Gauss-Bonnet on polyhedral manifolds

Classical GB ; Transgressions of higher order; GB for polyh. manif. Examples.

$(M^2, g) \rightsquigarrow k =$ Gauss curvature , $dg =$ volume form $\rightsquigarrow k\,dg \in \Omega^2(M)$

If M is compact, $\partial M = \emptyset \implies 2\pi\chi(M) = \int_M k\,dg$

$R =$ curvature tensor , $\{e_1, e_2\} =$ ON frame $\quad R = k\, e^1 \wedge e^2 \otimes \underbrace{\begin{cases} e_2 \mapsto e_1 \\ e_1 \mapsto -e_2 \end{cases}}_{\in \mathrm{End}^-(TM)}$

$R \rightsquigarrow$ double form $k\,dg \otimes \overset{\text{2-form}}{dg} \in \Omega^2(M) \otimes \Omega^2(M)$

$A \in \mathrm{End}^-(T_pM) \qquad \omega_A(X,Y) = \langle X, AY \rangle$

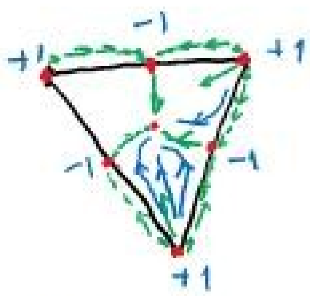
The Berezin integral : alg. operation of erasing dg

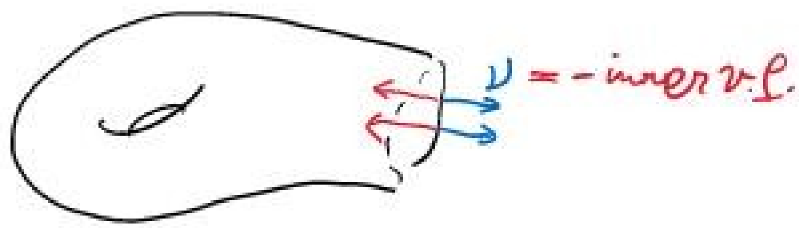
$$2\pi\chi(M) = \int_M B(R)$$

$\chi(M) = V_0 - V_1 + V_2 =$ # vertices − # edges + # faces in a triangulation

$$= \sum_{p \in \mathrm{zero}(X)} \mathrm{index}_p(X)$$ where X is a vector field w. isolated non-deg zeroes

Triangulation $\rightsquigarrow$ vector field X w. zero set at the vertices of the baricentric subdivision





$\partial M \neq 0$

$\nu =$ unit outer vect. field $\rightsquigarrow$ geodesic curvature function

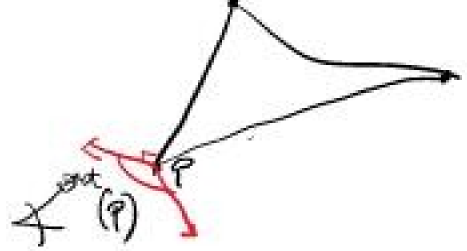
$$\nabla^M_X Y = \nabla^{\partial M}_X Y + \mathrm{II}(X,Y)\cdot\nu \quad \text{for every } X, Y \text{ tangent to } \partial M$$

length elem.

$$2\pi\chi(M) = \int_M B(R) - \int_{\partial M} a \cdot dl + \sum_{p\ \mathrm{corner}} \underbrace{\measuredangle^{\mathrm{outer}}(p)}_{\in(0,\pi)}$$

. $\partial M \neq 0$, corners : M is locally modeled by $[0,\infty) \times [0,\infty)$

$$a = \mathrm{II}(X,X) \text{ for } |X| = 1$$

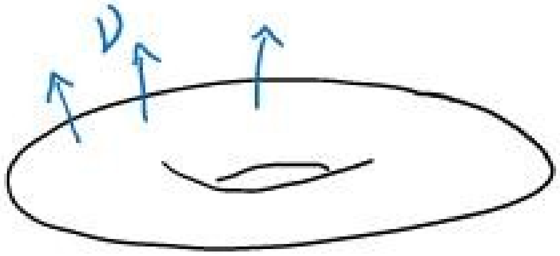


Higher dim: $(M^{2n}, g) \overset{\text{isom. embedding}}{\hookrightarrow} \mathbb{R}^{2n+1}$ compact, no boundary

Hopf: $(2\pi)^n \chi(M) = \int_M \underbrace{Pf(R)}_{\frac{1}{n!}B(R^n)}$

$R \in \Omega^2(M, \mathrm{End}^-(TM)) \longleftrightarrow \Omega^2(M) \otimes \Omega^2(M)$ = space of double forms.

$R^n \in \Omega^{2n}(M) \otimes \Omega^{2n}(M)$ product in the tensor product of the algebra $\Omega^*(M)$ with itself.



Gauss map $G : M \longrightarrow S^{2n}$

$p \longmapsto \nu_p$

Topological fact: $\chi(M) = 2 \deg(G : M \longrightarrow S^{2n})$

$M = \partial N \implies 0 = \chi(2N) = \chi(N^+) - \chi(M) + \chi(N^-)$

$\implies \chi(M) = 2\chi(N)$

$2N^{2n+1}$ = compact, no boundary

$2N = N^+ \cup N^-$

$\deg G = \frac{1}{\mathrm{vol}\, S^{2n}} \int_M G^*(dg_{S^{2n}}) \overset{\text{mod. constants}}{\equiv} \int_M (\det G_*)\, dg_M$

$G_* X = -AX \quad A =$ Weingarten : $= \int_M B(\mathrm{II}^{2n}) = \int_M B(R^n)$

$= X(\nu)$. $\overset{\text{def}}{=} \int_M Pf(R)$

$\mathrm{II} \in \Omega^1(M) \otimes \Omega^1(M)$ product in $\Lambda^* \otimes \Lambda^*$

$\mathrm{II} = \sum \lambda_j e^j \otimes e^j \implies \mathrm{II}^{2n} = (2n)! \prod_{j=1}^{2n} \lambda_j \cdot dg \otimes dg$

Gauss eq: $\langle R^M_{XY} T, Z \rangle = \mathrm{II}(X,Z)\,\mathrm{II}(Y,T) - \mathrm{II}(X,T)\,\mathrm{II}(Y,Z)$

Gauss eq: $\langle R^M_{XY}T, Z\rangle = \mathrm{II}(X,Z)\,\mathrm{II}(Y,T) - \mathrm{II}(X,T)\,\mathrm{II}(Y,Z)$

$$R^M = \tfrac{1}{2}\mathrm{II}^2 \text{ in } \Omega^*\otimes\Omega^*$$

$$\left(= \tfrac{1}{2}\,\mathrm{II} \owedge \mathrm{II} \quad \text{Kulkarni-Nomizu}\right)$$

$$R \in \Omega^2(M)\otimes\Omega^2(M) = R_{ijkl}\, e^i\wedge e^j \otimes e^k\wedge e^l$$

$$R^n = \prod_{\alpha\geq 1}^{n} R_{i_\alpha j_\alpha k_\alpha l_\alpha}\, dg\otimes \cancel{dg} \overset{B}{\longmapsto} \Omega^{2n}(M)$$

$$\prod R_{ijkl}\cdot \underbrace{|dg|}$$

$$(V^{2n}, \langle\cdot,\cdot\rangle)_{/\mathbb{R}} \Longrightarrow \Lambda^{2n}V \xrightarrow{\cong} \mathbb{R}$$

$A\in \mathrm{End}^-(V) \rightsquigarrow \omega_A : (X,Y)\longmapsto \langle X, AY\rangle \qquad \omega_A \in \Lambda^2(V^*)$

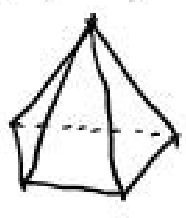
$\mathrm{Pf}(A) = \frac{1}{n!} B(\omega_A^n) \in \mathbb{R}$. Pf = polyn. of deg n in the entries of A

$$\mathrm{Pf}(A)^2 = \boxed{\begin{array}{l} {}^tA = -A \\ \det A \geq 0 \\ \text{Exercise.}\end{array}}$$

$$\left[e^{\omega_A}\right]_{\Lambda^{2n}(V^*)}$$

Allendoerfer (1940): Hopf's formula hold for $M^{2n} \hookrightarrow \mathbb{R}^{2n+k}$, $k\geq 1$.

Allendoerfer-Weil (1943): Extension to Riemannian polyhedra



$\Longrightarrow$ deduce Hopf's formula in general (no embedding assumption)

New: contribution from faces. (extend 2-d formula)

Chern (1944): New proof using diff. geom. (connections, sphere bundles ...) for closed M and M comp. w. boundary.

Extend Chern's proof to polyhedral manifolds.

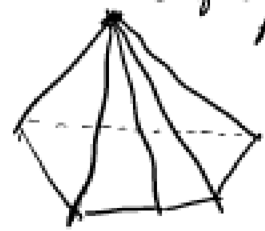
$V_{/\mathbb{R}}$ vector space, $\alpha_1,\dots,\alpha_k \in V^* \rightsquigarrow P = \{v\in V;\ \alpha_1(v)<0,\dots,\alpha_k(v)<0\}$

P = linear open polyhedron = open cone k can be $>\dim V$!!

Ex. $[0,\infty)^k \times \mathbb{R}^n$ = linear polyh. in $\mathbb{R}^{k+n}$.

Affine polyhedron: $\{ v \in V ; \alpha_1(v) < a_1, \dots, \alpha_k(v) < a_k \}$

Polytope = compact affine polyhedron



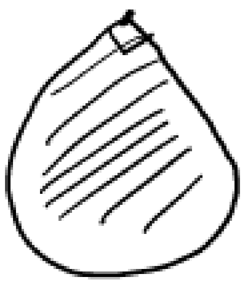
Def. A polyhedral manifold M is a compact top. space locally homeo to linear closed polyhedra, with a smooth atlas. ← can be locally extended smoothly

Ex. Manifolds w. boundary: locally modeled by $[0,\infty) \times \mathbb{R}^{n-1}$

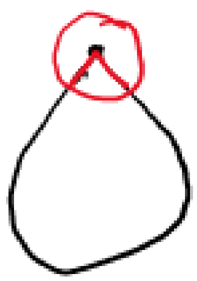
Manif. w. corners: $[0,\infty)^n$

Riem. metric

Find $\chi(M)$ for (M,g) polyhedral manif. with metric



polyh. manif.

$\partial M =$  $\neq$ polyh. manif.

Transgressions $\quad (E^{2n}, \nabla, g)$
$\downarrow$
M = open C^∞ manif.

X^ℓ = polyhedral manif. (typically X = simplex) of dim ℓ

Assume $V : X \longrightarrow \{ \text{length-1 sections in } E \to M \}$

$V : X \times M \longrightarrow \pi_2^* E \qquad X \times M \xrightarrow{\pi_2} M$

$d^X V = \sum \frac{\partial v}{\partial x_i} dx^i \in \Omega^1(X) \otimes C^\infty(M,E)$

$\nabla V = C^\infty(X, \Omega^1(M,E))$

V = family of sections in $E \to M$

$\Omega^{p,q} = \Omega^p(X \times M, \Lambda^q E)$

For integers l, k, n satisfying $0 \le l \le 2k+1 \le 2n-1$, define a universal constant $c(n,k,l)$ by

$$c(n,k,l) = \frac{2^k k!}{(n-1-k)!(2k+1-l)!} \in \mathbb{Q}.$$

Definition 5.1. For $l \ge 0$, we define the $(l+1)^{\text{th}}$ transgression of the Pfaffian with respect to the family $V : X \to \Omega^{(0,1)}(M)$ by

$$\mathcal{T}_V^{(l+1)} = \sum_{l \le 2k+1 \le 2n-1} \frac{c(n,k,l)}{l!} \int_X \mathcal{B}\left[V (d^X V)^l \, (\nabla V)^{2k+1-l} \, R^{n-1-k} \right] \in \Omega^{2n-l-1}(M).$$

Here $(d^X V)^l \in \Omega^l(X) \otimes \Omega^{0,l}(M,E)$, $R \in \Omega^{2,2}(M,E)$, and $\nabla V \in C^\infty(X, \Omega^{1,1}(M,E))$, so $\mathcal{B}$ is applied to a volume form on X tensored with a form of degree $2n-l-1$ on M.

$V \in \Omega^{0,1}$ $\quad d^X V \in \Omega^{1,1}$ $\quad R \in \Omega^{2,2}$ $\quad \Omega^*(X \times M) \otimes \Lambda^* E$

$V \in \Omega^0(X \times M) \otimes C^\infty(X \times M, \Lambda^1 E)$

<u>Thm.</u> $$d\,T^{(l+1)}_{(X,V)} = \begin{cases} -\mathrm{Pf}(R) & \text{for } l=0, \ X=\{*\} \quad l=\dim X \\ -T^{(l)}_{(\partial X, V)} & \text{for } l \ge 1. \end{cases}$$

<u>Chern</u>: There exists always a non-zero section in

where SE = sphere bundle of E

$\{ v \in E ; |v| = 1 \}$

$$\begin{tikzcd} E \arrow[d] & \pi^* E \arrow[l] \arrow[d] \\ M & SE \arrow[l, "\pi"'] \end{tikzcd}$$

SE

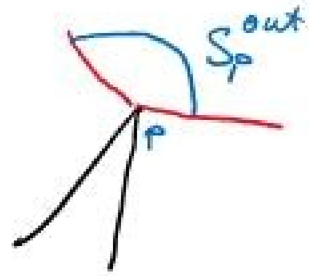
Theorem 6.1. *Let M^{2n} be a compact Riemannian polyhedral manifold. Then*

$$(2\pi)^n \chi(M) - \int_M \mathrm{Pf}(R) = \sum_{l=1}^{2n} \sum_{k=\lceil \frac{l}{2} \rceil}^{n} \frac{(-1)^l 2^{k-1}(k-1)!}{(n-k)!(2k-l)!}$$
$$\sum_{Y \in \mathcal{F}^{(l)}(M)} \int_{S^{\mathrm{out}} Y} \mathcal{B}_Y \left[(R^Y - \tfrac{1}{2} g(A,A))^{n-k} (A^*)^{2k-l} \right] |dg|.$$

$S^{out} Y = ?$ $\quad Y \subset M$ is a face $\quad$ A, A^* are obtained from $\mathrm{II}_{Y \hookrightarrow M}$

$\mathrm{Cone}^{out}_p(Y) = \{ v \in T_p M ; \langle v, X \rangle < 0, \ \forall X = \dot{\gamma} \text{ inner} \}$

$S^{out}_p Y = \{ v \in C^{out}_p Y ; |v| = 1$



Theorem 6.5. *Let (N, g) be a compact Riemannian polyhedral manifold of odd dimension $2n-1$. Then*

$$(2\pi)^n \chi(N) = \sum_{l=1}^{2n-1} \sum_{k=\lceil \frac{l-1}{2} \rceil}^{n-1} \frac{(-1)^{l-1}\pi(2k-1)!!}{(n-1-k)!(2k+1-l)!}$$
$$\cdot \sum_{Y \in \mathcal{F}^{(l)}(N)} \int_{S^{\mathrm{out}}Y} \mathcal{B}_Y \left[(R^Y - \tfrac{1}{2}g(A,A))^{n-1-k} (A^*)^{2k+1-l} \right] |dg|.$$

By convention, $(-1)!! = 0! = 0!! = 1$, and the 0^{th} power of a double form is always 1.

Y = boundary face of codim ℓ = conn. comp. of the set of points of depth ℓ $\quad [0,\infty)^{\ell} \times \mathbb{R}^{\dim M - \ell}$

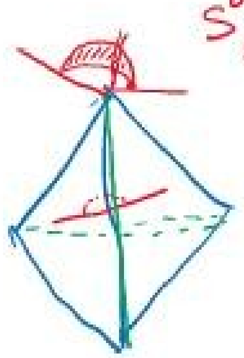
volume of an ideal hyperbolic 4-simplex is given by

$$\mathrm{vol}(M) = -2\pi^2 + \frac{\pi}{3} \sum_{Y \in \mathcal{F}^{(2)}(M)} \angle^{\mathrm{out}}(Y)$$

where $\angle^{\mathrm{out}}(Y)$ is the outer dihedral angle of the ideal triangle Y in M, i.e., the angle between the outer normals to the two hyperfaces containing Y.

Simple formulas when M is of const. sect. curvature, all faces are geodesic.

dim = 2



S_P^{out} = spherical polygon

$\dfrac{\mathrm{Area}(S^{\mathrm{out}})}{\mathrm{Area}(S^{2})}$