathrm{d}\,\eta_0 = \omega_{\mathcal{X}_1^J} = \frac{k}{y^2}\,\mathrm{d}\,x \wedge \mathrm{d}\,y + 2\nu\,\mathrm{d}\,q \wedge \mathrm{d}.$ $(\tilde{\mathcal{X}}_1^J, \eta_0)$ *is a contact manifold.*
a) For η_0 *and* $g_{\tilde{\mathcal{X}}_1^J}$ *there is no* Φ *s. th.* $(\tilde{\mathcal{X}}_1^J, \Phi, \xi, \eta_0, g_{\tilde{\mathcal{X}}_1^J})$, $\xi = \frac{1}{\sqrt{\delta}}\frac{\partial}{\partial\kappa}$ *is an ACM.*
b) The C $(\tilde{\mathcal{X}}_1^J, \eta_0)$ *can be endowed with an ACM structure* $(\tilde{\mathcal{X}}_1^J, \Phi, \xi, \eta_0, g'_{\tilde{\mathcal{X}}_1^J})$. *The six components* $\Phi_{xx}, \Phi_{xy}, \Phi_{xq}, \Phi_{xp}, \Phi_{yx}, \Phi_{qq}$ *of* Φ *can be expressed as function of the four remaining independent variables* $\Phi_{yq}, \Phi_{yp}, \Phi_{qp}, \Phi_{pq}$, *while the rest of the components of* Φ *are obtained with ... The fundamental quadratic form associated to the metric tensor* g' *... must be positive definite.*

$\tilde{\mathfrak{X}}^J_1$ with C structure

Proposition

$(\tilde{\mathfrak{X}}^J_1, \eta_0)$ is a contact Hamiltonian system.
The equations of motion:

$$\dot{x} = \frac{y^2}{k}\frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{y^2}{k}\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial \kappa},$$
$$\dot{q} = \frac{1}{2\nu}\frac{\partial H}{\partial p}, \quad \dot{p} = -(\frac{1}{2\nu}\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial \kappa}),$$
$$\dot{\kappa} = (-y\frac{\partial H}{\partial y} + p\frac{\partial H}{\partial p} - H)\frac{\partial H}{\partial \kappa}.$$

The Jacobi bracket: $\{f,g\} = \{f,g\}_P + f_e\frac{\partial g}{\partial \kappa} - g_e\frac{\partial f}{\partial \kappa}$,
the Poisson bracket:
$\{f,g\}_P = \frac{1}{k}\frac{y^2+1}{y^2}(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}) + \frac{1}{2\nu}[\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} + \frac{1}{y^2}(\frac{\partial f}{\partial q}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial y})]$.
the Euler operator: $f_e = f + \frac{1}{y^3}\frac{\partial f}{\partial y} - p\frac{\partial f}{\partial p}$.

Remark

Remark

If in the equations of motion on $\tilde{\mathfrak{X}}^J_1$ expressed in the S-variables (x, y, p, q, κ) on the generalized transitive almost cosymplectic manifold $(\tilde{\mathfrak{X}}^J_1, \theta, \omega)$ (respectively, on the contact manifold $(\tilde{\mathfrak{X}}^J_1, \eta_0)$) we ignore the "red" (respectively "green") parts we get the equations of motion on $\mathfrak{X}^J_1$ in (x, y, p, q).

Linear Hamiltonian in the generators of $G_1^J(\mathbb{R})$

Int. J. Geom. Methods Mod. Phys. (2013) S. B. considered a linear Hermitian Hamiltonian $\boldsymbol{H}$ in the generators of the Jacobi group G_1^J

$(\star)$ $\boldsymbol{H} = \epsilon_a \boldsymbol{a} + \bar{\epsilon}_a \boldsymbol{a}^\dagger + \epsilon_0 \boldsymbol{K}_0 + \epsilon_+ \boldsymbol{K}_+ + \epsilon_- \boldsymbol{K}_-, \quad \bar{\epsilon}_+ = \epsilon_-, \quad \bar{\epsilon}_0 = \epsilon_0.$

Notation: $\epsilon_a := a + \mathrm{i}\, b,\ \epsilon_+ := m - \mathrm{i}\, n,\ \epsilon_0 := 2c, \quad a, b, c, m, n \in \mathbb{R}.$

The energy function $\mathcal{H}$ associated to the linear Hamiltonian in (η, v) splits into the sum of two independent functions

$\mathcal{H}(\eta, v) = \mathcal{H}(\eta) + \mathcal{H}(v), (\star\star) \quad v = x + \mathrm{i}\, y,\ y > 0,\ \eta = q + \mathrm{i}\, p$, where

$$\mathcal{H}(q,p) = \nu[(m+c)q^2 + (c-m)p^2 + 2nqp + 2(aq+bp)],$$

$$\mathcal{H}(x,y) = k\{\frac{1}{y}[(m+c)(x^2+y^2) - 2(nx+cy)] + 3c - m\}.$$

Continuation

We particularize eqs GTACOS to the linear Ham.
$\mathcal{H}=\mathcal{H}(p,q)+\mathcal{H}(x,y)+h(\kappa)$,

Proposition

The eqs of motion on the extended Siegel-Jacobi upper half-plane organized as GTACOS $g(\tilde{\mathcal{X}}^J_1,\theta,\omega)$ corresponding to the energy function $\mathcal{H}$

$$\dot{x} = (c+m)(-x^2+y^2)+mx-c+m, \quad \dot{y} = -2(c+m)y^2+2ny,$$

$$\dot{q} = -(m+c)q-np-a-\frac{q}{2\nu}\frac{\partial h}{\partial \kappa}, \quad \dot{p} = qn+(c-m)p+b-\frac{p}{2\nu}\frac{\partial h}{\partial \kappa},$$

$$\dot{\kappa} = (c+m)qa^2+(-c+m)p^2+(m-n)pq+nq+bp-\frac{1}{\sqrt{\delta}}h.$$

Remark: If in eqs mot. on $\tilde{\mathcal{X}}^J_1$ generated by the linear Hamiltonian we ignore the "red parts" $\Rightarrow$ matrix Riccati equation in (x,y) and the linear system of differential equations in (p,q) generated by the linear Hamiltonian $(\star\star)$ on $\mathcal{X}^J_1$ found in 2013 are reobtained.

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MANY THANKS