The duality covariant formulation of Abelian gauge theories on Riemannian four-manifolds

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Outline

1. Paracomplex structures and parahermitian spaces

2. Paratamed symplectic vector bundles

3. Abelian gauge theory on Riemannian 4-manifolds

Paracomplex numbers

Let $\mathbb{A} = \mathbb{R}[\epsilon]/(\epsilon^2 - 1) = \mathbb{R} \oplus \epsilon\mathbb{R}$ be the commutative $\mathbb{R}$-algebra of paracomplex (a.k.a. dual) numbers. For any $a = x + \epsilon y \in \mathbb{A}$ with $x, y \in \mathbb{R}$, define:

$$\mathfrak{Re}(a) \stackrel{\text{def.}}{=} x \ \ , \ \ \mathfrak{Im}(a) \stackrel{\text{def.}}{=} y \ \ , \ \ \bar{a} = x - \epsilon y \ \ .$$

We have the neutral metric and symplectic pairing on $\mathbb{A}$:

$$\langle a, a' \rangle_{\mathbb{A}} \stackrel{\text{def.}}{=} \mathfrak{Re}(a\bar{a}') = xx' - yy' \ \ , \ \ \omega_{\mathbb{A}}(a, a') \stackrel{\text{def.}}{=} -\mathrm{Im}(a\bar{a}') = xy' - x'y \ \ ,$$

as well as the seminorm and signature:

$$||a|| \stackrel{\text{def.}}{=} \sqrt{|\langle a, a \rangle_{\mathbb{A}}|} = \sqrt{|a\bar{a}|} = \sqrt{|x^2 - y^2|} \ \ , \ \ \forall a = x + \epsilon y \in \mathbb{A}$$

$$\epsilon(a) \stackrel{\text{def.}}{=} \mathrm{sign}\langle a, \bar{a} \rangle_{\mathbb{A}} = \mathrm{sign}(x^2 - y^2) \ \ ,$$

which are morphisms of groups from $(\mathbb{A}^\times, \cdot)$ to $\mathbb{R}_{>0}$ and $\mathbb{G}_2 = \{-1, 1\} \simeq \mathbb{Z}_2$.

Proposition

An element $a \in \mathbb{A}$ is invertible iff $||a|| \neq 0$. Moreover, $\mathbb{A}^\times$ is isomorphic with $\mathbb{Z}_2 \times \mathbb{R}_{>0} \times \mathbb{R} \simeq \mathbb{R}^\times \times (\mathbb{R}, +) \simeq \mathbb{R}^\times \times \mathbb{R}^\times$. For any $a \in \mathbb{A}^\times$, we have $a^{-1} = \epsilon(a) \frac{\bar{a}}{||a||}$ and:

$$a = \epsilon^{\frac{1-\epsilon(a)}{2}} \, ||a|| \, (\cosh(\theta) + \epsilon \sinh(\theta)) \ \ ,$$

where the argument $\theta = \theta(a) \in \mathbb{R}$ is uniquely determined by a and gives a morphism of groups $\theta : \mathbb{A}^\times \to (\mathbb{R}, +)$.

Paracomplex spaces

Definition

A *paracomplex* $2n$*-space* is a pair (V, K), where V is a $2n$-dimensional $\mathbb{R}$-vector space and K is a *paracomplex structure* on V, i.e. an endomorphism $K \in \mathrm{End}_{\mathbb{R}}(V)$ which satisfies the conditions:

1. $K^2 = \mathrm{id}_V$ (i.e. K is a product structure on V)
2. The eigenspaces $V_+ \stackrel{\text{def.}}{=} \ker(K - \mathrm{id}_V)$ and $V_- \stackrel{\text{def.}}{=} \ker(K + \mathrm{id}_V)$ of K have dimension equal to n.

A $2n$-dimensional paracomplex space (V, K) is a free left $\mathbb{A}$-module of rank n when endowed with the external multiplication:

$$a \bullet x \stackrel{\text{def.}}{=} \mathfrak{Re}(a)x + \mathfrak{Im}(a)Kx \quad \forall a \in \mathbb{A} \quad \forall x \in V \quad . \tag{1}$$

The rank $n = \frac{1}{2}\dim_{\mathbb{R}} V$ of this module is the *paracomplex rank* of (V, K).

Definition

A *morphism of paracomplex spaces* (or *paracomplex linear map*) $f : (V, K) \to (V', K')$ is a morphism of the underlying $\mathbb{A}$-modules, i.e. an $\mathbb{R}$-linear map $f : V \to V'$ which satisfies:

$$f(Kx) = K' f(x) \quad \forall x \in V \quad .$$

Paracomplex spaces

The *standard* $2n$*-dimensional paracomplex space* is the paracomplex space $(\mathbb{R}^{2n}, K_{2n})$, where the *standard paracomplex structure* K_{2n} of $\mathbb{R}^{2n}$ is:

$$K_{2n}(x,y) \stackrel{\text{def.}}{=} (y,x) \quad \forall x, y \in \mathbb{R}^n$$

and has matrix in the canonical basis given by:

$$\hat{K}_{2n} = \left[\begin{array}{cc} 0 & I_n \\ I_n & 0 \end{array}\right] \quad .$$

Definition

A *paracomplex basis* of a paracomplex $2n$-space (V, K) is a basis of the corresponding free A-module, i.e. an ordered system of vectors $v_1, \ldots, v_n \in V$ such that $K(v_1), \ldots, K(v_n), v_1, \ldots, v_n$ is a basis of V over $\mathbb{R}$.

Any choice of paracomplex basis gives an isomorphism $(V, K) \simeq (\mathbb{R}^{2n}, K_{2n})$.

Definition

The *paracomplex general linear group* $\mathrm{GL}(n, \mathbb{A})$ is the group of automorphisms of the standard paracomplex space $(\mathbb{R}^{2n}, K_{2n})$.

We have:

$$\mathrm{GL}(n,\mathbb{A}) = \left\{ \left[\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right] \;\middle|\; A, B \in \mathrm{GL}(n,\mathbb{R}) \right\} \simeq \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \quad .$$

Definition

A basis $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ of a paracomplex space (V, K) is *non-degenerate* if $f_1, \ldots, f_n$ is a basis of V over $\mathbb{A}$, i.e. if $K(f_1), \ldots, K(f_n), f_1, \ldots, f_n$ is a basis of V over $\mathbb{R}$. The *period matrix* $\sigma := \sigma_{\mathcal{E}}(K) = (\sigma_{ij})_{i,j=1,\ldots,n} \in \mathrm{Mat}(n, \mathbb{A})$ of K in this basis is given by the expansion $e_i = \sum_{j=1}^{n} \sigma_{ji} \bullet f_j \ \ \forall i = 1, \ldots, n$.

Proposition

Let $\sigma_{\mathcal{E}}(K) = \sigma_R + \epsilon\sigma_I$ (where $\sigma_R, \sigma_I \in \mathrm{Mat}(n, \mathbb{R})$) be the period matrix of (V, K) in a non-degenerate basis $\mathcal{E}$ of V over $\mathbb{R}$. Then σ_I is invertible and the matrix of K in the basis $\mathcal{E}$ is $\hat{K}_{\mathcal{E}} = \left[\begin{array}{cc} \sigma_I^{-1}\sigma_R & \sigma_I^{-1} \\ \sigma_I - \sigma_R\sigma_I^{-1}\sigma_R & -\sigma_R\sigma_I^{-1} \end{array}\right]$.

Proposition

Let $\mathcal{E}, \mathcal{E}'$ be non-degenerate bases of (V, K) and $M = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \in \mathrm{GL}(2n, \mathbb{R})$ $(A, B, C, D \in \mathrm{Mat}(n, \mathbb{R}))$ be the base change matrix from $\mathcal{E}$ to $\mathcal{E}'$. Then $B^T\sigma_{\mathcal{E}}(K) + D^T$ is invertible and $\sigma_{\mathcal{E}'}(K) = (A^T\sigma_{\mathcal{E}}(K) + C^T)(B^T\sigma_{\mathcal{E}}(K) + D^T)^{-1}$.

Comparison to periods of complex vector spaces

Definition

A basis $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ over $\mathbb{R}$ of a complex vector space (V, J) is *non-degenerate* if $(f_1, \ldots, f_n)$ is a basis of V over $\mathbb{C}$, i.e. if $J(f_1), \ldots J(f_n), f_1, \ldots, f_n$ is a basis of V over $\mathbb{R}$. The *period matrix* $\tau := \tau_{\mathcal{E}}(J) = (\tau_{ij})_{i,j=1,\ldots,n} \in \mathrm{Mat}(n, \mathbb{C})$ of J in this basis is given by the expansion $e_i = \sum_{j=1}^n \tau_{ji} f_j$ for all $i = 1, \ldots, n$.

Proposition

Let $\tau_{\mathcal{E}}(J) = \tau_R + \mathbf{i}\tau_I$ (with $\tau_R, \tau_I \in \mathrm{Mat}(n, \mathbb{R})$) be the period matrix of a complex vector space (V, J) in a non-degenerate basis $\mathcal{E}$ of V over $\mathbb{R}$. Then τ_I is invertible and the matrix of J in this basis is:

$$\hat{J}_{\mathcal{E}} = \begin{bmatrix} \tau_I^{-1}\tau_R & \tau_I^{-1} \\ -\tau_I - \tau_R\tau_I^{-1}\tau_R & -\tau_R\tau_I^{-1} \end{bmatrix} .$$

Parahermitian spaces

Definition

A *parahermitian space* is a triplet (V, ω, K), where (V, ω) is a symplectic $\mathbb{R}$-vector space and K is a *parataming* of (V, ω), i.e. a paracomplex structure on V which satisfies:

$$\omega(Kx, Ky) = -\omega(x, y) \Longleftrightarrow \omega(Kx, y) = -\omega(x, Ky) \quad \forall x, y \in V \quad .$$

Definition

A *morphism of parahermitian spaces* $f : (V, \omega, K) \to (V', \omega', K')$ is a morphism of symplectic spaces from (V, ω) to (V', ω') which is also a morphism of paracomplex spaces from (V, K) to (V', K').

The *standard parahermitian* $2n$*-space* is $(\mathbb{R}^{2n}, \omega_n, K_{2n})$, where K_{2n} is the parataming of ω_n given by the standard paracomplex structure K_{2n} of $\mathbb{R}^{2n}$.

Definition

The *paraunitary group* $\mathrm{U}(n, \mathbb{A})$ is the group of automorphisms of the $2n$-dimensional standard parahermitian space $(\mathbb{R}^{2n}, \omega_n, K_{2n})$, which is given by:

$$\mathrm{U}(n, \mathbb{A}) = \left\{ \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} \;\middle|\; A \in \mathrm{GL}(n, \mathbb{R}) \right\} \simeq \mathrm{GL}(n, \mathbb{R}) \quad .$$

The induced neutral form. Positive and negative Lagrangian subspaces

Given a parahermitian space (V, ω, K), let h be the neutral metric induced by ω and K on V:

$$h(x, y) \stackrel{\text{def.}}{=} \omega(Kx, y) \ \ \forall x, y \in V \quad .$$

K is h-antisymmetric. For any subspace $W \subset V$, let $h_W \stackrel{\text{def.}}{=} h|_{W \times W}$.

Definition

A Lagrangian subspace $L \subset V$ is called *positive* or *negative* if h_L is positive-definite resp. negative-definite.

Proposition-Definition

Any parahermitian space (V, ω, K) admits a positive Lagrangian subspace. If L_+ is such a subspace, then $L_- \stackrel{\text{def.}}{=} K(L_+)$ is a negative Lagrangian subspace and we have $V = L_+ \oplus L_-$. Moreover, this decomposition is h-orthogonal and called a *special Lagrangian decomposition* of (V, ω, K).

Proposition-Definition

Let L_+ be a positive Lagrangian subspace of (V, ω, K) and $L_- \stackrel{\text{def.}}{=} K(L_+)$. Then the endomorphism of V given by $J := J_{L_+}(K) \stackrel{\text{def.}}{=} K|_{L_+} \oplus (-K|_{L_-})$ satisfies $J(L_+) = L_-$ and is a taming of (V, ω), called the *special taming* determined by L_+.

Relation between tamings and paratamings

Proposition

Let (V, ω, J) be a Hermitian space and L_+ be any Lagrangian subspace of (V, ω). Then $L_- \stackrel{\text{def.}}{=} J(L_+)$ is a Lagrangian complement of L_+ and $K := K_{L_+}(J) \stackrel{\text{def.}}{=} J|_{L_+} \oplus (-J|_{L_-})$ is a paratamingof ω. Moreover, L_+ is a positive Lagrangian subspace for (V, ω, K) and we have $K(L_+) = L_-$ and $J_{L_+}(K) = J$.

Let $T(V, \omega)$ be the set of tamings of ω. For any Lagrangian subspace L, let $P_L(V, \omega)$ be the set of those paratamings of ω for which L is positive.

Corollary

The map $\Psi : P_L(V, \omega) \to T(V, \omega)$ defined through $\Psi(K) \stackrel{\text{def.}}{=} J_L(K)$ is a bijection whose inverse is given by $\Psi^{-1}(J) = K_L(J)$.

Definition

Let $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a symplectic basis of the parahermitian space (V, ω, K) and consider the complementary Lagrangian subspaces:

$$L_-(\mathcal{E}) \stackrel{\text{def.}}{=} \langle e_1, \ldots, e_n \rangle \quad , \quad L_+(\mathcal{E}) \stackrel{\text{def.}}{=} \langle f_1, \ldots, f_n \rangle \quad .$$

The symplectic basis $\mathcal{E}$ is called:

- *positive*, if L_+ is a positive Lagrangian subspace.
- *special*, if L_+ and L_- give a special Lagrangian decomposition of (V, ω, K).
- *adapted* if it is both special and a paracomplex basis (meaning a basis over $\mathbb{A}$) i.e. if it is special and satisfies $K(e_i) = f_i$ for all $i = 1, \ldots, n$.

Any adapted basis is special and any special basis is positive.

Proposition

Let $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a special symplectic basis of the parahermitian space (V, ω, K). Then $\mathcal{E}$ is positive and non-degenerate.

Proposition

Let $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a non-degenerate symplectic *basis of a parahermitian space (V, K, ω). Then $\sigma_{\mathcal{E}}$ is a symmetric matrix. If $\mathcal{E}$ is also positive, then $\mathfrak{Im}(\sigma_{\mathcal{E}})$ is positive-definite.*

Definition

Let $\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a positive and non-degenerate symplectic basis of (V, ω, K). The *complex period matrix* $\tau_{\mathcal{E}}(K) \in \mathrm{Mat}(n, \mathbb{C})$ of K in the basis $\mathcal{E}$ is the period matrix of the complex structure $J_{L_+(\mathcal{E})}(K)$ defined by K relative to the positive Lagrangian subspace $L_+(\mathcal{E}) \stackrel{\mathrm{def.}}{=} \langle f_1, \ldots, f_n \rangle$:

$$\tau_{\mathcal{E}}(K) \stackrel{\mathrm{def.}}{=} \tau_{\mathcal{E}}(J_{L_+(\mathcal{E})}) \in \mathbb{SH}_n \quad .$$

The expansion:

$$e_a = \tau_{ab} f_b = \left[\mathrm{Re}(\tau)_{ab} + \mathrm{Im}(\tau)_{ab} J_{L_+(\mathcal{E})}\right] f_b = \left[\mathrm{Re}(\tau)_{ab} + \mathrm{Im}(\tau)_{ab} K\right] f_b \qquad (2)$$

shows that we have:

$$\tau_{\mathcal{E}} \stackrel{\mathrm{def.}}{=} \mathfrak{Re}(\sigma_{\mathcal{E}}) + \mathbf{i}\mathfrak{Im}(\sigma_{\mathcal{E}}) \quad .$$

Let M be a manifold.

Definition

A *paratamed symplectic vector bundle* on M is a triplet $(\mathcal{S}, \omega, \mathcal{K})$, where $(\mathcal{S}, \omega)$ is a symplectic vector bundle on M and $\mathcal{K}$ is a *paratamіng* of $(\mathcal{S}, \omega)$, i.e. an endomorphism of $\mathcal{S}$ such that $\mathcal{K}_m$ is a parataming of $(\mathcal{S}_m, \omega_m)$ for all $m \in M$.

A parataming $\mathcal{K}$ of $(\mathcal{S}, \omega)$ gives complementary Lagrangian sub-bundles $\mathcal{S}_\pm \stackrel{\mathrm{def.}}{=} \ker(\mathcal{K} \mp \mathrm{id}_\mathcal{S})$ of $(\mathcal{S}, \omega)$.

Theorem

The following statements are equivalent for a symplectic vector bundle $(\mathcal{S}, \omega)$:

- (a) $(\mathcal{S}, \omega)$ *admits a parataming.*
- (b) $(\mathcal{S}, \omega)$ *admits a Lagrangian sub-bundle.*
- (c) $(\mathcal{S}, \omega)$ *admits a hyper-parahermitian structure, i.e. there exist a taming $\mathcal{J}$ and a parataming $\mathcal{K}$ of $(\mathcal{S}, \omega)$ such that $\mathcal{K} \circ \mathcal{J} = -\mathcal{J} \circ \mathcal{K}$.*
- (d) *The structure group of $\mathcal{S}$ reduces from $\mathrm{Sp}(2n, \mathbb{R})$ to $\mathrm{O}(n, \mathbb{R})$.*

In this case, the odd Chern classes $c_{2k+1}(\mathcal{S}, \omega)$ vanish for all k and we have:

$$c_{2k}(\mathcal{S}, \omega) = (-1)^k p_k(\mathcal{S}_+) = (-1)^k p_k(\mathcal{S}_-) \quad \forall k \geq 0 \quad .$$

Corollary

Suppose that $(\mathcal{S},\omega)$ admits a Lagrangian sub-bundle $\mathcal{L}$. Then the set $\mathcal{T}(\mathcal{S},\omega)$ of tamings of $(\mathcal{S},\omega)$ is in natural bijection with the set $\mathcal{P}_{\mathcal{L}}(\mathcal{S},\omega)$ consisting of those paratamings of $(\mathcal{S},\omega)$ for which $\mathcal{L}$ is positive.

Definition

A *paratamed duality structure* defined on M is a system $\Theta = (\mathcal{S},\omega,\mathcal{D},\mathcal{K})$, where $(\mathcal{S},\omega,\mathcal{D})$ is a duality structure on M and $\mathcal{K}$ is a paratamming of $(\mathcal{S},\omega)$.

Let (M,g) be an oriented Riemannian 4-manifold with Hodge operator $*_g$.

Definition

The *parapolarized Hodge operator* of a paratamed symplectic vector bundle $(\mathcal{S},\omega,\mathcal{K})$ defined on M is the operator:

$$\star_{g,\mathcal{K}} \stackrel{\mathrm{def.}}{=} *_g \otimes \mathcal{K} \in \mathrm{End}(\wedge(M,\mathcal{S})) \quad ,$$

where $\wedge(M,\mathcal{S}) \stackrel{\mathrm{def.}}{=} \wedge T^*M \otimes \mathcal{S}$.

We have $\star^2_{g,\mathcal{K}}|_{\wedge^2(M,\mathcal{S})} = \mathrm{id}_{\wedge^2(M,\mathcal{S})}$.

Definition

Let (M, g) be an oriented Riemannian 4-manifold and $\Theta = (\Delta, \mathcal{K})$ be a paratamed duality structure on M with underlying duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. The space of *Abelian field strength configurations* defined by Δ is:

$$\mathrm{Conf}(M, \Delta) = \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S}) \quad .$$

The space of *Abelian field strengths* defined by Θ is:

$$\mathrm{Sol}(M, g, \Theta) = \{\mathcal{V} \in \mathrm{Conf}(M, \Delta) | \star_{g,\mathcal{K}} \mathcal{V} = -\mathcal{V}\} \quad .$$

We can now define *parapolarized Siegel bundles* $\mathbf{P} = (P, \mathcal{K})$ and consider the affine space $\mathfrak{Conf}(P) = \mathrm{Conn}(P)$ of gauge configurations (=principal connections on P) and the set $\mathfrak{Sol}(M, g, \mathbf{P}) \subset \mathfrak{Conf}(P)$ of gauge potentials (=those principal connections $\mathcal{A} \in \mathrm{Conn}(P)$ whose adjoint curvature $\mathcal{V}_{\mathcal{A}}$ is parapolarized *anti*-self-dual). The details are very similar to what we did in the Lorentzian signature case.