Mathematical supergravity and its applications in differential geometry

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Summary and main goal

I will give a pedagogical introduction to the mathematical theory of four-dimensional ungauged supergravity and its Killing spinor equations, mentioning some of its relations with diverse mathematical areas such as complex geometry, spin geometry or differential topology and providing a general overview on the current status of the mathematical foundations of supergravity. Work in collaboration with Calin Lazaroiu and Vicente Cortés.

Supergravity References

- Supersymmetric Field Theories, Sergio Cecotti.
- Gravity and Strings, Tomás Ortín.
- Supergravity, Antoine Van Proeyen, Daniel Z. Freedman.

A mathematical introduction in the form of proceedings [arXiv:2006.16157,C. Lazaroiu and CSS].

Supergravity and String Theory

- Supergravity theories arise in the *low-energy limit* of string theory $\rightarrow$ Encode the effective dynamics of the string theory massless sector.
- Supergravity theories are especially well adapted to characterize and classify string theory compactification backgrounds, in particular supersymmetric.
- Supergravity theories describe the effective dynamics of string theory compactifications to low dimensions.
- Supergravity theories contain important information about the non-perturbative spectrum of string theory and its various dualities.
- Given the close connection with string theory and supersymmetry, supergravity theories are at the heart of many of the outstanding applications and interactions of string theory with differential/algebraic geometry/topology $\rightarrow$ various remarkable mathematical applications.

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Supergravity theories are supersymmetric theories of gravity which extend and unify, using (crucially) supersymmetry as the guiding fundamental principle, three cornerstones of differential geometry, namely:

- Harmonic/wave maps
- Einstein metrics
- Yang-Mills connections

Supergravity and Mathematics

In order to explore the mathematical problems posed by supergravity and its potential applications to differential geometry and topology, in the spirit for instance of the role played by Yang-Mills and Seiberg-Witten theory in low-dimensional topology, we need a mathematical, differential-geometric, theory of supergravity:

- A geometric model for supergravity consists of a system of partial differential equations determined in terms of global differential operators defined on a fixed geometric object, such as a spinor bundle, a principal bundle, a gerbe, a Courant algebroid, a categorified principal bundle or a combination thereof.

Supergravity an Differential/Complex/Algebraic Geometry

- Mirror symmetry and its potential extension to non-Kähler manifolds; Candelas, Graña, Green, Louis, Minasian, Ossa, Parkes, Waldram, Yau.
- Projective Special-Kähler geometry, through $N = 2$ sigma models and moduli spaces of Calabi-Yau structures, Strominger, Freed, Cortés, Hitchin.
- Quaternionic-Kähler geometry, in the context of the supergravity celebrated c-map; Ferrara, Haydys, Hitchin, Sabharwal, Salamon.
- Riemannian geometry with torsion, motivated by supersymmetric manifolds in Type-II supergravity; Agricola, Friedrich, Ivanov, Papadopoulos.
- Spin geometry, harmonic, parallel or Killing spinors; Bär, Baum, Figueroa-O'Farrill, Friedrich.
- Hitchin's generalized geometry and its various Courant-algebroid extensions in the context of supergravity compactifications and moduli spaces. Hitchin, Gualtieri, Cavalcanti, García, Rubio, Tipler.
- Special Hermitian metrics on non-Kähler complex manifolds in the context of Heterotic supergravity and its compactifications. Fu, Li, Streets, Yau.
- Categorified gauge theory in differential geometry. Baraglia, Fiorenza, Heckmati, Rubio, Sati, Schreiber, Tipler.
- G_2 and Spin(7) manifolds Acharya, Babalic, Braun, Lazaroiu, Witten, Yau.

Supergravity: current status

- The local structure of the theory has been extensively studied in the literature since the 70's.
- The local structure of all ungauged supergravities has essentially been classified by dimension and amount supersymmetry preserved.
- The local gaugings of supergravity are being intensively studied: still a lot of work to be done before arriving to a complete classification which clarifies their potential String/M-theory origin.
- Various formalisms, such as generalized geometry and double field theory, have have been developed to further explore the local structure of supergravity, uncovering interesting mathematical structures.
- Local supersymmetric solutions of supergravity have been systematically studied in the literature and are in the process of being classified.
- Intense activity regarding the study of the *quantum properties* of supergravity as a QFT.
- Supergravity is a fundamental ingredient in various areas of string theory, such as string cosmology and string phenomenology.

There is a well-established program to study the **local!** structure of supergravity, its zoo of solutions and various physical applications.

The problem of developing the mathematical theory and foundations of supergravity has not been systematically addressed in the literature!

The supergravity Lagrangian

- Supergravity is defined locally via a Lagrangian $\mathcal{L}[\phi^b, \phi^f]$ depending on a given number of *bosonic fields* ϕ^b and a given number of *fermionic fields* ϕ^f.
- The fermionic fields are the variables that take values in a spinor bundle: theory of Lipschitz structures, [math/9901137; T. Friedrich, A. Trautman], [arXiv:1606.07894, arXiv:1711.07765; C. Lazaroiu, CSS].
- The set of bosonic fields always contains a metric and the set of fermionic fields always contains a *gravitino*.
- There are many inequivalent supergravity theories, involving different geometric structures. Partially determined by the signature and number of supersymmetries.

The equations of motion associated to $\mathcal{L}[\phi^b, \phi^f]$ are invariant under certain infinitesimal transformations of the type:

$$\delta_\epsilon \phi^b = \mathscr{B}(\epsilon, \phi^f)\,, \qquad \delta_\epsilon \phi^f = D\epsilon + (\phi^b + \mathscr{B}(\phi^f, \phi^f))\epsilon\,.$$

These are the supersymmetry transformations of the theory.

Developing the mathematical theory of supergravity requires understing the theory mathematically in the framework of global supergeometry, a quest which seems to be currently out of reach for matter coupled supergravities. Nonetheless, since supergravity is a fundamental theory of gravity, for most of its physical and mathematical applications in differential geometry and topology, such as:

- The global Lorentzian and Riemannian structure of supergravity gravitational solutions.
- The evolution problem, constraint equations and allowed Cauchy surfaces of globally hyperbolic supergravity solutions.
- The study and classification of supersymmetric compactification backgrounds.
- The study of the moduli spaces of supersymmetric solutions and its applications to differentiable geometry and topology.

we must consider *bosonic solutions* (non-zero fermions a priori inconsistent!), whence ony the *bosonic sector* is relevant: we need to truncate the theory! (compare with supersymmetric field theory)

Crucially, truncating to bosonic supergravity does not *erase* supersymmetry, there is a *remnant*: the Killing spinor equations, which define the notion of susy solution.

Set $\phi^f = 0$ in $\mathcal{L}[\phi^b, \phi^f]$ (always consistent!). Then, $\mathcal{L}[\phi^b, 0]$ defines a local system of equations for ϕ^b, the bosonic equations of motion. Given $(\phi^b, 0)$ we can apply a supersymmetry transformation. Using that $\delta_\epsilon \phi^b = 0$ we obtain:

$$(\phi^b, \phi^f = 0) \xrightarrow{\delta_\epsilon} (\phi^b, \delta_\epsilon \phi^f)\,.$$

The susy transform of $(\phi^b, 0)$ my no longer be bosonic, since in general $\delta_\epsilon \phi^f \neq 0$!

This leads to the notion of supersymmetric solution: a bosonic solution that is preserverd by supersymmetry, that is:

$$D\epsilon + \phi^b \epsilon = 0\,.$$

This is a remarkable system of spinorial partial differential equations!

Local supergravity IV: general remarks

In contrast to the full supergravity, the mathematical theory of bosonic supergravity and its Killing spinor equations can be completely and rigorously developed in a classical differential-geometric framework, using standard differential geometric structures and objects together with their higher categorifications.

Because of its fundamental importance and in order to develop the mathematical theory behind electromagnetic duality and its interplay with U-duality, which is specific of four dimensions, I will focus on four-dimensional supergravity.

Research goal: develop the mathematical theory of bosonic four-dimensional supergravity and its Killing spinor equations and develop the theory of globally hyperbolic supersymmetric solutions.

Let M be an oriented and spin four-manifold equipped with a spin structure. For every Lorentzian metric g on M, denote by S_g the associated bundle of irreducible Clifford modules (real of rank four).

- The configuration space $\mathrm{Conf}(M)$ of the theory is the space of Lorentzian metrics on M. The theory is defined through the Lagrangian $\mathcal{L}\colon \mathrm{Conf}(M) \to \Omega^4(M)$:

  $$\mathcal{L}[g] = (\mathrm{R}^g + 6\lambda^2)\,\nu_g\,.$$

  Hence, the solution space is $\mathrm{Sol}(M) = \left\{g \in \mathrm{Conf}(M)\,|\,\mathrm{Ric}^g = -3\lambda^2 g\right\}$.
- Killing spinor equations: $g \in \mathrm{Sol}(M)$ is supersymmetric if $\exists \epsilon \in \Gamma(S_g)$ such that:

  $$\nabla^g_v \epsilon = \frac{\lambda}{2} v \cdot \epsilon\,, \qquad \forall v \in TM\,.$$
- Hence, supersymmetric solutions are non-positive curvature Lorentzian Einstein manifolds admitting *real* Killing spinor.
- Well-known example: AdS_4 *space-time* and exact gravitational waves!

Let M be an oriented and spin four-manifold equipped with a spin structure and fix $w \in \mathbb{C}$. For every Lorentzian metric g on M, denote by $S_g = S_g^+ + S_g^-$ the associated bundle of irreducible Clifford modules (complex of rank four).

- The configuration space $\mathrm{Conf}(M)$ of the theory is the space of Lorentzian metrics on M. The theory is defined through the Lagrangian $\mathcal{L} \colon \mathrm{Conf}(M) \to \Omega^4(M)$:

$$\mathcal{L}[g] = (\mathrm{R}^g + |w|^2)\, \nu_g \,.$$

  Hence, the solution space is $\mathrm{Sol}(M) = \left\{ g \in \mathrm{Conf}(M) \,|\, \mathrm{Ric}^g = -\frac{|w|^2}{2} g \right\}$.

- Killing spinor equations: $g \in \mathrm{Sol}(M)$ is susy if $\exists \epsilon \in \Gamma(S_g^+)$ such that:

$$\nabla^g_v \epsilon = w\, v \cdot c(\epsilon) \,, \qquad \forall v \in TM \,,$$

  where $c \colon S^+ \to S^-$ is the canonical conjugation morphism.

- Susy solutions are non-positive curvature Lorentzian Einstein manifolds admitting a generalized Killing spinor, not a standard Killing spinor!

M oriented four-manifold equipped admitting $\mathrm{Spin}^c_0(3,1)$ structures and fix a complex manifold $\mathcal{M}$ equipped with a negative Hermitean line bundle. For every Lorentzian metric g on M, denote by $S_g = S_g^+ + S_g^-$ the associated spinor bundle.

- The configuration space is the space of Lorentzian metrics on M and smooth maps $\varphi\colon M \to \mathcal{M}$. Lagrangian:

$$\mathcal{L}[g,\varphi] = (\mathrm{R}^g - |\mathrm{d}\varphi|^2_{g,\mathcal{G}})\,\nu_g\,.$$

Solution space: $\mathrm{Sol} = \left\{(g,\varphi)|\, \mathrm{G}^g = \varphi^*\mathcal{G} - \frac{1}{2}|\mathrm{d}\varphi|^2_{g,\mathcal{G}}\, g\,,\ \ \mathrm{Tr}_g\nabla\mathrm{d}\varphi = 0\right\}$.

- Killing spinor equations: $(g,\varphi) \in \mathrm{Sol}$ is susy if $\exists\epsilon \in \Gamma(S_g^+)$ such that:

$$\nabla^{\varphi}\epsilon = 0\,, \qquad \mathrm{d}\varphi^{0,1}\cdot\epsilon = 0\,,$$

where ∇^{φ} is the canonical spinorial lift of ∇^g and the pull-back by φ of the Chern connection on $\mathcal{L}$ and we understand $\mathrm{d}\varphi \in \Omega^1(T\mathcal{M}^{\varphi})$.

Supersymmetric solutions are thus particular cases of $\mathrm{Spin}^c_0(3,1)$ manifolds admitting a parallel spinor. Such manifolds have been studied by Moroianu (Riemannian case) and Ikemakhen (Lorentzian case). Adapting Ikemakhen's results:

Proposition

Let M be a geodesically complete and simply-conneted Lorentzian four-manifold admitting a supersymmetric solution (g, φ, ϵ) to $\mathcal{N} = 1$ chiral supergravity with vanishing superpotential. Then we can have at most the following possibilities:

1. (M, g) is isometric to four-dimensional flat Minkowski space.
2. (M, g) is isometric to $(M, g) \simeq (\mathbb{R}^2 \times X, \delta_{1,1} \times h)$.
3. The holonomy group H of (M, g) contained in $\mathrm{SO}(2) \ltimes \mathbb{R}^2 \subset \mathrm{SO}_0(3,1)$.

Every such solution must be as above. The converse may not be true, since a supersymmetric solution requires (M, g) to admit a parallel spinor with respect to the specific connection ∇^{φ}, which is coupled to the scalar map φ, which is in turn required to satisfy its corresponding Killing spinor equation.

General remarks

The previous examples can be all understood as particular cases of the general $\mathcal{N} = 1$ supergravity whose global mathematical structure and formulation is currently ongoing work, for more details see [arXiv:1810.12353, V. Cortés, C. Lazaroiu, CSS].

Even only considering the previous particular examples we obtain a plethora of unanswered mathematical questions regarding for instance globally hyperbolic supersymmetric solutions:

- Study the initial value problem of the Killing spinor equations: examples evolving solutions.
- Well-posedness of the initial value problem?
- Study of the constraint equations for the initial data: moduli of solutions?
- Allowed topologies of compact Cauchy surfaces?
- Reduction to a Riemannian problem.

We have considered some of these questions in [arXiv:2011.02423; A. Murcia, CSS], using the theory of *spinorial polyforms*. This is a novel spinorial framework [arXiv:1911.08658; V. Cortés, C. Lazaroiu, CSS] specially well-adapted to study supergravity Killing spinor equations associated to a Lipschitz structure.

Let $U \subset \mathbb{R}^4$ contractible and oriented open set. Fix non-negative integers n_s, n_v. The configuration space is defined as the set of triples (g, ϕ, A) consisting of:

- A Lorentzian metric g defined on U.
- An $\mathbb{R}^{n_s}$-valued function $\phi\colon U \to \mathbb{R}^{n_s}$ defined on U with components $\phi^i\colon U \to \mathbb{R}$, $i = 1, \dots, n_s$ and fix an oriented open subset $V \subset \mathbb{R}^{n_s}$ containing $\phi(U)$. The real functions $\{\phi^i\}$ are the (locally-defined) *scalar fields* of the theory. Such functions $\phi\colon U \to \mathbb{R}^{n_s}$ as *scalar maps*.
- An $\mathbb{R}^{n_v}$-valued one-form $A \in \Omega^1(U, \mathbb{R}^{n_v})$ with components of A by $A^\Lambda \in \Omega^1(U)$, with $\Lambda = 1, \dots, n_v$, which correspond to the local $\mathrm{U}(1)$ *gauge fields* of the theory. We set $F \stackrel{\text{def.}}{=} \mathrm{d}A \in \Omega^2(U, \mathbb{R}^{n_v})$ with components $F^\Lambda = \mathrm{d}A^\Lambda \in \Omega^2(U)$.

The *local* bosonic sector of four-dimensional supergravity is defined through the following action functional:

$$\mathrm{S}_l[g,\phi,A] = \int_U \left\{-\mathrm{R}_g + \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j + \mathcal{R}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{ab} * F^{\Sigma\, ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{ab}F^{\Sigma\, ab}\right\} \nu_g$$

- $\mathcal{G} \in \Gamma(T^*V \odot T^*V)$ is a Riemannian metric on V. We denote by:

  $$\mathcal{G}(\phi) \stackrel{\text{def.}}{=} \mathcal{G}\circ\phi \colon U \to \mathrm{Sym}(n_s,\mathbb{R})\,,$$

  the composition of $\mathcal{G}$ with ϕ and by $\mathcal{G}_{ij}(\phi)$ the components of $\mathcal{G}(\phi)$.
- $\mathcal{R},\mathcal{I}\colon V \to \mathrm{Sym}(n_v,\mathbb{R})$ are smooth functions on V valued in the space of $n_v\times n_v$ square symmetric matrices with real entries. We denote by:

  $$\mathcal{R}(\phi) \stackrel{\text{def.}}{=} \mathcal{R}\circ\phi \colon U \to \mathrm{Sym}(n_v,\mathbb{R})\,, \qquad \mathcal{I}(\phi) \stackrel{\text{def.}}{=} \mathcal{I}\circ\phi \colon U \to \mathrm{Sym}(n_v,\mathbb{R})\,,$$

  the compositions of $\mathcal{R}$ and $\mathcal{I}$ with ϕ with components by $\mathcal{I}_{\Lambda\Sigma}(\phi)$, $\mathcal{R}_{\Lambda\Sigma}(\phi)$ the entries of the corresponding symmetric matrices. ($\mathcal{I}$ positive definite).

Local bosonic sector of supergravity on (U, V) is uniquely determined by a choice of Riemannian metric $\mathcal{G}$ on V and matrix-valued functions $\mathcal{I}$ and $\mathcal{R}$ as above.

$$\mathrm{S} = \mathrm{S}^e_l + \mathrm{S}^s_l + \mathrm{S}^v_l \,,$$

where $\mathrm{S}^e_l[g] \stackrel{\text{def.}}{=} -\int_U \mathrm{R}_g \nu_g$ is the Einstein-Hilbert action,

$$\mathrm{S}^s_l[g, \phi] \stackrel{\text{def.}}{=} \int_U \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j \,\nu_g \,,$$

is a local non-linear sigma model with target space metric $\mathcal{G}$, and

$$\mathrm{S}^v_l[g, \phi, A] \stackrel{\text{def.}}{=} \int_U \left\{\mathcal{R}_{\Lambda\Sigma}(\phi)F^\Lambda_{ab} * F^{\Sigma\, ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{ab}F^{\Sigma\, ab}\right\} \nu_g \,,$$

is a local abelian Yang-Mills theory coupled to the scalars $\{\phi^i\}_{i=1,\dots,n_s}$.

- S^e_l defines the *gravity sector* of the theory.
- S^s_l defines the *scalar sector* of the theory.
- S^v_l defines the *gauge sector* of the theory.

Supersymmetry constrains the local isometry type of the Riemannian manifold $(V, \mathcal{G})$ that can be considered as the target space of the non-linear sigma model of a given supergravity theory. Depending on the amount $\mathcal{N}$ of supersymmetry preserved, the local isometry type of $(V, \mathcal{G})$ is:

- $\mathcal{N} = 1$: Kähler-Hodge.
- $\mathcal{N} = 2$: Local product of a projective special Kähler and a QK manifolds.
- $\mathcal{N} = 3$: $\mathrm{SU}(3, n)/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(n))$.
- $\mathcal{N} = 4$: $, \mathrm{SU}(1, 1)/\mathrm{U}(1) \times \mathrm{SO}(6, n)/\mathrm{S}(\mathrm{O}(6) \times \mathrm{O}(n))$.
- $\mathcal{N} = 5$: $\mathrm{SU}(1, 5)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5))$.
- $\mathcal{N} = 6$: $\mathrm{SO}^*(12)/(\mathrm{U}(1) \times \mathrm{SU}(6))$.
- $\mathcal{N} = 8$: $\mathrm{E}_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$.

The equations of motion that follow from the bosonic supergravity action associated to a given triple $(\mathcal{G}, \mathcal{R}, \mathcal{I})$ are:

- The *Einstein equations*:

$$\mathrm{G}^g_{ab} = \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial_b\phi^j - \frac{1}{2}g_{ab}\mathcal{G}_{ij}(\phi)\partial_c\phi^i\partial^c\phi^j$$
$$+2\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{ac}F^{\Sigma c}_b - \frac{1}{2}g_{ab}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{cd}F^{\Sigma cd}\,.$$

- The *scalar equations*:

$$\nabla^g_a(\mathcal{G}_{ik}(\phi)\partial^a\phi^i) = \frac{1}{2}\partial_k\mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j + \frac{1}{2}\partial_k\mathcal{R}_{\Lambda\Sigma}(\phi)F^\Lambda_{ab} * F^{\Sigma ab} + \frac{1}{2}\partial_k\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{ab}F^{\Sigma ab}$$

- The *Maxwell equations*:

$$\nabla^g_a(\mathcal{R}_{\Lambda\Sigma}(\phi) * F^{\Sigma ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Sigma ab}) = 0$$

We have defined the theory locally; the goal is now to obtain a global geometric model that we can study mathematically. How do we proceed? We need to understand Maxwell equations geometrically!

The variables of the supergravity equations consist on Lorentzian metrics g on U, n_s scalars $\{\phi^i\}$ and n_v closed two-forms $\{F^\Lambda\}$. Conditions $\mathrm{d}F^\Lambda = 0$ are the *Bianchi identities*, and ensure that $F = \mathrm{d}A$. The Maxwell equations are equivalent to:

$$\mathrm{d}(\mathcal{R}_{\Lambda\Sigma}(\phi)F^\Sigma) = \mathrm{d}(\mathcal{I}_{\Lambda\Sigma}(\phi) * F^\Sigma)\,.$$

Define now the two-forms:

$$G_\Lambda(\phi) \stackrel{\mathrm{def.}}{=} \mathcal{R}_{\Lambda\Sigma}(\phi)F^\Sigma - \mathcal{I}_{\Lambda\Sigma}(\phi) * F^\Sigma \in \Omega^2(U)\,, \qquad \Lambda = 1, \dots, n\,.$$

Then, the Bianchi identities and Maxwell equations become:

$$\mathrm{d}F^\Sigma = 0\,, \qquad \mathrm{d}G_\Lambda = 0\,, \qquad \Lambda = 1, \dots, n$$

which in turn can be equivalently written simply as:

$$\mathrm{d}\mathcal{V}(\phi) = 0\,,$$

where $\mathcal{V}(\phi) \in \Omega^2(U, \mathbb{R}^{2n})$ denotes the following vector of two-forms:

$$\mathcal{V}(\phi) = \begin{pmatrix} F \\ G(\phi) \end{pmatrix} \in \Omega^2(U, \mathbb{R}^{2n_v})\,.$$

Lemma

A vector-valued two-form $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})$ *can be written as:*

$$\mathcal{V} = \begin{pmatrix} F \\ G(\phi) \end{pmatrix} ,$$

for $F \in \Omega^2(U, \mathbb{R}^{n_v})$*, where* $\phi\colon U \to \mathbb{R}^{n_s}$*,* $G(\phi) = \mathcal{R}(\phi) F - \mathcal{I}(\phi) * F$ *and* $(\mathcal{R}, \mathcal{I})$ *is fixed, if and only if:*

$$*\mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V}) ,$$

where $\mathcal{J}\colon V \to \mathrm{Gl}(2n_v, \mathbb{R})$ *is the matrix-valued map defined as follows*

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} : V \to \mathrm{Gl}(2n_v, \mathbb{R}) ,$$

and $\mathcal{J}(\phi) \stackrel{\text{def.}}{=} \mathcal{J} \circ \phi\colon U \to \mathrm{Gl}(2n_v, \mathbb{R})$*. In particular, we have* $\mathcal{J}^2 = -1$*.*

Lemma

Let ω be the standard symplectic form on $\mathbb{R}^{2n}$. A matrix-valued map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n})$ can be written as:

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} : V \to \mathrm{Aut}(\mathbb{R}^{2n}) \tag{1}$$

for a local electromagnetic structure $(\mathcal{R}, \mathcal{I})$ if and only if $\mathcal{J}|_p$ is a compatible taming of ω for every $p \in V$.

Hence, we can equivalently define the configuration space of the universal bosonic sector as follows:

$$\mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) =$$
$$\left\{(g, \phi, \mathcal{V}) \mid \ast_g \mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V}),\ g \in \mathrm{Lor}(U),\ \phi \in C^\infty(U, V),\ \mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})\right\}$$

Understanding the group of duality transformations of the local equations is crucial to construct the global geometric model of supergravity.

- *Duality transformations*: symmetries of the local supergravity equations not involving diffeomorphisms of U.

Set $G = \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$. We have a natural action:

$$\mathbb{A}\colon G \times \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v}) \to \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v})$$
$$(f, T, g, \phi, \mathcal{V}) \mapsto (g, f \circ \phi, T\,\mathcal{V})\,.$$

This action does not preserve in general the configuration space $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ of a given local supergravity associated to $(\mathcal{G}, \mathcal{J})$. By definition, the *U-duality group* is the group of $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ that preserves both the configuration and solution spaces of the given supergravity. Recall the natural action:

$$\mathcal{J} \mapsto T\,(\mathcal{J} \circ f^{-1})\,T^{-1}\,,$$

for every $(f, T) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and every taming map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$. Given $(f, T) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and a taming map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$, set:

$$\mathcal{J}^f_T = T\,(\mathcal{J} \circ f^{-1})\,T^{-1}\,.$$

Theorem

For every $(f, T) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$, *the map* $\mathbb{A}_{f,T}$ *induces by restriction a bijection:*

$$\mathbb{A}_{f,T} \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}_U(f_* \mathcal{G}, \mathcal{J}^f_A)\,,$$

such that it further restricts to a bijection of the corresponding spaces of solutions:

$$\mathbb{A}_{f,T} \colon \mathrm{Sol}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Sol}_U(f_* \mathcal{G}, \mathcal{J}^f_T)\,,$$

where $f_* \mathcal{G}$ *is the push-forward of* $\mathcal{G}$ *by* $f \colon V \to V$ *and* $\mathcal{J}^f_T \stackrel{\mathrm{def.}}{=} T\,(\mathcal{J} \circ f^{-1}) T^{-1}$

Definition

The **electromagnetic U-duality group**, or **U-duality group** for short, of the local bosonic supergravity associated to $(\mathcal{G}, \mathcal{J})$ is given by:

$$\mathrm{U}(\mathcal{G}, \mathcal{J}) \stackrel{\mathrm{def.}}{=} \left\{(f, T) \in \mathrm{Iso}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R}) \mid T\,\mathcal{J}\,T^{-1} = \mathcal{J} \circ f\right\}$$

$$1 \to \mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) \to \mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \to 1\,,$$

Consider a four-manifold M and fix a locally constant Cech cocycle u_{ab} valued in the U-duality group $\mathrm{U}(\mathcal{G}, \mathcal{J})$. Consider a good open cover $M \subset \{U_a\}$. On each U_a we consider a local supergravity as define above and we glue them all together using u_{ab}, which gives a consistent result since u_{ab} acts by simetries of the local theories. Task: interpret the global result through the implementation of the Dirac-Schwinger-Zwanziger quantization condition.

Let M be an oriented and connected four-manifold.

Definition

A *scalar bundle* of rank n_s on M is a triple $(\pi, \mathcal{H}, \mathcal{G})$ consisting of:

- A smooth submersion $\pi \colon X \to M$, where X is a connected and oriented differentiable manifold of dimension $n_s + 4$.
- A complete Ehresmann connection $\mathcal{H} \subset TX$ on π.
- A *vertical Euclidean metric*, i.e. a Euclidean metric $\mathcal{G}$ defined on the vertical bundle $\mathcal{V} \subset TX$ of π which is preserved by the parallel transport of $\mathcal{H}$.

Set $\mathrm{Aff}_I \stackrel{\text{def.}}{=} \mathrm{U}(1)^{2n_v} \rtimes \mathrm{Sp}_I(2n, \mathbb{Z})$ where $I \in \mathbb{Z}^{n_v}$ is the fixed type of a symplectic lattice in $\mathbb{R}^{2n_v}$. The group Aff_I identifies with the set $\mathrm{U}(1)^{2n_v} \times \mathrm{Sp}_I(2n_v, \mathbb{Z})$ equipped with the multiplication rule:

$$(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 a_2, \gamma_1\gamma_2)\,, \quad \forall\, a_1, a_2 \in \mathrm{U}(1)^{2n_v}\,,\ \ \gamma_1, \gamma_2 \in \mathrm{Sp}_I(2n_v, \mathbb{Z})\,.$$

Moreover, Aff_I is the group of affine symplectomorphisms of the $2n_v$-dimensional symplectic torus $\mathbb{R}^{2n_v}/\Lambda_I$.

Definition

A *Siegel bundle* P_l of rank n_v and type $l \in \mathbb{Z}^{n_v}$ on M is a principal bundle defined on M with structure group Aff_l.

Let $(\pi, \mathcal{H}, \mathcal{G})$ be a scalar bundle of rank n_v and type $l \in \mathbb{Z}^{n_v}$ over X. The adjoint bundle of P_l admits a natural structure of integral duality bundle of type l, denoted by $\Delta(P_l) = (\mathcal{S}, \omega, \mathcal{D})$. By definition, a vertical taming $\mathcal{J}$ of P_l is a vertical taming of $\Delta(P_l)$. Given a scalar section $s \in \Gamma(\pi)$, we denote by P^s_l the pullback of P_l by s, which becomes a Siegel bundle over M. Denote by $\Delta(P^s_l) = (\mathcal{S}^s, \omega^s, \mathcal{D}^s)$ the integral duality defined by P^s. Let $\mathrm{Conn}(P^s_l)$ be the space of connections on P^s_l. The curvature $\mathcal{F}_{\mathcal{A}} \in \Omega^2(M, \mathcal{S}^s)$ of a connection $\mathcal{A} \in \mathrm{Conn}(P^s_l)$ is a two-form valued in $\mathcal{S}^s$ which is $\mathrm{d}_{\mathcal{D}^s}$-closed by the Bianchi identity since all connections on P^s induce the same connection on $\Delta(P^s_l)$, which coincides with the connection induced by $\mathcal{D}^s$ on the adjoint bundle of P^s_l. Hence:

$$\mathrm{d}_{\mathcal{D}^s} \mathcal{F}_{\mathcal{A}} = 0 \,.$$

For every Lorentzian metric g on M and every scalar section $s \in \Gamma(\pi)$, consider the isomorphism of vector bundles:

$$\star_{g,\mathcal{J}^s} \colon \wedge T^*M \otimes \mathcal{S}^s \to \wedge T^*M \otimes \mathcal{S}^s \, ,$$

defined through $\star_{g,\mathcal{J}^s} = *_g \otimes \mathcal{J}^s$. Since both $*_g$ and $\mathcal{J}^s$ square to minus the identity, this restricts to an involutive automorphism:

$$\star_{g,\mathcal{J}^s} \colon \wedge^2 T^*M \otimes \mathcal{S}^s \to \wedge^2 T^*M \otimes \mathcal{S}^s$$

which gives a direct sum decomposition into eigenbundles:

$$\wedge^2 T^*M \otimes \mathcal{S}^s = (\wedge^2 T^*M \otimes \mathcal{S}^s)_+ \oplus (\wedge^2 T^*M \otimes \mathcal{S}^s)_- \, .$$

The spaces of smooth global sections of these sub-bundles are called polarized (anti)-self-dual two-forms with respect to $\mathcal{J}^s$.

- This is the notion of *self-duality* required by supergravity in four-dimensional Lorentzian signature!

Definition

A *polarized scalar-Siegel bundle* of rank n_v and type $I \in \mathrm{Div}^{n_v}$ over M is a tuple $\zeta = (\pi, \mathcal{H}, \mathcal{G}, P_I, \mathcal{J})$, where $(\pi, \mathcal{H}, \mathcal{G})$ is a scalar bundle over M, P_I is a Siegel bundle of rank n_v and type $I \in \mathrm{Div}^{n_v}$ over X and $\mathcal{J}$ is a vertical taming on $\Delta(P_I)$.

Theorem (C. Lazaroiu, CSS)

The universal bosonic sector of four-dimensional supergravity is completely determined by a polarized scalar-Siegel bundle over M.

Definition

Let $(\pi, \mathcal{H}, \mathcal{G}, P_I, \mathcal{J})$ be a polarized scalar-Siegel bundle over M. The *configuration space* of the bosonic supergravity defined by ζ is the set:

$$\mathrm{Conf}(\zeta) = \{(g, s, \mathcal{A}) \mid g \in Lor(M)\,,\ s \in \Gamma(\pi)\,,\ \mathcal{A} \in \mathrm{Conn}(P_I^s)\}\ .$$

Definition

The *universal bosonic sector* of four-dimensional supergravity determined on M by ζ is defined through the following system of partial differential equations for triples $(g, s, \mathcal{A}) \in \mathrm{Conf}(\zeta)$:

- The Einstein equations:

$$\mathrm{Ric}^g - \frac{g}{2}\mathrm{R}^g = \frac{1}{2}\mathrm{Tr}_g(s_{\mathcal{C}}^*\mathcal{G})\, g - s_{\mathcal{C}}^*\mathcal{G} + 2\mathcal{F}_{\mathcal{A}} \oslash_{Q^s} \mathcal{F}_{\mathcal{A}}\,.$$

- The scalar equations:

$$\nabla^{\Phi(g,s)}\mathrm{d}^{\mathcal{C}} s = \frac{1}{2}(*\mathcal{F}_{\mathcal{A}}, \Psi^s\mathcal{F}_{\mathcal{A}})_{g,Q^s}\,.$$

- The Maxwell equations:

$$\star_{g,\mathcal{J}^s}\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}\,,$$

  whose set of solutions we denote by $\mathrm{Sol}(\zeta) \subset \mathrm{Conf}(\zeta)$.

Connections $\mathcal{A}$ satisfying equation solving the Maxwell equations are *polarized self-dual*. The Maxwell equations of the bosonic gauge sector of local supergravity are given by a system of second-order partial differential equations for a number n_v of *electric* local gauge potentials whose curvatures satisfy a generalization of the standard Maxwell equations. The Bianchi identity and polarized self-duality condition imply that the gauge potential of any solution $(g, s, \mathcal{A}) \in \mathrm{Sol}(\zeta)$ automatically satisfies the following second order differential equation of Yang-Mills type:

$$\mathrm{d}_{\mathcal{D}^s} \star_{g,\mathcal{J}} \mathcal{F}_{\mathcal{A}} = 0 \,.$$

These differ from the usual Yang-Mills equations since $\mathcal{F}_{\mathcal{A}}$ involves both electric and magnetic degrees of freedom while the equations themselves involve the pulled-back taming $\mathcal{J}^s$.

More details:

- The mathematical theory of universal bosonic supergravity [arXiv:2101.07778; C. Lazaroiu and CSS].
- Abelian gauge theory [arXiv:2101.07236; C. Lazaroiu and CSS].

Conclusions

- We have discussed general aspects of supergravity theories and their local supersymmetry transformations.
- We have justified the study of the bosonic supergravity sector and its Killing spinor equations, providing several examples.
- We have explained that the universal bosonc sector of four-dimensional ungauged supergravity is uniquely determined by a polarized Siegel bundle defined on the total space of a submersion equipped with an Ehresmann connection and a vertical Riemannian metric.
- We have explained that the universal bosonc sector of four-dimensional ungauged supergravity is defined through a system of partial differential equations for triples $(g, s, \mathcal{A})$, mixing thus special Lorentzian metrics, with scalar sections and principal connections.

Future directions

- Implement the constraints imposed by supersymmetry on the scalar bundle of the theory: Kähler-Hodge, Quaternionic-Kähler, Projective Special-Kähler.
- Construct the Killing spinor equations for each $\mathcal{N}$.
- Develop the theory of globally hyperbolic supersymmetric solutions; well-posedness, initial data conditions and evolution problem.
- Develop techniques in spinorial geometry to study supersymmetric supergravity solutions, in the spirit of .
- Study moduli spaces of supersymmetric solutions reduced to a three-manifold and their potential applications in differential geometry.
- Develop the mathematical theory of U-duality.

Thanks!