Generalised Volterra systems; singularities and discrete solitonic phenomenology

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- Conclusions and future problems

Motivation

Volterra equations are fundamental in biological modelling. It appears on many very different forms:

- in evolutionary biology as replicator equation (a kind of *"Schrodinger equation"* for biology)
- at the molecular level, genetic circuits and neuronal circuits are modelled by Volterra equations
- virus dynamics, population (prey-predator) dynamics, epidemic dynamics
- in fluids and plasma recently the incoherent discrete solitons are modelled by Volterra equations

In general, mathematical models in biology are given by equations which are strongly nonlinear and dissipative. So, one cannot have symmetries and conservatuions laws. So we are forced to find patterns and regularities. Some Volterra equations can be considered *perturbed integrable equations*.

Simple examples:

- population dynamics. Let us consider an infinite/finite/periodic chain of species of individuals, indexed by n having the $N_n(t)$ number of individuals. They increase when interact with the individuals in the species $n+1$ and decrease when interact with ones in the $n-1$ according to the rule:

$$\frac{dN_n}{dt} = (1+N_n)(N_{n+1} - N_{n-1})$$

or in general

$$\frac{dN_n}{dt} = F(N_n)(N_{n+1} - N_{n-1})$$

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- genetic network/circuit: the gene n is activated by the gene $n+1$ and repressed by the gene $n-1$. The equation that governs the time evolution of the protein production for the gene n is given by:

$$\frac{dp_n}{dt} = \frac{a+bp_{n+1}}{1+p_{n+1}} + \frac{b+ap_{n-1}}{1+p_{n-1}} - \lambda p_n$$

  where

$$f(p_i) = \frac{a+bp_i}{1+p_i}$$

  is the promoter-activity function of the gene "i" and it is activated for $b>a$ and repressed for $b<a$. Also λ is the protein degradation rate. Making the substitution $M_n = 1/(1+p_n)$ we find

$$dM_n/dt = (b-a)M_n^2(M_{n+1} - M_{n-1}) + \mathrm{perturbation(b, a, \lambda)}$$

Question: how do we study these types of equations? They are differential-difference form and hard to apply methods of hamiltonian mechanics (to get invariants for example).
On the other hand we want patterns, regular dynamics (nonlinear equations have in general *chaotic* dynamics),

Integrability vs Singularities

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In order to avoid *butterfly effect* on the Riemann sheets of some branch points of the solutions we impose that in t the singular part to be at most *poles*

On the other hand the equations are discrete. It means that we have iterations. If the iterations does not develop indeterminacies and after a finite number of iterations the singular behaviour is confined then we are in a situation of a possible non-chaotic equation which obey the singularity confining criterion.

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On the other hand the equations are discrete. It means that we have iterations. If the iterations does not develop indeterminacies and after a finite number of iterations the singular behaviour is confined then we are in a situation of a possible non-chaotic equation which obey the singularity confining criterion. To ilustrate this let us consider,

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1})$$

which can be written as a 2-point mapping,

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni (u_n, v_n) \to (u_{n+1}, v_{n+1}) \in \mathbb{P}^1 \times \mathbb{P}^1$$

whose points are depeding on t:

$$u_{n+1} = v_n \tag{1}$$

$$v_{n+1} = \frac{\dot{v}_n}{v_n} + u_n \tag{2}$$

It is obvious that if (u_n, v_n) have no movable critical singularities, then the same will be true for (u_{n+1}, v_{n+1}). Let us consider the simplest case, in a neighbourhood of t, to have a simple zero for v_n and regular u_n. Thus the curve $(u_n, 0)$ goes to a point $(0, \infty)$ which means loosing a degree of freedom (curve blow-down process). More precisely, starting as above from $(\tau = t - t_0)$,

$$u_n = a_0 + a_1\tau + O(\tau^2), v_n = \alpha\tau + \beta\tau^2 + O(\tau^3)$$

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$$\begin{pmatrix} a_0 \\ \alpha\tau + ... \end{pmatrix} \rightarrow \begin{pmatrix} \alpha\tau + ... \\ \tau^{-1} + \beta/\alpha + a_0 + ... \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} \tau^{-1} + \beta/\alpha + a_0 + ... \\ -\tau^{-1} + \beta/\alpha + a_0 + ... \end{pmatrix} \rightarrow \begin{pmatrix} -\tau^{-1} + \beta/\alpha + a_0 + ... \\ \gamma(a_0, \alpha, \beta)\tau + ... \end{pmatrix} \rightarrow \begin{pmatrix} \gamma(a_0, \alpha, \beta)\tau + ... \\ f(a_0, \alpha, \beta) + ... \end{pmatrix}$$

where γ, f are some finite expressions containing the parameters a_0, α, β etc. So in a small neighbourhood of t_0 (where $\tau \approx 0$) we can write

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$$... \to \text{regular} \to \begin{pmatrix} a_0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} \infty \\ -\infty \end{pmatrix} \to \begin{pmatrix} -\infty \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ f(a_0, \alpha, \beta) \end{pmatrix} \to \text{regu}$$

So the initial curve blows down to three points and then blows up to another curve containing initial parameters. In this way the singularity confinement is satisfied.

This singularity pattern is crucial. It helps us to *find a substitution* which *solves explicitly the equation*
Indeed one can see immediately that for both u_n, v_n the pattern is

$$u_n(t) : \text{...regular} \rightarrow 0 \rightarrow \infty \rightarrow \infty \rightarrow 0 \rightarrow \text{regular...}$$

$$v_{n-1}(t) : \text{...regular} \rightarrow 0 \rightarrow \infty \rightarrow \infty \rightarrow 0 \rightarrow \text{regular...}$$

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So exist a function F_n which is *holomorphic* and u_n, v_n are expressed as ratios of products of such functions. Hence let us consider that u_n has a function F_n in the numerator and this F_n passes through 0, so $u_n = 0$. Because u_{n+1}, u_{n+2} are infinite then the denominator of u_n must have F_{n-1}, F_{n-2}. Then $u_{n+3} = 0$ so at the numerator we have F_{n-3}. Accordingly one can write

$$u_n = \frac{F_n F_{n-3}}{F_{n-1} F_{n-2}}$$

and introducing in the equation we find the *bilinear form*:

$$E(F_n \cdot F_n) \equiv (\partial_t F_{n-1}) F_{n-2} - F_{n-1}(\partial_t F_{n-2}) - F_n F_{n-3} + F_{n-1} F_{n-2} = 0$$

F_n is holomorphic, equation is bilinear so the solutions will be a **series of exponentials**

Let:

$$F_n(t) = 1 + \epsilon f_n + \epsilon^2 g_n + \epsilon^3 h_n + ...$$

- $$O(\epsilon) : \partial_t(f_{n-1} - f_{n-2}) - f_n - f_{n-3} + f_{n-1} + f_{n-2} = 0$$
- $$O(\epsilon^2) : \partial_t(g_{n-1} - g_{n-2}) - g_n - g_{n-3} + g_{n-1} + g_{n-2} = E(f_n \cdot f_n)$$
- $$O(\epsilon^3) : \partial_t(h_{n-1} - h_{n-2}) - h_n - h_{n-3} + h_{n-1} + h_{n-2} = E(f_n \cdot g_n)$$

So at the first order take $f_n = e^{kn+\omega t}$.
We get the *dispersion* relation

$$\omega(k) = (1 + e^{-3k} - e^{-k} - e^{-2k})/(e^{-k} - e^{-2k}) = 2\sinh k$$

and **all** other terms can be zero. So one solution is

$$F_n = 1 + e^{kn+\omega t}$$

so, the Volterra equation will have the **1-discrete soliton solution**

$$u_n = \frac{(1 + e^{kn+\omega t})(1 + e^{k(n-3)+\omega t})}{(1 + e^{k(n-1)+\omega t})(1 + e^{k(n-2)+\omega t})}$$

For 2-modes solution $f_n = e^{k_1 n + \omega_1 t} + e^{k_2 n + \omega_2 t}$, we can no longer take $g_n = 0$. In fact this imposes $g_n = e^{A_{12}} e^{(k_1+k_2)n + (\omega_1+\omega_2)t}$ with

$$e^{A_{12}} = \left(\frac{\sinh(k_1 - k_2)/4}{\sinh(k_1 + k_2)/4)} \right)^2$$

which means we have interaction of modes and the **2-discrete soliton solution**

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For $N = 3$

$$F_n = 1 + \sum_{i=1}^{3} e^{k_i n + \omega_i t} + \sum_{i<j} e^{A_{ij}} e^{(k_i+k_j)n + (\omega_i+\omega_j)t} + e^{A_{12}+A_{13}+A_{23}} e^{(k_1+k_2+k_3)n + (\omega_1+\omega_2+\omega_3)t}$$

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For N-modes the solution will be

$$F_n(t) = \sum_{\mu_1,\ldots,\mu_N\in\{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i(k_i n + \omega_i(k_i)t) + \sum_{i<j}^{N} A_{ij}\mu_i\mu_j\right)$$

and all interaction terms are **factorized in two-modes interactions** i.e. *bilinear integrability*$\equiv$ existence of discrete **N-soliton solution**.

From the structure of 2-soliton solution we can extract the information about the scattering process. This is done in a frame comoving with the one of the solitons and letting the other soliton to go to spatial infinity as $|t| \to \infty$ Let us write the 2-soliton as:

$$F_n = 1 + e^{\eta_1} + e^{\eta_2} + e^{A_{12}} e^{\eta_1 + \eta_2}$$

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- $\eta_2 \to -\infty$, $F_n \to 1 + e^{\eta_1}$
- $\eta_2 \to \infty$, $F_n \to e^{\eta_2}(1 + e^{\eta_1-\eta_2} + e^{-\eta_2} + e^{\eta_1+A_{12}}) \equiv 1 + e^{A_{12}+\eta_1}$

The soliton speed is $-\omega_1/k_1 = -2\sinh k_1/k_1$ and the location of the soliton 1 changes with $\Delta n_1 = -A_{12}/k_1$.
The interaction is in fact *atractive*, namely the faster one advances and the slower one is retarded. Moreover, during the collisions the amplitude decreases.
This shows that the full interaction process is nonlinear.

We want to study higher order (generalisations) of Volterra equations. We are going to consider the following examples

$$\dot{u}_n = u_n(\sum_{j=1}^{N} u_{n+j} - \sum_{j=1}^{N} u_{n-j}) \tag{3}$$

then the first multiplicative Bogoyavlenski equation (mB1),

$$\dot{u}_n = u_n(\prod_{j=1}^{N} u_{n+j} - \prod_{j=1}^{N} u_{n-j}) \tag{4}$$

the second multiplicative Bogoyavlenski one (mB2)

$$\dot{u}_n = u_n^2(\prod_{j=1}^{N} u_{n+j} - \prod_{j=1}^{N} u_{n-j}). \tag{5}$$

and finally the third multiplicative Bogoyavlensii lattice (B3) which is a new equation

$$\dot{u}_n = u_n(u_n - a)(\prod_{j=1}^{N} u_{n+j} - \prod_{j=1}^{N} u_{n-j}) \tag{6}$$

Take $N = 3$ for the first equation. The system can be written as:

$$u_{1,n+1} = u_{2,n}, \quad u_{2,n+1} = u_{3,n} \tag{7}$$

$$u_{3,n+1} = u_{4,n} \tag{8}$$

$$u_{4,n+1} = u_{5,n} \tag{9}$$

$$u_{5,n+1} = u_{6,n} \tag{10}$$

$$u_{6,n+1} = \dot{u}_{4,n}/u_{4,n} + u_{1,n} + u_{2,n} + u_{3,n} - u_{5,n} - u_{6,n} \tag{11}$$

Singularities and bilinear forms

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and consider all $u_{i,n}, i = 1, 2, 3, 5, 6$ are regular and initially have the finite values $u_{n,i} = f_i + O(\tau)$, except $u_{4,n} = \alpha\tau + O(\tau^2)$.

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$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ f_5 \\ f_6 \end{pmatrix} \to \begin{pmatrix} * \\ * \\ 0 \\ * \\ * \\ \infty \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ * \\ * \\ \infty \\ \infty \end{pmatrix} \to \begin{pmatrix} 0 \\ * \\ * \\ \infty \\ \infty \\ * \end{pmatrix} \to \begin{pmatrix} * \\ * \\ \infty \\ \infty \\ * \\ * \end{pmatrix} \to$$

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This is a confined pattern since the codimensions of subvarities are the in the following sequence:

$$1 \to 2 \to 3 \to 3 \to 2 \to 3 \to 3 \to 2 \to 1$$

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and accordingly one can write:

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In general

$$u_{1,n+1} = u_{2,n} \tag{12}$$

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$$\cdots \tag{14}$$

$$u_{2N-1,n} = u_{2N,n} \tag{15}$$

$$u_{2N,n} = \frac{\dot{u}_{N+1,n}}{u_{N+1,n}} + \sum_{k=1}^{N} u_{k,n} - \sum_{k=N+2}^{2N} u_{k,n} \tag{16}$$

Now if $u_{i,n} = f_i + O(\tau)$ regular for $i = 1, .., N, N+2, ...2N$ and singularity enters through $u_{N+1,n} = a\tau + O(\tau^2)$ we will obtain a strictly confining pattern in $2N+1$ steps.

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ 0 \\ \vdots \\ f_{2N} \end{pmatrix} \to \begin{pmatrix} f_2 \\ f_3 \\ \vdots \\ 0 \\ f_{N+2} \\ \vdots \\ \infty \end{pmatrix} \to \ldots \to \begin{pmatrix} 0 \\ f_{N+2} \\ f_{N+3} \\ \vdots \\ \infty \\ -\infty \\ \vdots \end{pmatrix} \to \ldots \to \begin{pmatrix} * \\ * \\ \vdots \\ 0 \\ \vdots \\ * \\ * \end{pmatrix}$$

The bilinear form can be recovered from the substitution given by singularity analysis

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$$D_tF_{n+1} \cdot F_n + F_nF_{n+1} = F_{n+1+N}F_{n-N} \tag{17}$$

where $D_t^m a \cdot b = \partial_y^m a(t+y)b(t-y)|_{y=0}$ is the Hirota bilinear operator.
Down-shifting a half integer and taking into account that
$\exp(mD_n)a \cdot b = a(n+m)b(n-m)$ we can write in a nicer way,

$$\left(D_t e^{\frac{1}{2}D_n} + e^{\frac{1}{2}D_n} - e^{(\frac{1}{2}+N)D_n}\right) F_n \cdot F_n = 0$$

The general M-discrete soliton solution (we say M to avoid confusion with N the number of terms in the equation) has the form:

$$F_n(t) = \sum_{\mu_1,\ldots,\mu_M \in \{0,1\}} \exp\left(\sum_{i=1}^{M} \mu_i(k_in + \omega_it) + \sum_{i<j}^{M} A_{ij}\mu_i\mu_j\right) \tag{18}$$

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with the dispersion relation and interaction phase given by

$$\omega_i(k_i) = \frac{2\sinh(k_iN/2)\sinh(k_i(N+1)/2)}{\sinh(k_i/2)}$$

$$\exp A_{ij} = \frac{-\cosh((k_i-k_j)/2) + \cosh(1/2(k_i-k_j)(1+2N)) + (-\omega_i+\omega_j)\sinh((k_i-k_j)/2)}{\cosh((k_i+k_j)/2) - \cosh(1/2(k_i+k_j)(1+2N)) + (\omega_i+\omega_j)\sinh((k_i+k_j)/2)}$$

Multiplicative mB1 equation:
Here we have o more complicated situation where the confined patterns are not visible when starting from simple singularities. Let us consider **the even N case** and take $N = 2$. The equation has the form

$$\dot{u}_n = u_n(u_{n+1}u_{n+2} - u_{n-1}u_{n-2})$$

which can be written as a mapping from $(\mathbb{P}^1)^4$ to itself:

$$U_{1,n+1} = U_{2,n} \tag{19}$$
$$U_{2,n+1} = U_{3,n} \tag{20}$$
$$U_{3,n+1} = U_{4,n} \tag{21}$$
$$U_{4,n+1} = \frac{\partial_t U_{3,n} + U_{1,n}U_{2,n}U_{3,n}}{U_{3,n}U_{4,n}} \tag{22}$$

The simplest sources of singularities can be simple zeros of $U_{3,n}$ or $U_{4,n}$ so we can write initially,

$$U_{3,n} = a_0\tau + O(\tau^2)$$

or $U_{4,n} = \alpha_0\tau + O(\tau^2)$ and $U_{1,n}$, $U_{2,n}$, $U_{4,n}$ or $U_{3,n}$ are regular.

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or $U_{4,n} = \alpha_0\tau + O(\tau^2)$ and $U_{1,n}$, $U_{2,n}$, $U_{4,n}$ or $U_{3,n}$ are regular. In the first case we find the following anti-confining pattern (where $(a_1, a_2, a_0\tau, a_4)^T$ is the initial condition)

$$\dots\begin{pmatrix}0\\ \infty\\ 0\\ \infty\end{pmatrix}\to\begin{pmatrix}\infty\\ 0\\ \infty\\ *\end{pmatrix}\to\begin{pmatrix}0\\ \infty\\ *\\ 0\end{pmatrix}\to\begin{pmatrix}\infty\\ *\\ 0\\ *\end{pmatrix}\to\begin{pmatrix}*\\ 0\\ *\\ *\end{pmatrix}\to\dots$$

$$\to\begin{pmatrix}a_1\\ a_2\\ 0\\ a_4\end{pmatrix}\to\begin{pmatrix}*\\ 0\\ *\\ \infty\end{pmatrix}\to\begin{pmatrix}0\\ *\\ \infty\\ 0\end{pmatrix}\to\begin{pmatrix}*\\ \infty\\ 0\\ \infty\end{pmatrix}\to\begin{pmatrix}\infty\\ 0\\ \infty\\ 0\end{pmatrix}\to\dots.$$

Starting with U_4 we go to essentially the same situation, an anti-confining pattern. This picture can be easily generalised for any N. These type of patterns usually are related to *linearisable* systems and they have *no use in solving the equation*.

$$\dots \begin{pmatrix} 0 \\ \infty \\ 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ \infty \\ * \end{pmatrix} \to \begin{pmatrix} 0 \\ \infty \\ * \\ 0 \end{pmatrix} \to \begin{pmatrix} \infty \\ * \\ 0 \\ * \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ * \\ * \end{pmatrix} \to \dots$$

$$\to \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ a_4 \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ * \\ \infty \end{pmatrix} \to \begin{pmatrix} 0 \\ * \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} * \\ \infty \\ 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ \infty \\ 0 \end{pmatrix} \to \dots.$$

Starting with U_4 we go to essentially the same situation, an anti-confining pattern. This picture can be easily generalised for any N. These type of patterns usually are related to *linearisable* systems and they have *no use in solving the equation*. In the **case of N odd** the situation is even worse. The patterns are totally non-confined For example in the case of $N = 3$

$$U_{1,n+1} = U_{2,n}, \quad U_{2,n+1} = U_{3,n} \tag{23}$$

$$U_{3,n+1} = U_{4,n}, \quad U_{4,n+1} = U_{5,n} \tag{24}$$

$$U_{5,n+1} = U_{6,n} \tag{25}$$

$$U_{6,n+1} = \frac{\partial_t U_{4,n} + U_{1,n}U_{2,n}U_{3,n}U_{4,n}}{U_{4,n}U_{5,n}U_{6,n}} \tag{26}$$

If the source of singularity is either $U_{6,n} = a_0\tau + O(\tau^2)$ or $U_{5,n} = a_1\tau + O(\tau^2)$ or $U_{4,n} = a_2\tau + O(\tau^2)$ we have *nonconfined patterns* with forward evolution to zeros and infinities in the column and backward evolution with finite elements i.e. forward evolution is:

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$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ 0 \end{pmatrix} \to \begin{pmatrix} * \\ * \\ * \\ * \\ 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} * \\ * \\ * \\ 0 \\ \infty \\ * \end{pmatrix} \to \begin{pmatrix} * \\ * \\ 0 \\ \infty \\ * \\ * \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ \infty \\ * \\ * \\ \infty \end{pmatrix} \to$$

$$\to \begin{pmatrix} 0 \\ \infty \\ * \\ * \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} \infty \\ * \\ * \\ \infty \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} * \\ * \\ \infty \\ 0 \\ 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} * \\ \infty \\ 0 \\ 0 \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ 0 \\ \infty \\ 0 \\ 0 \end{pmatrix} \to \text{...points}$$

Interaction of singularities and recovering of bilinear forms

Very recently was shown the lattice KdV equation can have *many types of singularities and superposition or interaction of singularities were analysed*. In order to see if this happens in our examples, we are going to analyse again the mB1 lattice for $N = 2$ (system (30)-(33)). Let us consider the following expansions:

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$$U_{4,n} = \frac{a_4}{\tau} + b_4 + c_4\tau + O(\tau^2)$$

So one singularity is produced from $U_{3,n}$=0 and another one from $U_{4,n} = \infty$. We will find the following **strictly confining singularity pattern**

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$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \infty \end{pmatrix} \rightarrow \begin{pmatrix} * \\ 0 \\ \infty \\ * \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \infty \\ * \\ \infty \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} \infty \\ * \\ \infty \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} * \\ \infty \\ 0 \\ * \end{pmatrix} \rightarrow \begin{pmatrix} \infty \\ 0 \\ * \\ * \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix} \rightarrow \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \rightarrow \ldots.$$

In the case of $N = 3$ for the system (34)-(39) we start again from the "interacting" expansions:

$$U_{1,n} = f_1 + g_1\tau + h_1\tau^2 + O(\tau^3)$$
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$$U_{3,n} = f_3 + g_3\tau + h_3\tau^2 + O(\tau^4)$$
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$$U_{6,n} = f_6 + g_6\tau + h_6\tau^2 + O(\tau^2)$$

and we find the following pattern:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ \infty \\ f_6 \end{pmatrix} \to \begin{pmatrix} * \\ * \\ 0 \\ \infty \\ * \\ * \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ \infty \\ * \\ * \\ \infty \end{pmatrix} \to \begin{pmatrix} 0 \\ \infty \\ * \\ * \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} \infty \\ * \\ * \\ \infty \\ 0 \\ * \end{pmatrix} \to$$

$$\to \begin{pmatrix} \infty \\ * \\ * \\ \infty \\ 0 \\ * \\ * \end{pmatrix} \to \begin{pmatrix} * \\ \infty \\ 0 \\ * \\ * \\ * \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ * \\ * \\ * \\ * \end{pmatrix} \to \text{...regular}$$

In the even case the orbit of let us say $U_{1,n}$ we find

$$0 \to \infty \to * \to \infty \to 0$$

showing exactly the bilinear substitution for the bilinear form

$$U_{1,n} = \frac{F_{n-2}F_{n+2}}{F_{n-1}F_{n+1}}$$

In the odd case the orbit of $U_{1,n}$ gives the following pattern,

$$* \to 0 \to \infty \to * \to * \to \infty \to 0 \to *$$

and the corresponding bilinear substitution:

$$U_{1,n} = \frac{F_{n-2}F_{n+3}}{F_{n-1}F_{n+2}}$$

These substitutions can be easily generalised to any N.

So for the *general odd case*, $N = 2\nu + 1$ we have as above the following substitution:

$$u_n(t) = \frac{F_{n+\nu+2} F_{n-\nu-1}}{F_{n-\nu} F_{n+\nu+1}} \tag{27}$$

and one can see immediately that,

$$\prod_{i=1}^{2\nu+1} u_{n+i} = \frac{F_{n-\nu-1} F_{n+3\nu+3}}{F_{n+\nu+1}^2}, \quad \prod_{i=1}^{2\nu+1} u_{n-i} = \frac{F_{n-3\nu-2} F_{n+\nu+2}}{F_{n-\nu}^2}.$$

Introducing in (mB1) we get:

$$\partial_t \left(\frac{F_{n+\nu+2} F_{n-\nu-1}}{F_{n-\nu} F_{n+\nu+1}} \right) = \frac{F_{n-\nu}^2 F_{n-\nu-1} F_{n+3\nu+3} - F_{n+\nu+1}^2 F_{n+\nu+2} F_{n-3\nu-2}}{F_{n-\nu}^2 F_{n+\nu+1}^2} \tag{28}$$

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The LHS can be written in the form:

$$\partial_t \left(\frac{F_{n+\nu+2}F_{n-\nu-1}}{F_{n-\nu}F_{n+\nu+1}} \right) = \frac{F_{n-\nu-1}F_{n-\nu}D_tF_{n+\nu+2} \cdot F_{n+\nu+1} + F_{n+\nu+2}F_{n+\nu+1}D_tF_{n-\nu-1} \cdot F_{n-\nu}}{F^2_{n-\nu}F^2_{n+\nu+1}}$$

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Introducing in (28) we find the following quadrilinear expression:

$$F_{n-\nu-1}F_{n-\nu}(D_t F_{n+\nu+2} \cdot F_{n+\nu+1} + F_{n-\nu}F_{n+3\nu+3}) = \tag{29}$$

$$= F_{n+\nu+2}F_{n+\nu+1}(D_t F_{n-\nu-1} \cdot F_{n-\nu} + F_{n+\nu+1}F_{n-3\nu-2})$$

This quadrilinear expression is correct provided the following bilinear forms holds (showing that first factors in LHS is equal with the second factor in RHS and viceversa):

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These two bilinear forms are in fact identical since the second is the down-shift of the first with $(2\nu + 1)$ steps.

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These two bilinear forms are in fact identical since the second is the down-shift of the first with $(2\nu + 1)$ steps. So down-shifting the first one with $(\nu + 1)$ steps we get the final bilinear form of mB1:

$$D_t F_n \cdot F_{n+1} + F_{n-2\nu-1} F_{2\nu+2} - F_n F_{n+1} = 0 \tag{30}$$

In the case of $N = 2\nu$ even we consider the following substitution:

$$u_n(t) = \frac{F_{n+\nu+1}F_{n-\nu-1}}{F_{n-\nu}F_{n+\nu}} \tag{31}$$

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$$u_n(t) = \frac{F_{n+\nu+1}F_{n-\nu-1}}{F_{n-\nu}F_{n+\nu}} \tag{31}$$

Using the same procedures as above we find

$$F_{n+\nu+1}F_{n+\nu}(D_tF_{n-\nu-1} \cdot F_{n-\nu} + F_{n+\nu}F_{n-3\nu-1}) =$$

$$F_{n-\nu-1}F_{n-\nu}(D_tF_{n+\nu} \cdot F_{n+\nu+1} + F_{n-\nu}F_{n+3\nu+1})$$

Finally the bilinear form is

$$D_tF_n \cdot F_{n+1} + F_{n-2\nu}F_{n+2\nu+1} - F_nF_{n+1} = 0 \tag{32}$$

One can see immediately that the bilinear forms are identical for N odd and even and it is also the same with the one of (aB) (up to a sign of time):

$$D_tF_n \cdot F_{n+1} + F_{n-N}F_{n+N+1} - F_nF_{n+1} = 0 \tag{33}$$

The multisoliton solution is the same as in the additive case.

In the case of mB2 lattice we have the same situation. We take the case $N=2$

$$U_{1,n+1} = U_{2,n}, \quad U_{2,n+1} = U_{3,n}, \quad U_{3,n+1} = U_{4,n} \tag{34}$$

$$U_{4,n+1} = \frac{\partial_t U_{3,n} + U_{1,n} U_{2,n} U_{3,n}^2}{U_{3,n}^2 U_{4,n}} \tag{35}$$

and consider the following initial "interacting" singularity pattern:

$$U_{1,n} = a_1 + O(\tau), \quad U_{2,n} = a_2 + O(\tau)$$

$$U_{3,n} = a_3\tau + b_3\tau^2 + c_3\tau^3 + O(\tau^4), U_{4,n} = \frac{a_4}{\tau} + b_4 + c_4\tau + O(\tau^2).$$

We will find the following strictly confining pattern:

$$\ldots \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \infty \end{pmatrix} \to \begin{pmatrix} * \\ 0 \\ \infty \\ \infty \end{pmatrix} \to \begin{pmatrix} 0 \\ \infty \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} \infty \\ \infty \\ 0 \\ * \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ * \\ * \end{pmatrix} \to$$

$$\to \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix} \to \ldots\text{regular}$$

One can easily see the orbit of $U_{2,n}$ (or anyone of them) having the form,

$$* \to 0 \to \infty \to \infty \to 0 \to *$$

which gives

$$U_{2,n} = \frac{F_{n-1}F_{n+2}}{F_n F_{n+1}}$$

For the solution we consider the following nonlinear substitution

$$u_n = \frac{F_{n-1}F_{n+N}}{F_nF_{n+N-1}}. \tag{36}$$

which gives the following relations:

$$u_n^2 \prod_{i=1}^{N} u_{n+i} = \frac{F_{n-1}^2 F_{n+2N} F_n}{F_n^2 F_{n+N-1}^2}$$

$$u_n^2 \prod_{i=1}^{N} u_{n-i} = \frac{F_{n+N}^2 F_{n-N-1} F_{n+N-1}}{F_n^2 F_{n+N-1}^2}$$

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Also

$$\dot{u}_n = \partial_t \left(\frac{F_{n-1}F_{n+N}}{F_nF_{n+N-1}} \right) = \frac{F_{n-1}F_nD_tF_{n+N} \cdot F_{n+N-1} + F_{n+N}F_{n+N-1}D_tF_{n-1} \cdot F_n}{F_n^2F_{n+N-1}^2}$$

Introducing all these relations in the mB2 we find:

$$F_{n-1}F_n(D_tF_{n+N} \cdot F_{n+N-1} - F_{n-1}F_{n+2N}) = F_{n+N}F_{n+N-1}(D_tF_n \cdot F_{n-1} - F_{n+N}F_{n-N-1})$$

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Accordingly the bilinear form will be

$$D_t F_{n-1} \cdot F_n - F_{n+N}F_{n-N-1} - F_n F_{n-1} = 0 \tag{37}$$

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For the solution we consider the following nonlinear substitution

$$u_n = \frac{F_{n-1}F_{n+N}}{F_nF_{n+N-1}}. \qquad (36)$$

which gives the following relations:

$$u_n^2 \prod_{i=1}^{N} u_{n+i} = \frac{F_{n-1}^2 F_{n+2N} F_n}{F_n^2 F_{n+N-1}^2}$$

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$$F_{n-1}F_n(D_tF_{n+N} \cdot F_{n+N-1} - F_{n-1}F_{n+2N}) = F_{n+N}F_{n+N-1}(D_tF_n \cdot F_{n-1} - F_{n+N}F_{n-N-1})$$

Accordingly the bilinear form will be

$$D_tF_{n-1} \cdot F_n - F_{n+N}F_{n-N-1} - F_nF_{n-1} = 0 \qquad (37)$$

with the following multisoliton solution

$$F_n(t) = \sum_{\mu_1,\ldots,\mu_M \in \{0,1\}} \exp\left(\sum_{i=1}^{M} \mu_i(k_in + \omega_it) + \sum_{i<j}^{M} A_{ij}\mu_i\mu_j \right),$$

Bilinear Integrability of mB3 lattice

$$\dot{u}_n = u_n(u_n - a)(\prod_{j=1}^{N} u_{n+j} - \prod_{j=1}^{N} u_{n-j}) \tag{38}$$

In the case $N = 2$ the system is completely integrable but for general N is not known. For simplicity consider the case $N = 2$:

$$U_{1,n+1} = U_{2,n}, U_{2,n+1} = U_{3,n}$$

$$U_{3,n+1} = U_{4,n}$$

$$U_{4,n+1} = \frac{\dot{U}_{3,n} + U_{1,n}U_{2,n}U_{3,n}(U_{3,n} - a)}{U_{4,n}U_{3,n}(U_{3,n} - a)} \tag{39}$$

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Now we have **two possible sources of singularities** $z = a, t = 0, z = 0$. We are interested in finding strictly confining patterns to obtain bilinear forms. So let us start with singularity entering through $U_{3,n} = a$. If $\tau = t - t_0(n)$ is the singularity manifold and

$$U_{1,n} = a_0 + a_1\tau + ..., \quad U_{2,n} = b_0 + b_1\tau + ...,$$

$$U_{3,n} = a + c_1\tau + c_2\tau^2 + ...$$

$$U_{4,n} = d_0 + d_1\tau + ...$$

then we will have the following confining singularity pattern

$$\begin{pmatrix} x_0 \\ y_0 \\ a \\ t_0 \end{pmatrix} \to \begin{pmatrix} * \\ a \\ * \\ \infty \end{pmatrix} \to \begin{pmatrix} a \\ * \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} * \\ \infty \\ 0 \\ * \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ * \\ * \end{pmatrix} \to \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ a \\ t_0 \end{pmatrix} \to \begin{pmatrix} * \\ a \\ * \\ \infty \end{pmatrix} \to \begin{pmatrix} a \\ * \\ \infty \\ 0 \end{pmatrix} \to \begin{pmatrix} * \\ \infty \\ 0 \\ * \end{pmatrix} \to \begin{pmatrix} \infty \\ 0 \\ * \\ * \end{pmatrix} \to \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix}$$

For the other source entering through $U_{4,n} = 0$.

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$$U_{3,n} = c_0 + c_1\tau + c_2\tau^2 + ...$$

$$U_{4,n} = d_1\tau + d_2\tau^2 + ...$$

we have the following singularity pattern

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So one can see that we have to main confined singularity patterns (for the orbit of $U_{3,n}$ and for simplicity we call it u_n):

$$(a, *, \infty, 0) \quad \text{and} \quad (0, \infty, *, a)$$

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From the first pattern we have that $u_n = \infty, u_{n-1} = \text{finite}, u_{n-2} = a$. This means that exists a tau-function F whose zeroes give the infinite value of u_n. So,

$$u_n = a + p\frac{F_{n+2}}{F_n}$$

. On the other hand because $u_n = \infty, u_{n+1} = 0$ we have

$$u_n = q\frac{F_{n-1}}{F_n}$$

where p and q are for the moment arbitary functions depending on the *second* tau function G_n. So

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Now for the second pattern we introduce the second tau-function G_n in the same way

$$u_n = r\frac{G_{n+1}}{G_n} = a + s\frac{G_{n-2}}{G_n}$$

where r, s are functions depending on the first tau-function F_n. These two representations are compatible if

$$u_n = a + \mu\frac{G_{n-2}F_{n+2}}{G_nF_n} = \nu\frac{G_{n+1}F_{n-1}}{G_nF_n}$$

where μ, ν are now arbitrary constants.

For simplicity we put $\mu = a, \nu = 2a$ and we get the first bilinear equation

$$G_{n-2}F_{n+2} + G_nF_n - 2G_{n+1}F_{n-1} = 0 \tag{40}$$

In order to get the second bilinear equation we introduce in the nonlinear equation the two representations of u_n and we find

$$a\frac{d}{dt}(1+\frac{G_{n-2}F_{n+2}}{G_nF_n}) = a\frac{d}{dt}(\frac{G_{n-2}}{F_n}\frac{F_{n+2}}{G_n}) = a\frac{F_{n+2}}{G_n}\frac{D_tG_{n-2}\cdot F_n}{F_n^2} - a\frac{G_{n-2}}{F_n}\frac{D_tG_n\cdot F_{n+2}}{G_n^2} =$$

$$= 8a^4\frac{G_{n+1}F_{n-1}}{G_nF_n}\frac{G_{n-2}F_{n+2}}{G_nF_n}\left(\frac{G_{n+2}F_nG_{n+3}F_{n+1}}{G_{n+1}F_{n+1}G_{n+2}F_{n+2}} - \frac{G_nF_{n-2}G_{n-1}F_{n-3}}{G_{n-1}F_{n-1}G_{n-2}F_{n-2}}\right)$$

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After simplification (and rescaling time with $1/8a^3$) we get the following cuadrilinear form:

$$G_nF_{n+2}(D_tG_{n-2}\cdot F_n + G_{n+1}F_{n-3}) = G_{n-2}F_n(D_tG_n\cdot F_{n+2} + G_{n+3}F_{n-1})$$

From this we find the bilinear form by identifying the parantheses with the opposite factors in both sides.

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From this we find the bilinear form by identifying the parantheses with the opposite factors in both sides.

$$D_tG_{n-1}\cdot F_{n+1} + G_{n+2}F_{n-2} = G_{n-1}F_{n+1} \qquad (41)$$

which is the second bilinear equation.

We can do the same machinery in the general case (arbitary N) with the analog ansatz:

$$u_n = a + \mu \frac{G_{n-N}F_{n+N}}{G_nF_n} = \nu \frac{G_{n+1}F_{n-1}}{G_nF_n}$$

And we find the following general bilinear system:

$$G_{n-N}F_{n+N} + G_nF_n - 2G_{n+1}F_{n-1} = 0 \qquad (42)$$

$$D_tG_{n-N} \cdot F_n + G_{n+1}F_{n-N-1} = G_{n-N}F_n \qquad (43)$$

This system admits at least the 3-soliton solution

$$G = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + \sum_{i<j} A_{ij}e^{\eta_i+\eta_j} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}$$

$$F = 1+e^{\eta_1+\phi_1}+e^{\eta_2+\phi_2}+e^{\eta_3+\phi_3}+\sum_{i<j} A_{ij}e^{\eta_i+\eta_j+\phi_i+\phi_j}+A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3+\phi_1+\phi_2+\phi_3}$$

provided that

$$\omega_i = \frac{e^{-Nk_i}(-1+e^{Nk_i})(-1+e^{k_i+Nk_i})}{2(-1+e^{k_i})}, e^{\phi_i} = \frac{e^{k_i-Nk_i}(-1-e^{Nk_i}+2e^{k_i+Nk_i})}{(-2+e^{k_i}+e^{k_i+Nk_i})}$$

The interaction term is:

$$A_{ij} = \frac{(e^{k_i} - e^{k_i(1+N)} - e^{k_j} + e^{k_j+k_i(1+N)} + e^{k_j(1+N)} - e^{k_i+k_j(1+N)})}{(-1+e^{k_i}+e^{k_jN} - e^{k_jN+k_i(1+N)} - e^{k_i+k_j(1+N)} + e^{(k_i+k_j)(1+N)})} \times$$
$$\times \frac{(e^{k_iN} - e^{k_i(1+N)} - e^{k_jN} + e^{k_jN+k_i(1+N)} + e^{k_j(1+N)} - e^{k_iN+k_j(1+N)})}{(-1+e^{k_iN}+e^{k_j} - e^{k_j+k_i(1+N)} - e^{k_iN+k_j(1+N)} + e^{(k_i+k_j)(1+N)})}$$

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2. Singularity analysis is much harder when the order is high. Nonconfining and weakly confining patterns may appear. However the intreraction of singularities may give confining patterns.
3.The more singularity patterns appear the bigger number of tau-functions are involved.

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