Classical and quantum integrable dynamics of nonlinear rotons on spherical fluids; from nuclei to neutron stars

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Nov, 2022

Outline

1. Motivation
2. Fluid drop with rigid core - model equations
   - Liquid drop with rigid core - model equations
   - Defocusing nonlinear Schrodinger equation on the sphere
   - Periodic dark multisoliton solution; Dubrovin's approach
   - Quantization: Algebraic Bethe Anstaz
3. Azimutal rotons on the neutron star surface
   - Fluid equations of neutron star
   - Korteweg de Vries equation as an asymptotic model for azimutal nonlinear rotons
   - Multi-phase solution of Korteweg de Vries equation
4. Conclusions

Motivation

- *Integrability on curved surfaces;*
  It is extremely complicated since equations are in general non-autonomous; Lame coefficients and metric tensors usually appear in the coefficinets of the equations
- Astroseismology and generation of gravitational waves;
  The majority of stars can be modelled as spherical fluids with a rich nonlinear dynamics. Moreover binary neutron stars can display tidal deformations propagating as nonlinear waves on their surface and producing gravitational waves in the surrounding space-time.
- *Nuclear physics;*
  liquid drop models of heavy and super-deformed nuclei, nuclear breakup and fission
- *Biological systems;*
  motile cells, cellular division etc

Nonlinear terms from fluid equations (Navier-Stokes,Euler) and from the boundary usually introduce couplings between modes of oscillations. Nonlinear terms coming from geometry of curved, closed surfaces provide additional couplings - general connections between kinematics and shape.

Liquid drop with rigid core; model equations

We consider a fluid spherical deformable surface Σ of radius R_0 which surrounds a rigid core of radius $R_0 - h, h << R_0$. The fluid is considered to be inviscid and incompressible and is described by irrotational field velocity $\vec{v} = \vec{v}(r, \theta, \phi, t)$ and constant mass density ρ. The surface and its deformation (which is the nonlinear wave propagating on it) are described by

$$r = R_0(1 + \xi(\theta, \phi, t)) \quad \xi(\theta, \phi, t) = g(\theta)\zeta(\phi, t)$$

and we will assume that the $\theta-$dependence is very slow (almost constant in θ)

Euler's equations

From the irrotational condition we can introduce the velocity potential and Laplace equation $\vec{v} = \vec{\nabla}\Phi, \Delta\Phi = 0$ and Euler equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} P}{\rho} + \frac{\vec{f}}{\rho}$$

where P is the pressure and $\vec{f}$ volume density of the Coulombian (gravitational) force proportional to gradient of electrostatic or gravitational potential Ψ Introducing the velocity potential we get the *scalar* equation:

$$\Phi_t + \frac{1}{2}|\vec{\nabla}\Phi|^2)|_\Sigma = -\frac{P}{\rho} - \frac{\Psi}{\rho}$$

$$\vec{\nabla} = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi})$$

Now we have to write the boundary conditions in order to determine the functions Φ and ξ. We have two closed surfaces: the external free one and the inner one. The kinematic boundary condition is:

$$v_r \equiv \frac{\partial \Phi}{\partial r}|_\Sigma = \frac{dr}{dt}|_\Sigma = (\frac{\partial r}{\partial t} + \frac{\partial r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial r}{\partial \phi}\frac{d\phi}{dt})|_\Sigma = R_0(\xi_t + \frac{\xi_\theta}{r^2}\Phi_\theta + \frac{\xi_\phi}{r^2 \sin^2}\Phi_\phi)|_\Sigma$$

Also at the rigid core the velocity is zero

$$v_r \equiv \frac{\partial \Phi}{\partial r}|_{r=R_0-h} = 0$$

For Laplace equation we consider the following formal expansion:

$$\vec{\nabla}^2\Phi = 0, \quad \Phi = \sum_{n=0}^{\infty}\left(\frac{r-R_0}{R_0}\right)^n f_n(\theta, \phi, t)$$

For convergence we assume that $\frac{r-R_0}{R_0} \leq \max|\xi| = \epsilon$. So the amplitude of the nonlinear waves will be small. Also the thickness of the shell will be the same magnitude $h/R_0 = O(\epsilon)$
So from the rigid core boundary condition we get:

$$\sum_{n=1}^{\infty} n\left(\frac{h}{R_0}\right)^{n-1} f_n = 0$$

so *in the first order in* $\epsilon = h/R_0$

$$f_1 = 2\frac{h}{R_0}f_2$$

The truncation of the Laplace expansions up to order two are:

$$\frac{\partial \Phi}{\partial r}|_{\Sigma} = \frac{f_1}{R_0} + 2\xi \frac{f_2}{R_0} + O(\epsilon^2)$$

$$\frac{\partial \Phi}{\partial \phi}|_{\Sigma} = f_{0,\phi} + \xi f_{1,\phi} + O(\epsilon^2)$$

$$\frac{\partial \Phi}{\partial \theta}|_{\Sigma} = f_{0,\theta} + \xi f_{1,\theta} + O(\epsilon^2)$$

Introducing in the kinematic boundary condition for the free surface we get and neglecting the θ dependence we get

$$f_1 + 2\xi f_2 = R_0^2 \xi_t + \frac{\xi_\phi (1 - 2\xi)}{\sin^2 \theta}(f_{0,\phi} + \xi f_{1,\phi}) + O(\epsilon^3)$$

From linearisation of kinematic boundary (restriction to collective radial vibrations) we take

$$f_1 = R_0^2 \xi_t + O(\epsilon^2)$$

and introducing above we get

$$2\xi f_2 = \frac{1}{\sin^2 \theta}(-\xi_\phi f_{0,\phi} + \xi \xi_\phi (f_{1,\phi} - 2 f_{0,\phi}))$$

Using the rigid boundary condition we get finally

$$f_{0,\phi} = \frac{R_0^2 \sin^2 \theta}{\epsilon} \frac{\xi \xi_t}{\xi_\phi}(1 + 2\xi) + O(\epsilon^3)$$

Velocity and Pressure

The defining relation of Σ gives

$$\frac{1}{r^n} = \frac{1}{R_0^n} \sum_{k \geq 0} (-1)^k ((n-1)k + 1)\xi^k$$

Velocity and Pressure in the Euler equation are

$$v_\phi = \frac{\Phi_\phi}{r \sin\theta} = \frac{f_{0,\phi}}{R_0 \sin\theta} + O(\xi^3)$$

$$P|_\sigma = 2\sigma H$$

where σ is the surface tension coefficient of the fluid and H is the mean curvature given by the coefficients of the first and second fundamental forms. In the spherical coordinates these expressions are very complicated and we expand them up to order two. We get

$$P = 2\sigma H = \frac{\sigma}{R_0}(-2\xi - 4\xi^2 - \Delta_\Omega \xi) + O(\xi^3)$$

where the angular part of the laplacian operator is

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

We can add the Coulomb or gravitational potential in the form expanded up to some order but we neglect it for the moment

Mean curvature is given in differential geoemtric terms

$$H = \frac{h_{uu} + h_{vv} + h_v^2 h_{uu} - 2h_u h_v h_{uv} + h_u^2 h_{vv}}{(1 + h_u^2 + h_v^2)^{3/2}}$$

where the coordinates are (u, v) and $h = h(u, v)$ is the shape function. In the spherical coordinates $(u, v) = (\theta, \phi)$ and $r = R_0 + \xi(\theta, \phi)$. Accordingly

$$H = \frac{B - ((R_0 + \xi)^2 + \xi_\theta^2)\xi_\theta \sin\theta\cos\theta + C\sin^2\theta}{2((R_0 + \xi)^2 + \xi_\theta^2 + \xi_\phi^2/\sin^2\theta)^2 \sin^2\theta}$$

where

$$B = 3R_0\xi_\phi^2 + 3\xi\xi_\phi^2 - R_0^2\xi_{\phi\phi} - 2R_0\xi\xi_{\phi\phi} - \xi^2\xi_{\phi\phi} -$$

$$-\xi_\theta^2\xi_{\phi\phi} + 2\xi_\theta\xi_\phi\xi_{\theta\phi} - \xi_{\theta\theta}\xi_\phi^2 - 2\xi_\theta\xi_\phi^2\cot\theta$$

$$C = (R_0 + \xi)(2(R_0 + \xi)^2 + 3\xi_\theta^2 - R_0\xi_{\theta\theta} - \xi\xi_{\theta\theta})$$

If the shape function is small compared to the radius ($\xi << R_0$) we expand in powers of ξ and we get as above:

$$H = -\xi - 2\xi^2 - \Delta_\Omega\xi + ...$$

Euler equation expressed in the variable ξ

Deriving with respect to ϕ the Euler equation we get (we neglect the θ derivatives)

$$\partial_t(f_{0,\phi} + \xi f_{1,\phi} + \ldots) + \partial_\phi\left(\frac{v_\phi^2}{2}\right) = \frac{2\sigma}{\rho R_0}\xi_\phi + \frac{\sigma}{\rho R_0}\Delta_\Omega \xi_\phi + \mathcal{O}(\xi^2), \tag{1}$$

we obtain the final equation for the nonlinear rotons $\xi(\phi, t)$ on the sphercal surface:

$$\partial_t\left(\frac{R_0^2 \sin^2\theta\, \xi\xi_t}{\epsilon\xi_\phi}\right) + \partial_\phi\left(\frac{R_0^2\sin^2\theta\,\xi^2\xi_t^2}{2\epsilon^2\xi_\phi^2}\right) - \frac{2\sigma\xi_\phi}{\rho R_0} - \frac{\sigma\xi_{\phi\phi\phi}}{\rho R_0 \sin^2\theta} = 0.$$

We havetwo types of nonlinearities; the geometrical one (red) and the physical one (blue).
Crucial remark: In the physical linear case (blue term removed) the equation admits plane travelling wave solution $\xi = A(\theta)e^{i(k\phi-\omega t)}$+c.c. with the Boussinesq-type dispersion relation

$$\omega^2 = \frac{\epsilon\sigma}{\rho R_0^3 \sin^2\theta}\left(2k^2 - \frac{k^4}{\sin^2\theta}\right).$$

Adding the physical nonlinearity the effect will be a **modulation of the amplitudes and higher armonics** of the linear wave which **in some slow variables** will be **balanced** by the dispersion relation. So in this case we can take the following approximation on a **slow angular-temporal scale** φ, τ

$$\frac{\xi\xi_t}{\xi_\phi} = -(\omega/k)\xi(\theta,\phi,t) \sim \sum_{n=-\infty}^{\infty} \varepsilon^{s_n} Q_n(\theta,\varphi,\tau)e^{in(k\phi-\omega t)}.$$

Here ε is a small parameter measuring the weak modulation of the amplitude in a slow space-time scale ($\varepsilon \sim h/R_0 \sim \max|\xi|$) and s_n are some exponents which **have to be determined form asymptotic balance**. Now we can define the slow variables

$$\varphi = \varepsilon \left(\phi - \frac{2\sigma t}{v\rho R_0^2 \sin^2\theta} + \frac{3\sigma k^2 t}{v\rho R_0^3 \sin^4\theta} \right),$$

$$\tau = -\varepsilon^2 k t,$$

and the amplitudes of the expansion:

$$s_0 = 2, \quad Q_0(\varphi,\theta,\tau) = g(\theta) V_0(\varphi,\tau),$$

$$s_2 = 2, \quad Q_2(\varphi,\theta,\tau) = g(\theta) V_2(\varphi,\tau),$$

$$s_n = n, \quad Q_n(\varphi,\theta,\tau) = g(\theta) V_n(\varphi,\tau), \quad n \neq 0,2, \quad V_{-n} = V_n^*.$$

Introducing these expressions from above in the Euler equation we obtain the following defocusing Nonlinear Schrödinger equation with dimensionless terms (using a long but straightforward calculation):

$$i\frac{A(\theta)}{3D(\theta)}\frac{\partial\zeta}{\partial\tau} + \frac{\partial^2\zeta}{\partial\varphi^2} - \frac{C(\theta)^2 g(\theta)^2}{36D(\theta)^2}|\zeta|^2\zeta = 0, \tag{2}$$

where we used the following notations:

$$\zeta(\varphi,\tau) = V_1/k, \quad A(\theta) = vR_0^2\sin^2\theta/h,$$

$$C(\theta) = v^2 R_0^4 \sin^4\theta/h^2, \quad D(\theta) = -\sigma/\rho R_0 \sin^2\theta.$$

The physical configuration is give by the parameterization equation

$$r = R_0(1 + g(\theta)(\epsilon k\zeta(\varphi,\tau)e^{i(k\phi+\omega t)} + \text{c.c.} + \epsilon^2(\frac{Cg}{6D}\zeta^2 - \frac{Cg}{3D}|\zeta|^2 e^{2i(k\phi+\omega t)} + \text{c.c}))).$$

Methodology

The procedure goes as following. In general if we have a scalar equation,

$$u_t + u_{xxxxx} + \partial_x(u^2) + \ldots = 0$$

and solution to be written as an infinite Fourier type series

$$u = \sum_{n\in\mathbb{Z}} u_n e^{in(kx+\omega t)}, \quad u_{-n} = u_n^*$$

Usually ω is computed from the dispersion of the *linearised equation* $\omega = k^5 + \ldots$ and k is an arbitrary number.

- first step:equating to zero the coeff of **every** $e^{in(kx+\omega t)}$ and we get from our example

$$(\partial_t + in\omega)u_n + (\partial_x + ink)^5 u_n + (\partial_x + ink)\left(\sum_{q=-\infty}^{\infty} u_q u_{n-q}\right) + \ldots. = 0$$

- it is sufficient to consider introduce the stretched variables $X = \epsilon(x + vt)$, $T = \epsilon^p t$ and $u_n = \epsilon^s U(X, T)$, p, s determined from the asymptotic balance.
- for various power of epsilons, and vor various n we get first the dispersion relation, then derivative of it and then some equations.

Periodic dark multisolitons; Dubrovin-Zagrodzinski method

The defocusing nonlinear Schrodinger is a *completely integrable system*. It possesses a Lax formulation and here it is understood as subjected to periodic boundary conditions. The inverse spectral transform for periodic boundary conditions makes can be used to compute the periodic solution. However it is very cumbersome and in order to get the solution we will implement a much more practical method due to Dubrovin and Zagrodzinski (called also "dispersion relation method"). The equation can be written in a normalised way (after scaling dependent and independent variables) as

$$i\zeta_t + \zeta_{xx} + \alpha|\zeta|^2\zeta = 0$$

The method is based on the fact that Riemann-theta function is the universal tau-function in the theory of integrable systems obeying the so called "addition property". Namely if $T : \mathbb{C}^g \to \mathbb{C}$ expressed by Riemann-Theta function then exists $G_\epsilon, F_\epsilon : \mathbb{C}^g \to \mathbb{C}$ such that $(g \in \mathbb{N})$:

$$T(z+y)T(z-y) = \sum_{\epsilon\in\mathbb{Z}_2^g} Z_\epsilon(z)W_\epsilon(y) = \sum_{\epsilon\in\mathbb{Z}_2^g} \Omega_\epsilon(y)T^2(z+\epsilon/2)$$

for every $z, y \in \mathbb{C}^g$. In the case of NLS equation we are interested in the so-called multi-phase (g-phase) periodic solution

$$\zeta(t,x) = \zeta(\kappa_1 x + \omega_1 t, \kappa_2 x + \omega_2 t, ..., \kappa_g x + \omega_g t; B) \equiv \zeta(z_1, z_2, ..., z_g; B)$$

.

We consider the following nonlinear substitution

$$\zeta(x,t) = qe^{(iwt-kx)}T(z+2s)/T(z) \equiv G/F$$

where $z = (z_1, ..., z_g), \omega_i = \omega(\kappa_i)$, w, k are real constants,q is a complex amplitude, s is a complex number and all must be determined. Introducing in the NLS equation we get:

Now, to be more specific let:

$$< z, n >:= \sum_{i=1}^{g} z_i n_i$$

and

$$T(z_1, ...z_g; B) = \Theta(z|B) \equiv \sum_{n\in\mathbb{Z}^g} \exp(i\pi(2 < z, n > + < n, Bn >))$$

be the Riemann-Theta function. Then for the addition property we may choose (there are many other possibilities and they give equivalent solutions up to modular transformations):

$$\Omega_\epsilon(y) = 2^{-g} \sum_{\mu\in\mathbb{Z}_2^g} (-1)^{<\epsilon,\mu>} e^{2\pi i<y,\epsilon>} \frac{\Theta(2y + B\epsilon|2B)}{\Theta(B\epsilon|2B)}, \quad \epsilon \in \mathbb{Z}_2^g$$

We introduce also the Hirota bilinear derivative $D_z^m f \cdot g = (\partial_y)^m f(z + y)g(z - y)|_{y=0}$ which will be essential in applying the addition property. Also immediately
$\partial_t = \sum_{j=1}^{g} \omega_j \partial_{z_j}, \partial_x^2 = \sum_{i,j=1}^{g} \kappa_i\kappa_j \partial_{z_i}\partial_{z_j}$

$$F((i\sum_{j=1}^{g} \omega_j D_{z_j} + \sum_{i,j=1}^{g} D_{z_i} D_{z_j} \kappa_i\kappa_j)G \cdot F) - G(\sum_{i,j=1}^{g} D_{z_i} D_{z_j} \kappa_i\kappa_j F \cdot F + \alpha|G|^2) = 0$$

Applying addition property and considering that the family of Riemann Theta functions $T^2(z + \epsilon/2)$ are independent we get the following system of coefficients ($\Omega_{\epsilon,i,j...} = \partial_{s_i,s_j,...}\Omega_\epsilon(s)$)

$$i\sum_j(\omega_j - 2k\kappa_j)\Omega_{\epsilon,j}(s) - \sum_{i,j} \kappa_i\kappa_j\Omega_{\epsilon,i,j}(s) - (\lambda_0 + k^2)\Omega_\epsilon(s) = 0$$

$$\sum_{i,j} \kappa_i\kappa_j\Omega_{\epsilon,i,j}(0) + \alpha|q|^2\Omega_\epsilon(0) - \lambda_0 = 0$$

This system provides the in principle the parameters $\omega_i = \omega(\kappa_i), s_i, w, k, \lambda_0$

. In general for $g = 1$ always solution exists. For $g \geq 2$ some constraints can appear on the matrix B of the Riemann Theta function (which for $g = 1$ is free).

So the nonlinear rotons are expressed as ratio of Riemann Theta functions. In the case $g = 1$ a particular solution it can be written as a Jacobi elliptic function

$$|\zeta(\varphi, \tau)| = H\mathrm{sn}\left(k\sqrt{\frac{C^2 g^2}{18k^2(m+1)}}\left(\varphi - \frac{6D\lambda}{A}\tau\right)\Big| m\right) \tag{3}$$

where $H = i\sqrt{2b_1 m / b_0(m+1)}$ and b_0, b_1, m are complicated expressions in terms of A, D, C

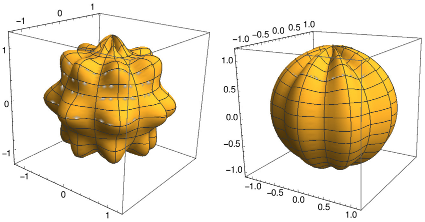

Figure: Periodic patterns for $g(\theta) = \cos^2(4\theta)$ and $\exp(-\theta^2)$

Quantization: Algebraic Bethe Ansatz

Since we have seen that our dynamics is given by the defocusig NLS equation now Our equation can be written in a hamiltonian form:

$$\partial_T \zeta = \frac{\delta}{\delta \zeta^\dagger} \int_0^{2\pi} \left(|\zeta_\varphi|^2 + c|\zeta|^4 \right) d\varphi$$

where we rescaled time $T = (A/3D)\tau$, and $c = C^2 g^2/36D^2$. In general the equation is completely integrable and has zero-curvature representation on the periodic interval $\phi \in (0, 2\pi)$:

$$[\partial_t - U, \partial_\phi + V] = 0$$

$$\partial_\phi f(\phi, t) = -V(\phi|\lambda) f(\phi, t)$$

$$\partial_t f(\phi, t) = U(\phi|\lambda) f(\phi, \lambda)$$

where

$$V(\phi|\lambda) = \frac{i\lambda}{2}\sigma_z + \Omega(\phi), \quad U(\phi|\lambda) = \frac{i\lambda^2}{2}\sigma_z + \lambda\Omega(\phi) + i\sigma_z(\partial_\phi \Omega + c|\zeta|^2)$$

with $\sigma_z = \mathrm{diag}(1, -1)$ and the matrix

$$\Omega(\phi) = \begin{pmatrix} 0 & i\sqrt{c}\zeta^* \\ -i\sqrt{c}\zeta & 0 \end{pmatrix}$$

The Poisson bracket $\{\zeta(\phi_1), \zeta^*(\phi_2)\} = i\hbar\delta(\phi_1 - \phi_2)$ and the other integrals of motion area

$$N = \int_0^{2\pi} |\zeta|^2 d\phi, \quad P = -i\hbar \int_0^{2\pi} \zeta^* \partial_\phi \zeta d\phi$$

These aspects do not touch the essential paradigm of quantisation: commutative algebra of classical observables should be deformed to noncommutative algebra of operators in a Hilbert space of states in such a way that the classical Poisson bracket is replaced by commutator/$i\hbar$
All intebrable systems have a specific feature. Instead of characterising integrability through Lax operators (V, U) one can equivalently characterize integrability through Lax operator V and the r-**matrix** such that:

$$\partial_\phi f(\phi, t) = -V(\phi|\lambda) f(\phi, t)$$

$$\{V(\phi_1|\lambda)\otimes, V(\phi_2|\mu)\} = \delta(\phi_1 - \phi_2)[r(\lambda, \mu), V(\phi_1|\lambda) \otimes I + I \otimes V(\phi_1|\mu)]$$

Now we put ζ to be operators in V, the vaccum state is $|0>$ and $\zeta(\varphi)|0>=0$. In order to do the quantization we put the system on a periodic lattice (polygon discretization) with M sides. Also the evolution variable will be the angle $\varphi := n$ which is increased/decreased by fixed step-angle h. Lax operator is transformed to a discrete one (with λ parameter)

$$L(n|\lambda) = \begin{pmatrix} 1 - \frac{i\lambda h}{2} & -ih\sqrt{c}\zeta_n^\dagger \\ ih\sqrt{c}\zeta_n & 1 + \frac{i\lambda h}{2} \end{pmatrix} = I - V(\phi_n|\lambda)h + O(h^2), \quad \phi_n = nh, n = 1, ..., M$$

and the quantum operators obey $[\zeta_n, \zeta_m^\dagger] = \delta_{nm}/h$ (we consider that $\hbar \equiv 1$). Now, because we have periodic boundary condition the Lax operator is transformed to monodromy operator (imposing periodicity we have transition from zero-curvature formulation to pure Lax formulation using the monodromy matrix).

$$T(\lambda) = L(M|\lambda)...L(1|\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

When we quantize, the classical matrix r is turned into matrix $R = I + \hbar r + O(\hbar^2)$ which "catch" the commutation relations of the monodromy matrix elements using the so-called RTT-relation

$$R(\lambda,\mu)(T(\lambda)\otimes T(\mu)) = (T(\mu)\otimes T(\lambda))R(\lambda,\mu),$$

where the matrix R is given by:

$$R = \begin{pmatrix} f(\lambda,\mu) & 0 & 0 & 0 \\ 0 & g(\lambda,\mu) & 1 & 0 \\ 0 & 1 & g(\lambda,\mu) & 0 \\ 0 & 0 & 0 & f(\lambda,\mu) \end{pmatrix},$$

with $f(\lambda,\mu) = 1 + ic/(\mu-\lambda), g(\lambda,\mu) = ic/(\mu-\lambda)$. Here $2c$ is the $\theta-$dependent coefficient of our NLS Eq. (2), $C^2 g^2/18D^2$. The action of elements of the monodromy matrix on the vacuum is

$$A(\lambda)|0> = a(\lambda)|0>, D(\lambda)|0> = d(\lambda)|0>,$$
$$C(\lambda)|0> = 0, B(\lambda)|0> = \text{free}.$$

As a result, one can see that in our case

$$a(\lambda) = \prod^{M}(1 - i\lambda h/2) = (1 - i\lambda h/2)^M,$$
$$\lim_{M\to\infty} a(\lambda) = e^{-i\lambda L/2}, d(\lambda) = (1 + i\lambda h/2)^M,$$
$$\lim_{M\to\infty}(1 + i\lambda h/2)^M = e^{i\lambda L/2}, \quad L = Mh - \text{finite}, h \to 0.$$

The quantum states are constructed by applying operator $B(\lambda_i)$ from the monodromy matrix. In the case of N parameters (usually called rapidities) we have:

$$\Psi(\lambda_1, ..., \lambda_N) = \prod_{j=1}^{N} B(\lambda_j)|0> .$$

Essential Property

The trace of the monodromy operator is the generating function of ALL the conservation laws. More precisely

$$\log\left(e^{i\lambda\pi}\mathrm{Trace}\,T(\lambda)\right)\to ic\left(\lambda^{-1}N+\lambda^{-2}(P-icN/2)+\lambda^{-3}(H-icP-c^2N/3)+O(\lambda^{-4})\right)$$

Now **imposing that this Ψ must be an eigenvector of the trace of the monodromy matrix** we find the following Bethe equations:

$$e^{2\pi i\lambda_m}=\prod_{j=1,j\neq m}^{N}\left(\frac{\lambda_m-\lambda_j+i\frac{C^2g^2}{36D^2}}{\lambda_m-\lambda_j-i\frac{C^2g^2}{36D^2}}\right),\tag{4}$$

and the eigenvalue of $\mathrm{Trace}\,T(\mu)$ are

$$\mathrm{Trace}\,T(\mu)\Psi=(A(\mu)+D(\mu))\Psi=\Lambda\Psi,$$

with

$$\Lambda(\mu,\lambda_1,...,\lambda_N)=e^{-i\mu\pi}\prod_{j=1}^{N}f(\mu,\lambda_j)+e^{i\mu\pi}\prod_{j=1}^{N}f(\lambda_j,\mu),$$

where, as it was shown above, $f(\mu,\lambda)=1+ic/(\lambda-\mu)$. It is easy to note that all the quasi-momenta λ_j are dimensionless.

Energy levels

The Bethe equations are the fundamental equations that gives the spectra of all the integrals of motion. From the trace formula we get the energy

$$E_N = \frac{\hbar^2}{R_0^2} \sum_{j=1}^{N} \lambda_j^2$$

- Bethe equations are transcendental and the solutions can be computed only numerically (has only real solutions and they are all periodic of period 1, so it is enough to consider the solutions $\lambda_j \in [0, 1]$).
- the structure of the energy spectra changes with N and with the parameter $c = C^2 g^2/36D^2$ and we have a $c = c_{crit}$ around which the energy spectrum becomes very dense. For values $c < c_{crt}$ equidistant spectral lines, while for larger c the spectrum similar with the spectrum for the rigid rotor. The larger the number of eigenvalues N, the smaller the value of c_{crt} is.
- possible expression of the coefficients C, D from liquid drop model were introduced in(Ludu, Draayer, PRL, 80, 2125, (1998)) it results that $c \sim 0$ at $\theta = 0, \pi$. In general, c is big for $\theta = \pi/2$. For certain combinations between h, σ, $c(\theta)$ can be very small for all polar angles θ resulting in a small probability to excite such soliton solutions. For droplets of the size of a medium-heavy nucleus $c \sim 100$ for almost all values of θ, i.e. larger probability to excite such periodic soliton excitations.
- small c the eq is dominant **linear** and we have **free bosons** while for c very big the equation is **strongly nonlinear** and we have **free fermions since the Bethe states obey the Pauli principle**.

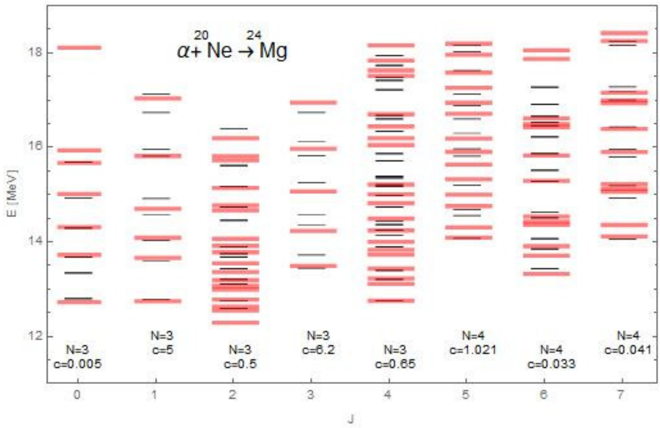

Figure: Black: Experimental energy spectra of positive- and negative-parity resonant states obtained in the collision of $\alpha-$particles on ^{20}Ne targets with formation of bound $\alpha-$cluster states in ^{24}Mg. The spectra are horizontally aligned by angular momentum J from $J = 0^+$ to $J = 7$. Red: The theoretical Bethe spectra are plotted for rapidities N $=$ 3 and 4 with the parameter c chosen to provide the best fit with experiments. The odd angular momentum states (labeled with higher placed text in the figure) provide a good fit for larger values of c, typically $c > c_{crit}$., while the best fit for even states occur for relative smaller c, shown in the figure under each column.

Azimutal rotons on the neutron star surface

- The interaction of stars in close binaries is very important; the tidal interaction is relevant for the production of gravitational waves.
- The asymmetry created in a neutron star by a tidal bulge can produce a certain amount of gravitational waves which can increase in intensity if the bulge can rotate fast.
- Since long duration of the movement of tides have been detected the occurence of solitary waves on their surface is to be expected.
- The neutron star oscillations (the so called r-modes) can be highly nonlinear and unstable
- numerical simulation showed that shock waves may appear due to unknown dissipative mechanisms.
- neutron stars are layered systems and accordingly surface waves can be very dispersive

We consider the following facts. The star is described by an incompresible fluid and surface nonlinear excitations are considered nonlinear and small. Accordingly we have the following equations:

$$\nabla \cdot \vec{v} = 0, \rightarrow \vec{v} = \nabla \Phi, \rightarrow \Delta \Phi = 0$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla P - \rho \nabla \Psi + \rho \vec{F}_{GR}$$

where P is the pressure, Ψ is the newtonian potential which obey the equation

$$\Delta \Psi = 4\pi G \rho$$

The gravitational reaction force due to time-varying current quadrupole is given by:

$$F^x_{GR} - iF^y_{GR} = -i\kappa(x+iy)(3v_z\partial_t^5 J_{22} + z\partial_t^6 J_{22})$$

$$F^z_{GR} = -\kappa \mathrm{Im}\left((x+iy)^2(3\frac{v_x + iv_y}{x+iy}\partial_t^5 J_{22} + \partial_t^6 J_{22})\right)$$

where

$$J_{22} = \int_D \rho r^2(\vec{v} \cdot (\vec{r} \wedge r\nabla Y^*_{22}/\sqrt{6}))d^3x$$

For pure azimutal nonlinear waves we approximate $F_{GR} = 0$.

Fluid equations of neutron star

The small nonlinear deformation of the surface Σ is taken to be described by

$$\Sigma : r = R_0(1 + \zeta(\theta, \phi, t))$$

where R_0 is the radius of the star at equilibrium. Also we consider that the star has a solid bulk of radius r_0. For this deformation the newtonian potential will have the form (we call q the constant $4\pi G\rho/3$)

$$\Psi = \frac{4}{3}\pi G\rho\zeta + O(\zeta^2) \equiv q\zeta + O(\zeta^2)$$

In the spherical coordinates the fluid equations together with boundary conditions are (we neglect θ derivatives):

$$2r\Phi_r + r^2\Phi_{rr} + \frac{1}{\sin^2\theta}\Phi_{\phi\phi} = 0 \quad r_0 < r < R_0(1 + \zeta) \tag{5}$$

$$\Phi_r|_\Sigma = R_0\left(\zeta_t + \frac{\zeta_\phi\Phi_\phi}{r^2\sin^2\theta}\right)|_\Sigma \tag{6}$$

$$\Phi_r|_{r=r_0} = 0 \tag{7}$$

$$\Phi_t + \frac{\Phi_\phi^2}{2r^2\sin^2\theta} + \Phi_r^2 + \frac{4}{3}\pi G\rho R_0\zeta = 0 \tag{8}$$

Now in order to do the asymptotic we consider the following stretched variables:

$$t_1 = \epsilon^{1/2} t, t_3 = \epsilon^{3/2} t, t_5 = \epsilon^{5/2} t, ...$$
$$\xi = \epsilon^{1/2} \phi$$
$$\zeta = \epsilon \hat{\zeta} = \epsilon(\zeta_0 + \epsilon\zeta_1 + \epsilon^2 \zeta_2 + ...)$$
$$\Phi = \epsilon^{1/2} \hat{\Phi} = \epsilon^{1/2}(\Phi_0 + \epsilon\Phi_1 + \epsilon^2 \Phi_2 + ...)$$

and of course

$$\partial_t = \epsilon^{1/2} \partial_{t_1} + \epsilon^{3/2} \partial_{t_3} + ...$$
$$\partial_\phi = \epsilon^{1/2} \partial_\xi$$

Now, using these scales the Laplace equation will be:

$$2r\hat{\Phi}_r + r^2 \hat{\Phi}_{rr} + \frac{\epsilon}{\sin^2\theta} \Phi_{\xi\xi} = 0 \quad r_0 < r < R_0(1+\zeta)$$
$$\hat{\Phi}_r|_{r=r_0} = 0$$

We solve them order by order and we get (M, F, G, H are arbitrary functions of $\xi, t_1, t_3, ...$) at $O(\epsilon^0)$ the solution $\Phi_0 = M/r + F$; but from the boundary condition at $r = r_0$ we remain with

$$\Phi_0 = F.$$
$$\Phi_1 = G - \frac{\operatorname{cosec}^2\theta(r_0 - r + r\log r)}{r} F_{\xi\xi},$$
$$\Phi_2 = H - \frac{\operatorname{cosec}^2\theta(r_0 - r + r\log r)}{r} G_{\xi\xi} + \frac{G^2_{\xi\xi}}{F_{\xi\xi\xi\xi}} +$$
$$+ \frac{\operatorname{cosec}^4\theta(5r - 6r_0 + \log r(-4r - 2r_0 + r\log r) + 4r_0\log r_0)}{2r} F_{\xi\xi\xi\xi},$$

and so on (we do not need more).

Now we go to the Euler equation (8) on the surface Σ. However the inverse of the squared radius at the surface will have the following expansion,

$$\frac{1}{r^2} = \frac{1}{R_0^2} - \frac{2\epsilon\hat{\zeta}}{R_0^2} + O(\epsilon^2)$$

and the Euler equation will have the form:

$$2q\hat{\zeta} + 2\hat{\Phi}_{t_1} + 2\epsilon\hat{\Phi}_{t_3} + \frac{\epsilon}{R_0^2}(1 - 2\epsilon\hat{\zeta})(\mathrm{cosec}^2\theta)\hat{\Phi}_\xi^2 + \hat{\Phi}_r^2 = 0$$

At $O(\epsilon^0)$ we have

$$2q\zeta_0 + 2\Phi_{0,t_1} = 0$$

which gives (using the solution for Φ_0)

$$\zeta_0 = -\frac{1}{q}F_{t_1} \tag{9}$$

Now we use the dynamic boudary condition (6)which in the stretched variables is:

$$\hat{\Phi}_r|_\Sigma = \epsilon R_0\hat{\zeta}_{t_1} + \epsilon^2\zeta_{t_3} + \frac{\epsilon^2\mathrm{cosec}^2\theta}{R_0}(1 - 2\epsilon\hat{\zeta})\hat{\Phi}_\xi\hat{\zeta}_\xi|_\Sigma$$

At $O(\epsilon^0)$ we have $\Phi_{0,r} = \partial_r F(\xi, t_1, ...) = 0$
At $O(\epsilon)$ we have

$$\Phi_{1,r}|_\Sigma = R_0\zeta_{0,t_1}|_\Sigma \tag{10}$$

But from the solution Φ_1 of the Laplace equation,

$$\partial_r\Phi_1|_{r=R_0(1+\epsilon\hat{\zeta})} = -\frac{R_0 - r_0}{R_0^2}\mathrm{cosec}^2\theta F_{\xi\xi} + O(\epsilon).$$

Defining the square of a speed,

$$v^2 := \frac{q(R_0 - r_0)}{R_0^3}\mathrm{cosec}^2\theta$$

we can write (10), (taking into account (9)) as a linear wave equation

$$F_{t_1,t_1} - v^2 F_{\xi\xi} = 0$$

which means that we can put $\xi \to \xi - vt_1$ and

$$F(\xi, t_1, t_3, t_5, ...) = F(\xi - vt_1, t_3....)$$

Now using (9) we have

$$\partial_{t_1} = -v\partial_\xi, \quad \zeta_0 = \frac{v}{q}F_\xi$$

Now we go back to Euler equation at order $O(\epsilon)$ (remember that $\Phi_0 = F$)

$$2q\zeta_1 + 2\Phi_{1,t_1} + F_{t_3} + \frac{\mathrm{cosec}^2\theta}{R_0^2}F_\xi^2 = 0 \tag{11}$$

We have to evaluate Φ_{1,t_1} at $r = R_0(1 + \epsilon\zeta)$. Exapnding we get

$$\Phi_1 = G + \frac{\mathrm{cosec}^2(\theta)(\mathrm{R_0} - \mathrm{r_0} - \mathrm{R_0}\log \mathrm{R_0})}{R_0}F_{\xi\xi} + O(\epsilon)$$

To simplfy the notation we denote

$$A := \frac{\mathrm{cosec}^2(\theta)(\mathrm{R_0} - \mathrm{r_0} - \mathrm{R_0}\log \mathrm{R_0})}{R_0}$$

so

$$\Phi_{t_1} = G_{t_1} + AF_{\xi\xi t_1} + O(\epsilon) = -vG_\xi - AvF_{\xi\xi\xi}$$

Introducing in (11) and making a derivative with respect to ξ we get:

$$\boxed{q\zeta_{1\xi} - vG_{\xi\xi} = qA\zeta_{0\xi\xi\xi} - \frac{q}{v}\zeta_{0t_3} - \frac{q^2\mathrm{cosec}^2(\theta)}{v^2R_0^2}\zeta_0\zeta_{0\xi}}$$

Now we go back to the $O(\epsilon^2)$ terms for the dynamic boundary condition, namely

$$\Phi_{2,r}|_\Sigma = R_0\zeta_{1t_1} + R_0\zeta_{0t_3} + \frac{\mathrm{cosec}^2(\theta)}{R_0}\Phi_{0\xi}\zeta_{0\xi}|_\Sigma \tag{12}$$

$$\Phi_{2,r}|_{r=R_0(1+\epsilon\zeta)} = -\frac{(R_0 - r_0)\mathrm{cosec}^2(\theta)}{R_0^2}G_{\xi\xi} + BF_{\xi\xi\xi\xi} + O(\epsilon)$$

where we make trhe notation:

$$B := \mathrm{cosec}^4(\theta)\frac{-2(R_0 - r_0 + r_0\log r_0) + (r_0 + R_0)\log R_0}{R_0^2}.$$

From the definition of the speed v^2 and from the fact that $(\partial_{t_1} = -v\partial_\xi, F_\xi = \Phi_{0\xi} = q\zeta_0/v$, the equation (12) will become after multiplying both members with $q/(vR_0)$:

$$\boxed{q\zeta_{1\xi} - vG_{\xi\xi} = \frac{q}{v}\zeta_{0t_3} + \frac{q^2\mathrm{cosec}^2(\theta)}{v^2R_0^2}\zeta_0\zeta_{0\xi} - \frac{q^2B}{v^2R_0}\zeta_{0\xi\xi\xi}}$$

Now one can see immediately that the two boxed equations have **the same left hand side**. Accordingly from both right hand sides we get the following **general Korteweg de Vries** equation:

$$\frac{2q}{v}\zeta_{0t_3} + \frac{2q^2\mathrm{cosec}^2(\theta)}{v^2R_0^2}\zeta_0\zeta_{0\xi} - \left(qA + \frac{q^2B}{v^2R_0}\right)\zeta_{0\xi\xi\xi} = 0$$

which can be scaled to $\zeta_{0t_3} + 6\zeta_0\zeta_{0\xi} + \zeta_{0\xi\xi\xi} = 0$

Multi-phase solution of Korteweg de Vries equation

In the physical variables (rescaling time with $2q/v$) our KdV equation is

$$\zeta_t + \frac{2q(\theta)R_0}{R_0 - r_0}\zeta_0\zeta_{0\xi} + \left(1 + \frac{r_0}{R_0} - \frac{2r_0}{R_0 - r_0}\ln\frac{R_0}{r_0}\right)\zeta_{0\xi\xi\xi} = 0$$

If difference between the too radii $\epsilon = R_0 - r_0$ is not very big then we are in the situation of the following equation:

$$\zeta_t + 2q(\theta)(1 + \frac{r_0}{\epsilon})\zeta_0\zeta_{0\xi} + \frac{\epsilon^2}{3r_0^2}\zeta_{0\xi\xi\xi} + O(\epsilon^3) = 0$$

whihc meand that the nonlinear terms is strongly dominant and we expect the appearance of shock waves (as it was observed numerically by Tohline et all PRL 2001...). We can estimate the solution in this limit using Whitham averaging method.

- for multi-phase solution of the **scaled** KdV equation we use Dubrovin's method: Take $\zeta_0(\xi, t_3) = 2(\log T)_{\xi\xi}$, $T = T(z_1, ..., z_g) \equiv T(\vec{\kappa}\xi + \vec{\omega}t_3), \vec{\kappa}, \vec{\omega} \in \mathbb{C}^g$ and calling $L = \log T$ we get:

$$\sum_{i,j=1}^{g}\omega_i\kappa_j L_{ij} + \sum_{i,j,k,l=1}^{g}\kappa_i\kappa_j\kappa_k\kappa_l(L_{ijkl} + 2(L_{ij}L_{kl} + L_{ik}L_{jl} + L_{il}L_{jk})) = c$$

  with c arbitrary constant and $L_{ij...} = \partial_{z_i z_j ...}L$. Applying addition property for the Riemann Theta function $T = \Theta(\kappa\xi + \omega t_3|M)$ we get the following **algebraic** system for $\vec{\kappa}, \vec{\omega}, c$.

$$\sum_{i,j=1}^{g}\omega_i\kappa_j W_{\epsilon,ij}(0) + \sum_{i,j,k,l=1}^{g}\kappa_i\kappa_j\kappa_k\kappa_l W_{\epsilon,ijkl}(0) = cW_\epsilon(0)$$

  with $\epsilon \in \mathbb{Z}_2^g$.

We have 2^g equations with $2g$ unknown quantities $(\kappa_j/c^{1/4}, \omega_j/c^{3/4})$. For $g=1$ we have the 1-cnoidal solution with the dispersion relation given by

$$\omega = -k^3 \frac{W_1(0)W_0(0)'''' - W_0(0)W_1(0)''''}{W_1(0)W_0(0)'' - W_0(0)W_1(0)''}$$

where

$$W_\epsilon(0) = \frac{1}{2}\Theta\left(\frac{\epsilon}{2}|\frac{M}{2}\right)$$

with M the period matrix for a suitable Riemann surface.
In the case of zero disperison and we have the fully nonlinear equation:

$$\zeta_{0t} + 2q(\theta)r_0\zeta_0\zeta_{0\xi} = 0$$

and for a given periodic initial condition $\zeta(\xi, 0) = F(\xi)$ we find the general solution as an implicit equation

$$\zeta(\xi, t) = F(\xi - 2\zeta(\xi, t)qr_0t)$$

. This solution develops a singularity. Indeed

$$\xi - 2qr_0t\zeta = F^{-1}(\zeta), \quad 1 - 2qr_0t\zeta_\xi = F_\zeta^{-1}\zeta_\xi$$

$$1 = (2qr_0t + F_\zeta^{-1})\zeta_\xi$$

Accordingly for

$$t > t_c(\theta) = \min_{\zeta\in[0,2\pi)}\left(-\frac{F_\zeta^{-1}}{2q(\theta)r_0}\right)$$

the solution breaks $(\zeta_\xi = \infty)$

In the case of *small* dispersion the solution is expressed as

$$\zeta_0(\xi, t) = 2\epsilon^3(\log \Theta((\kappa\xi + \omega t)/\epsilon^{3/2}|M))_{\xi\xi}$$

where κ, ω are no longer constants but varies slowly (Whitham modulation) according to a *completely integrable* dispersionless nonlinear system.

$$\partial_t \beta_i + \lambda_i \partial_\xi \beta_i = 0$$

where

$$\kappa = \pi\sqrt{\beta_1 - \beta_3}/K(m), \omega = 2\kappa(\beta_1 + \beta_2 + \beta_3)$$

$$\lambda_i = \prod_{i \neq k}(\beta_i - \beta_k)/\beta_i, m = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}$$

Here $K(m)$ is the elliptic integral of the first kind given by:

$$K(m) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m^2 \sin^2 x}}$$

and the period of the Riemann Theta function (with $z = (k\xi + \omega t)\epsilon^{-3/2}$)

$$\Theta(z|M) := \sum_{n=-\infty}^{\infty} e^{i\pi n^2 M + 2\pi i n z}$$

is $M = iK'(m)/K(m)$.

- We developed an asymptotic method to derive defocusing nonlinear Schrodinger equation for a thin spherical fluid shell surrounding a rigid core.
- The key ingredients were the interplay between geometric and physical nonlinearities and the azimutal approximation.
- Periodic solutions were obtained using dispersion relation method and the algebraic Bethe ansatz showed a good agreement beteween Bethe states energies and experimental spectra of α-particles in interaction with nuclei.
- in the context of neutron star, Korteweg de Vries equation has been obtained for *slow variables* (at large scales) together with the multi-phase solutions. It is expected that these solutions may have implications in dynamics of gravitational waves and in general stellar-seismology.

Main references:

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