Spinor squares, G-structures and Fierz potentials

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Outline

Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) and dimension $d = p + q$. Let $\Lambda(M) \stackrel{\text{def.}}{=} \wedge T^*M$ be the exterior bundle of M and consider the *geometric product* of (M, g):

$$\diamond \stackrel{\text{def.}}{=} \sum_{k=0}^{d} (-1)^{\left[\frac{k+1}{2}\right]} \Delta_k : \Lambda(M) \times \Lambda(M) \to \Lambda(M) \quad ,$$

where:

$$\Delta_k \stackrel{\text{def.}}{=} \frac{1}{k!} g^{a_1 b_1} \ldots g^{a_k b_k} \overleftarrow{\iota}_{a_k} \ldots \overleftarrow{\iota}_{a_1} \wedge \overrightarrow{\iota}_{b_1} \ldots \overrightarrow{\iota}_{b_k}$$

are the *generalized products*. $(\Lambda(M), \diamond)$ is a bundle of unital associative algebras (called the *Kahler-Atiyah bundle* of (M, g)) which is naturally isomorphic with the Clifford bundle $\mathrm{Cl}(T^*M, g^*)$ of the *cotangent* bundle of M through the *Chevalley-Riesz isomorphism* $\Psi : (\Lambda(M), \diamond) \xrightarrow{\sim} \mathrm{Cl}(T^*M, g^*)$.

Definition

The *Kähler-Atiyah trace* is the linear functional:

$$\mathcal{S} \colon \Lambda(M) \to \mathcal{C}^\infty(M) \quad , \quad \omega \mapsto 2^{\frac{d}{2}} \omega^{(0)} \quad .$$

which satisfies:

$$\mathcal{S}(1) = N \stackrel{\text{def.}}{=} 2^{\frac{d}{2}} \quad \text{and} \quad \mathcal{S}(\omega_1 \diamond \omega_2) = \mathcal{S}(\omega_2 \diamond \omega_1) \quad \forall \omega_1, \omega_2 \in \Lambda(M) \quad .$$

We say that spinors on (M, g) have *real* (or *normal simple*) *type* if the real Clifford algebra $\mathrm{Cl}(p, q)$ is isomorphic with a real matrix algebra, which happens iff $p - q \equiv_8 0, 2$ (which we assume from now on). In this case, d is even and we have $\mathrm{Cl}(p, q) \simeq \mathrm{Mat}(N, \mathbb{R})$. Moreover, there exists a unique (up to isomorphism) simple $\mathrm{Cl}(p, q)$-module Σ, which has dimension N and the corresponding representation is an isomorphism of unital algebras $\gamma : \mathrm{Cl}(p, q) \xrightarrow{\sim} \mathrm{End}(\Sigma)$. Let $V = \mathbb{R}^{p+q}$ and:

- π be the *standard automorphism* of $\mathrm{Cl}(p, q)$, which satisfies $\pi|_V = -\mathrm{id}_V$
- τ be the *standard anti-automorphism* of $\mathrm{Cl}(p, q)$, which satisfies $\tau|_V = \mathrm{id}_V$
- $\hat{\tau} \stackrel{\mathrm{def.}}{=} \pi \circ \tau = \tau \circ \pi$ be the twisted anti-automorphism.

Let (S, Γ) be a real spinor bundle (i.e. a bundle of irreducible Clifford modules) on (M, g), where $\Gamma : \mathrm{Cl}(T^*M, g^*) \to End(S)$ is the structure morphism which gives the (dualized) Clifford multiplication. Then Γ is an isomorphism since $p - q \equiv_8 0, 2$. Consider the isomorphism of bundles algebras:

$$\Psi_\Gamma \stackrel{\mathrm{def.}}{=} \Gamma \circ \Psi : (\Lambda(M), \diamond) \xrightarrow{\sim} (End(S), \circ) \quad .$$

Let $\mathrm{tr} : End(S) \to \mathbb{R}_M$ be the fiberwise trace.

Proposition

Let (S, Γ) be a real spinor bundle. Then:

$$\mathcal{S}(\omega) = \mathrm{tr}(\Psi_\Gamma(\omega)) \quad \forall \omega \in \Omega(M) \quad .$$

Theorem (Harvey)

Σ admits two non-degenerate bilinear pairings $\mathscr{B}_+ : \Sigma \times \Sigma \to \mathbb{R}$ and $\mathscr{B}_- : \Sigma \times \Sigma \to \mathbb{R}$ (each determined up to multiplication by a non-zero real number) such that:

$$\mathscr{B}_+(\gamma(x)(\xi_1), \xi_2) = \mathscr{B}_+(\xi_1, \gamma(\tau(x))(\xi_2)) ,$$
$$\mathscr{B}_-(\gamma(x)(\xi_1), \xi_2) = \mathscr{B}_-(\xi_1, \gamma(\hat{\tau}(x))(\xi_2))$$

for all $x \in \mathrm{Cl}(p, q)$ and $\xi_1, \xi_2 \in \Sigma$. The symmetry properties of $\mathscr{B}_\pm$ are:

k mod 4	0	1	2	3
$\mathscr{B}_+$	Symmetric	Symmetric	Skew-symmetric	Skew-symmetric
$\mathscr{B}_-$	Symmetric	Skew-symmetric	Skew-symmetric	Symmetric

where $k \stackrel{\text{def.}}{=} \frac{d}{2}$. In addition, if $\mathscr{B}_s$ (with $s \in \{-1, 1\}$) is symmetric, then it is of split signature unless $pq = 0$, in which case $\mathscr{B}_s$ is definite.

Definition

The sign factor s is called the *adjoint type* of $\mathscr{B}_s$. The symmetry type of an admissible bilinear form $\mathscr{B}$ is denoted by $\sigma \in \{-1, 1\}$. If $\sigma = +1$ then $\mathscr{B}$ is symmetric whereas if $\sigma = -1$ then $\mathscr{B}$ is skew-symmetric.

Definition

Let (S, Γ) be a real spinor bundle on (M, g), where $\Gamma : \mathrm{Cl}(T^*M, g^*) \to End(S)$ is the structure morphism which gives the (dualized) Clifford multiplication. A non-degenerate bilinear pairing $\mathscr{B}$ on S is called **admissible** if $\mathscr{B}_m : S_m \times S_m \to \mathbb{R}$ is an admissible pairing on the simple Clifford module (S_m, Γ_m) for all $m \in M$. A (real) *paired spinor bundle* on (M, g) is a triplet $\mathrm{S} = (S, \Gamma, \mathscr{B})$, where (S, Γ) is a real spinor bundle on (M, g) and $\mathscr{B}$ is an admissible pairing on S.

Since M is connected, the symmetry and adjoint type $\sigma, s \in \{-1, 1\}$ of the admissible pairings $\mathscr{B}_m$ are constant on M and are called the *symmetry type* and *adjoint type* of $\mathscr{B}$ and $(S, \Gamma, \mathscr{B})$.

Definition

We say that (M, g) is *strongly spin* if it admits a $\mathrm{Spin}_0(p, q)$-structure, which we call a *strong spin structure*. In this case, a real spinor bundle (S, Γ) on (M, g) is called *strong* if it is associated to a strong spin structure.

When (M, g) is strongly spin, then it is *strongly orientable* in the sense that its orthonormal coframe bundle admits a reduction to an $\mathrm{SO}_0(V^*, h^*)$-bundle.

Proposition

Suppose that (M, g) *is strongly spin and let* (S, Γ) *be a strong real spinor bundle on* (M, g)*. Then every admissible pairing on* (Σ, γ) *extends to an admissible pairing* $\mathscr{B}$ *on* (S, Γ)*. Moreover, the Levi-Civita connection* ∇^g *of* (M, g) *lifts to a unique connection on* S *(denoted* ∇^S *and called the* spinorial connection *of* S*), which acts by module derivations:*

$$\nabla^S_X(\alpha \cdot \epsilon) = (\nabla^g_X \alpha) \cdot \epsilon + \alpha \cdot (\nabla^S_X \epsilon) \quad \forall \alpha \in \Omega(M) \;\; \forall \epsilon \in \Gamma(S) \;\; \forall X \in \mathfrak{X}(M)$$

and is compatible with $\mathscr{B}$*:*

$$X[\mathscr{B}(\epsilon_1, \epsilon_2)] = \mathscr{B}(\nabla^S_X \epsilon_1, \epsilon_2) + \mathscr{B}(\epsilon_1, \nabla^S_X \epsilon_2) \quad \forall \epsilon_1, \epsilon_2 \in \Gamma(S) \;\; \forall X \in \mathfrak{X}(M) \quad .$$

Definition

The *signed squaring maps* of a paired vector bundle $(S, \mathscr{B})$ are the quadratic maps $\mathcal{E}_\pm \colon S \to End(S)$ defined through:

$$\mathcal{E}_\pm(\xi) = \pm \xi \otimes \xi^* \quad \forall \xi \in S \quad ,$$

where $\xi^* \stackrel{\text{def.}}{=} \mathscr{B}(-, \xi) \in S^*$ is the linear functional dual to ξ relative to $\mathscr{B}$. The map $\mathcal{E}_+$ is called the **positive squaring map** of $(\Sigma, \mathscr{B})$, while $\mathcal{E}_-$ is called the *negative squaring map* of $(\Sigma, \mathscr{B})$.

Definition

Let $\mathrm{S} = (S, \gamma, \mathscr{B})$ be a paired spinor bundle on M. The *signed spinor squaring maps* of S are the quadratic maps:

$$\mathcal{E}_{\mathrm{S}}^{\pm} \stackrel{\text{def.}}{=} \Psi_{\Gamma}^{-1} \circ \mathcal{E}_{\pm} : S \to \Lambda(M) \quad ,$$

where $\mathcal{E}_{\pm} : S \to End(S)$ are the signed squaring maps of $(S, \mathscr{B})$. Given a spinor $\xi \in \Gamma(M, S)$, the polyforms $\mathcal{E}_{\mathrm{S}}^{+}(\xi)$ and $\mathcal{E}_{\mathrm{S}}^{-}(\xi) = -\mathcal{E}_{\mathrm{S}}^{+}(\xi)$ are called the positive and negative *squares* of ξ relative to the admissible pairing $\mathscr{B}$. A polyform $\omega \in \Lambda(M)$ is called a *signed square* of ξ if $\omega = \mathcal{E}_{\mathrm{S}}^{+}(\xi)$ or $\omega = \mathcal{E}_{\mathrm{S}}^{-}(\xi)$.

The fiber maps $\mathcal{E}_{\mathrm{S}}^{\pm}$ are fiberwise quadratic and 2:1 away from the zero section of S (where they branch). Their images $Z^{\pm}(M)$ are subsets of the total space of $\Lambda(M)$ and fiber over M with cone fibers. We have $Z^{-}(M) = -Z^{+}(M)$ and $Z^{+}(M) \cap Z^{-}(M) = 0_{\Lambda(M)}$. The fiberwise sign action of $\mathbb{Z}_2$ on S permutes the sheets of these covers (fixing the zero section), hence $\mathcal{E}_{\mathrm{S}}^{\pm}$ give bijections from $S/\mathbb{Z}_2$ to $Z^{\pm}(M)$ as well as a single bijection:

$$\hat{\mathcal{E}}_{\mathrm{S}} : S/\mathbb{Z}_2 \xrightarrow{\sim} Z(M)/\mathbb{Z}_2 \quad ,$$

where $Z(M) \stackrel{\text{def.}}{=} Z^{+}(M) \cup Z^{-}(M)$ and $\mathbb{Z}_2$ acts by sign multiplication.

The sets $\dot{Z}^\pm(M) \stackrel{\mathrm{def.}}{=} Z^\pm(M)\setminus 0_{\Lambda(M)}$ are connected submanifolds of the total space of $\Lambda(M)$ and the restrictions:

$$\dot{\mathcal{E}}^\pm_{\mathrm{S}} : \dot{S} \to \dot{Z}^\pm(M) \tag{1}$$

of $\mathcal{E}^\pm_{\mathrm{S}}$ away from the zero section are 2:1 fiber surjections.
Let $\mathfrak{Z}^\pm(M) \stackrel{\mathrm{def.}}{=} \mathcal{E}^\pm_{\mathrm{S}}(\Gamma(M,S)) \subset \Omega(M)$ and set $\mathfrak{Z}(M) \stackrel{\mathrm{def.}}{=} \mathfrak{Z}^+(M)\cup\mathfrak{Z}^-(M)$. Then $\mathfrak{Z}^-(M) = -\mathfrak{Z}^+(M)$ and $\mathfrak{Z}^+(M)\cap\mathfrak{Z}^-(M) = \{0\}$ and we have strict inclusions $\mathfrak{Z}^\pm(M)\subset\Gamma(M,Z^\pm(M))$ and $\mathfrak{Z}(M)\subset\Gamma(M,Z(M))$. The signed spinor squaring maps restrict to two-to-one surjections:

$$\dot{\mathcal{E}}^\pm_{\mathrm{S}} : \dot{\Gamma}(M,S) \to \dot{\mathfrak{Z}}^\pm(M) \quad .$$

Moreover, $\mathcal{E}^\pm_{\mathrm{S}}$ induce the same bijection:

$$\hat{\mathcal{E}}_{\mathrm{S}} : \Gamma(M,S)/\mathbb{Z}_2 \stackrel{\sim}{\to} \mathfrak{Z}(M)/\mathbb{Z}_2 \quad .$$

Finally, let $\dot{\Gamma}(M,S) = \Gamma(M,\dot{S})$ be the set of nowhere-vanishing sections of S and $\dot{\mathcal{Z}}^\pm(M) \stackrel{\mathrm{def.}}{=} \dot{\Gamma}(M,Z^\pm(M)) = \Gamma(M,\dot{Z}^\pm(M))$ be the set of those polyforms in $Z^\pm(M)$ which are nowhere-vanishing and define $\dot{\mathcal{Z}}(M) \stackrel{\mathrm{def.}}{=} \dot{\mathcal{Z}}^+(M)\cup\dot{\mathcal{Z}}^-(M)$. Notice that $\dot{\mathcal{Z}}^+(M)\cap\dot{\mathcal{Z}}^-(M)=\emptyset$.

Proposition

Suppose that (M,g) is strongly spin and let $\mathsf{S} = (S, \Gamma, \mathscr{B})$ be a strong paired spinor bundle associated to a $\mathrm{Spin}_0(V^*, h^*)$-structure $\mathfrak{Q}$ on (M,g). Then every nowhere-vanishing polyform $\alpha \in \dot{\mathcal{Z}}(M)$ determines a cohomology class $c_{\mathfrak{Q}}(\alpha) \in H^1(M, \mathbb{Z}_2)$ which encodes the obstruction to existence of a globally-defined spinor $\epsilon \in \Gamma(M, S)$ (which is necessarily nowhere-vanishing) such that $\alpha \in \{\mathcal{E}^+_{\mathsf{S}}(\epsilon), \mathcal{E}^-_{\mathsf{S}}(\epsilon)\}$. More precisely, such ϵ exists iff $c_{\mathfrak{Q}}(\alpha) = 0$. In particular, we have:

$$\dot{\mathfrak{Z}}(M) = \{\alpha \in \dot{\mathcal{Z}}(M) \,|\, c_{\mathfrak{Q}}(\alpha) = 0\} \quad \text{and} \quad \dot{\mathfrak{Z}}^{\pm}(M) = \{\alpha \in \dot{\mathcal{Z}}^{\pm}(M) \,|\, c_{\mathfrak{Q}}(\alpha) = 0\} \quad .$$

Definition

The cohomology class $c_{\mathfrak{Q}}(\alpha) \in H^1(M, \mathbb{Z}_2)$ of the previous proposition is called the *spinor class* of the nowhere-vanishing polyform $\alpha \in \dot{\mathfrak{Z}}(M)$.

Proposition

Suppose that (M,g) is strongly spin and a let $\mathfrak{Q}$ be a $\mathrm{Spin}_0(V^*, h^*)$-structure on (M,g). For every nowhere-vanishing polyform $\alpha \in \dot{\mathcal{Z}}(M)$, there exists a unique $\mathrm{Spin}_0(V^*, h^*)$-structure $\mathfrak{Q}'$ such that $c_{\mathfrak{Q}'}(\alpha) = 0$.

Definition

Let (S, Γ) be a real spinor bundle on (M, g) and $\mathcal{D}$ be an arbitrary connection on S. A section $\epsilon \in \Gamma(M, S)$ is called *generalized Killing spinor (GKS) with respect to* $\mathcal{D}$ if:

$$\mathcal{D}\epsilon = 0 \quad . \tag{2}$$

A *linear constraint datum* for (S, Γ) is a pair $(\mathcal{W}, \mathcal{Q})$, where $\mathcal{W}$ is a real vector bundle over M and $\mathcal{Q} \in \Gamma(M, End(S) \otimes \mathcal{W}) \simeq \Gamma(M, Hom(S, S \otimes \mathcal{W}))$. Given such a datum, the condition:

$$\mathcal{Q}(\epsilon) = 0 \tag{3}$$

is called the *linear constraint* on ϵ defined by $\mathcal{Q}$. We say that ϵ is a (real) *constrained generalized Killing spinor (CGKS)* if it satisfies the system formed by (2) and (3).

Suppose that (M, g) is strongly spin and (S, Γ) is a strong real spinor bundle. Then $\mathcal{D} = \nabla^S - \mathcal{A}$ with $\mathcal{A} \in \Omega^1(End(S))$, where ∇^S is the spinorial connection on S. The CGKS equations become:

$$\nabla^S \epsilon = \mathcal{A}\epsilon \quad , \quad \mathcal{Q}(\epsilon) = 0$$

and their solutions are called CGK spinors *relative to* $(\mathcal{A}, \mathcal{W}, \mathcal{Q})$. The space of CGK spinors is finite-dimensional and such a spinor vanishes at a point iff it vanishes identically.

Theorem

Suppose that (M, g) is strongly spin and let $\mathrm{S} = (S, \Gamma, \mathscr{B})$ be a paired spinor bundle associated to the $\mathrm{Spin}_0(V^*, h^*)$-structure $\mathfrak{Q}$ and whose admissible form $\mathscr{B}$ has adjoint type s. Let $\mathcal{A} \in \Omega^1(M, End(S))$ and $(\mathcal{W}, Q)$ be a linear constraint datum for (S, Γ). Then the following statements are equivalent:

- (a) *There exists a nontrivial constrained generalized Killing spinor $\epsilon \in \Gamma(M, S)$ with respect to $(\mathcal{A}, \mathcal{W}, \mathcal{Q})$.*
- (b) *There exists a nowhere-vanishing polyform $\alpha \in \Omega(M)$ with vanishing cohomology class $c_{\mathfrak{Q}}(\alpha)$ which satisfies the following algebraic and differential equations for every polyform $\beta \in \Omega(M)$:*

$$\alpha \diamond \beta \diamond \alpha = \mathcal{S}(\alpha \diamond \beta)\, \alpha \quad , \quad (\pi^{\frac{1-s}{2}} \circ \tau)(\alpha) = \sigma_s\, \alpha \quad , \tag{4}$$

$$\nabla^g \alpha = \hat{\mathcal{A}} \diamond \alpha + \alpha \diamond (\pi^{\frac{1-s}{2}} \circ \tau)(\hat{\mathcal{A}}) \quad , \quad \hat{\mathcal{Q}} \diamond \alpha = 0 \tag{5}$$

or, equivalently, satisfies the equations:

$$\alpha \diamond \alpha = \mathcal{S}(\alpha)\, \alpha \quad , \quad (\pi^{\frac{1-s}{2}} \circ \tau)(\alpha) = \sigma_s\, \alpha \quad , \quad \alpha \diamond \beta \diamond \alpha = \mathcal{S}(\alpha \diamond \beta)\, \alpha \quad , \tag{6}$$

$$\nabla^g \alpha = \hat{\mathcal{A}} \diamond \alpha + \alpha \diamond (\pi^{\frac{1-s}{2}} \circ \tau)(\hat{\mathcal{A}}) \quad , \quad \hat{\mathcal{Q}} \diamond \alpha = 0 \quad , \tag{7}$$

for some fixed polyform $\beta \in \Omega(M)$ such that $\mathcal{S}(\alpha \diamond \beta) \neq 0$.

If $\epsilon \in \Gamma(M, S)$ is chiral of chirality $\mu \in \{-1, 1\}$, then we have to add the condition:

$$* (\pi \circ \tau)(\alpha) = \mu\, \alpha \quad .$$

The polyform α as above is determined by ϵ through the relation:

$$\alpha = \mathcal{E}^{\kappa}_{\mathrm{S}}(\epsilon)$$

for some $\kappa \in \{-1, 1\}$. Moreover, α satisfying the conditions above determines a nowhere-vanishing real spinor ϵ satisfying this relation, which is unique up to sign.

Let (M,g) be a Riemannian spin 8-manifold whose geometric product $\diamond$ we denote by juxtaposition. The volume form ν satisfies $\nu^2=1$ and is *twisted central*, i.e. we have $\nu\omega=\pi(\omega)\nu$ for all $\omega\in\Omega(M)$. A bundle S of simple real Clifford modules has rank $N=16$ and the structure morphism $\gamma:\Lambda(M)\to End(S)$ is an isomorphism. Up to constant scaling, S has two admissible pairings $\mathscr{B}_+$ and $\mathscr{B}_-$, which are symmetric and of opposite adjoint types. We work with the *fundamental pairing* $\mathscr{B}\stackrel{\mathrm{def.}}{=}\mathscr{B}_+$, which can be taken to be a scalar product. The adjointness condition amounts to:

$$\gamma(\omega)^t=\gamma(\tau(\omega))\ \ \forall\omega\in\Omega(M)\ \ ,$$

where ^{t} is the $\mathscr{B}$-transpose. We have $S=S^+\oplus S^-$, where $S^\pm$ are the bundles of spinors of chiralities ±1, which are the eigensubbundles of $\gamma(\nu)$.

Proposition

Giving a section $\xi\in\Gamma(M,S)$ which satisfies $\mathscr{B}(\xi,\xi)=1$ amounts to giving a global endomorphism $E\in\Gamma(M,End(S))$ which satisfies:

$$E^2=E\ \ ,\ \ E^t=E\ \ ,\ \ \mathrm{tr}(E)=1\ \ .$$

Namely, any such ξ defines such an endomorphism E and any such E defines such a ξ, which is determined up to sign by the condition:

$$E_{\xi,\xi}=E\ \ .$$

Define $\check{E} \stackrel{\text{def.}}{=} \gamma^{-1}(E) \in \Omega(M)$. We have:

$$\check{E}_{\xi,\xi} \stackrel{\text{def.}}{=} \gamma^{-1}(E_{\xi,\xi}) =_U \sum_{k=0}^{8} \frac{1}{k!} \mathscr{B}(\xi, \gamma_{a_1 \ldots a_k} \xi) e^{a_1 \ldots a_k} \quad .$$

Proposition

Giving a section $\xi \in \Gamma(M, S)$ which satisfies $\mathscr{B}(\xi, \xi) = 1$ amounts to giving an inhomogeneous form $\check{E} \in \Omega(M)$ which satisfies:

$$\check{E}^2 = \check{E} \quad , \quad \tau(\check{E}) = \check{E} \quad , \quad \mathcal{S}(\check{E}) = 1 \quad . \tag{8}$$

Namely, any $\xi \in \Gamma(M, S)$ defines such a form and any such form defines a section ξ, which is determined up to sign by the condition:

$$\check{E}_{\xi,\xi} = \check{E} \quad .$$

Furthermore, we have the equivalence

$$\gamma(\nu)\xi = \pm\xi \Longleftrightarrow \check{E}\nu = \pm\check{E} \quad ,$$

i.e. ξ has chirality ± 1 iff. $\check{E}$ is twisted (anti)self-dual.

Proposition

When ξ has positive chirality, relations (8) amount to the requirement that $\check{E}$ takes the form:

$$\check{E} = \frac{1}{16}(1 + \Phi + \nu) \quad ,$$

where $\Phi \in \Omega^4(M)$ is a self-dual four-form on M which satisfies:

$$\Phi^2 = 12\Phi + 14\nu + 14 \quad .$$

Moreover Φ has the following expansion in any local orthonormal frame defined on an open subset $U \subset M$:

$$\Phi =_U \frac{1}{4!}\mathscr{B}(\xi, \gamma_{a_1 \ldots a_4}\xi)e^{a_1 \ldots a_4} \in \Omega^4(M) \quad . \tag{9}$$

It is well-known that a positive-chirality Majorana-Weyl spinor of $\xi \in \Gamma(M, S^+)$ determines a $\mathrm{Spin}(7)_+$ structure on M which is compatible with the metric of M and whose calibration Φ is the selfdual four-form which is given in any local coordinate frame by relation (9). Conversely, any metric-compatible $\mathrm{Spin}(7)_+$ structure on M calibrated by Φ determines a positive chirality spinor $\xi \in \Gamma(M, S^+)$ (unique up to a sign) through the condition that Φ has the form (9) in any local orthonormal frame. Hence Proposition 4. implies:

Theorem

A four-form $\Phi \in \Omega^4(M)$ is the calibration of a metric-compatible $\mathrm{Spin}(7)_+$ *structure on (M, g) iff. it is self-dual and satisfies:*

$$\Phi^2 = 12\Phi + 14\nu + 14 \quad . \tag{10}$$

Expanding the geometric product gives:

$$\Phi^2 = \Phi \wedge \Phi = \Phi \Delta_2 \Phi + ||\Phi||^2 \quad .$$

Hence condition (10) amounts to the system of equations:

$$||\Phi||^2 = 14 \quad , \quad \Phi\Delta_2\Phi + 12\Phi = 0 \quad , \quad \Phi \wedge \Phi = 14\nu \quad . \tag{11}$$

Solutions of (10) are the critical points of the *Fierz potential* $\mathcal{W} : \Omega(M) \to \mathbb{R}$ defined through:

$$\mathcal{W}(\Phi) \stackrel{\text{def.}}{=} \mathrm{Tr}\Big[\frac{1}{3}\Phi^3 - 6\Phi^2 - 14(1+\nu)\Phi\Big] = \int_M \nu \Big[\frac{1}{3}\Phi^3 - 6\Phi^2 - 14(1+\nu)\Phi\Big]^{(0)} \quad ,$$

where $\mathrm{Tr} : \Omega(M) \to \mathbb{R}$ is given by:

$$\mathrm{Tr}(\omega) = \frac{1}{16}\int_M \mathcal{S}(\omega)\nu = \int_M \omega^{(0)}\nu \quad .$$

The framework of spinor squaring maps can be extended away from the case $p - q \equiv_8 0, 2$. This is considerably more complicated due to the fact that, in general, the so-called Schur bundle of a bundle of Clifford modules can be a complex or quaternionic line bundle rather than a real line bundle (the so-called complex and quaternionic cases). In general, this leads to a description of 'cosmooth stratified G-structures' as critical points of Fierz potentials defined on spaces of differential forms in terms of the geometric product. An example of this occurs for G_2-structures (in which case the Schur bundle is a complex line bundle).