The duality-covariant formulation of classical Abelian gauge theories

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Outline

1. Formulation through field strengths
2. Integral symplectic spaces and special affine symplectic tori
3. The Dirac integrality condition
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Let M be a connected d-manifold. Any symplectic vector bundle $(\mathcal{S}, \omega)$ of rank $2n$ is oriented by the section $\wedge^n \omega$ of $\det \mathcal{S}^* = \wedge^{2n} \mathcal{S}^*$. Since $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are homotopy equivalent to their common maximal compact subgroup $\mathrm{U}(n)$, the classification of symplectic, complex and Hermitian vector bundles defined on M are equivalent. Thus:

- Any complex vector bundle admits a Hermitian pairing
- Any symplectic vector bundle admits a complex structure (called a *taming*) which is compatible with its symplectic pairing and makes it into a Hermitian vector bundle.
- A real vector bundle of even rank admits a symplectic pairing iff it admits a complex structure; the conditions for this are well-known.

The classifying spaces $\mathrm{BSp}(2n, \mathbb{R})$ and $\mathrm{BU}(n)$ are homotopy equivalent, hence the fundamental characteristic classes of a symplectic vector bundle $(\mathcal{S}, \omega)$ are Chern classes, which we denote by $c_k(\mathcal{S}, \omega)$.

Definition

A *duality structure* is a flat symplectic vector bundle $\Delta \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$ on M. The *rank* of Δ is the rank of the vector bundle $\mathcal{S}$, which is necessarily even.

The Chern classes of the underlying symplectic vector bundle of a duality structure are torsion classes.

Duality structures

The following are equivalent for a symplectic vector bundle $(\mathcal{S}, \omega)$ of rank $2n$:

- a. $(\mathcal{S}, \omega)$ admits a flat symplectic connection $\mathcal{D}$.
- b. There exists a morphism $\rho : \pi_1(M) \to \mathrm{Sp}(2n, \mathbb{R})$ s.t. the structure group of $\mathcal{S}$ reduces from $\mathrm{Sp}(2n, \mathbb{R})$ to $\mathrm{im}\rho$.
- c. The lift of $(\mathcal{S}, \omega)$ to the universal cover of M is symplectically trivial.

For any duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$, we have:

$$c_k(\mathcal{S}, \omega) = \delta_{2k-1}(\hat{c}_k(\mathcal{D}))$$

where $\hat{c}_k(\mathcal{D}) \in H^{2k-1}(M, \mathrm{U}(1))$ are the Cheeger-Chern-Simons (CCS) invariants of $\mathcal{D}$ and δ_i are the connecting morphisms in the long exact sequence:

$$\ldots \to H^i(M, \mathbb{Z}) \to H^i(M, \mathbb{R}) \stackrel{\exp_*}{\to} H^i(M, \mathrm{U}(1)) \stackrel{\delta_i}{\to} H^{i+1}(M, \mathbb{Z}) \to H^{i+1}(M, \mathbb{R}) \to \ldots$$

induced by the exponential sequence:

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp} \mathrm{U}(1) \to 0 \quad .$$

The CCS invariants depend only on the gauge equivalence class of $\mathcal{D}$.

Definition

A *based isomorphism of duality structures* from $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ to $\Delta' = (\mathcal{S}', \omega', \mathcal{D}')$ is a based isomorphism of vector bundles $f : \mathcal{S} \xrightarrow{\sim} \mathcal{S}'$ which satisfies the conditions $\omega' \circ (f \otimes f) = \omega$ and $\mathcal{D}' \circ f = (\mathrm{id}_{T^*M} \otimes f) \circ \mathcal{D}$.

Duality structures

Let $H^*_{\mathcal{D}}(M, \mathcal{S})$ be the cohomology of the twisted de Rham complex:

$$0 \to \mathcal{C}^\infty(M, \mathcal{S}) \stackrel{\mathcal{D}}{\longrightarrow} \Omega^1(M, \mathcal{S}) \stackrel{\mathrm{d}_{\mathcal{D}}}{\longrightarrow} \ldots \stackrel{\mathrm{d}_{\mathcal{D}}}{\longrightarrow} \Omega^d(M, \mathcal{S}) \to 0 \quad .$$

We have a natural isomorphism of graded vector spaces:

$$H^*_{\mathcal{D}}(M, \mathcal{S}) \simeq H^*(M, \mathcal{C}^\infty_{\mathrm{flat}}(\mathcal{S})) \quad .$$

Proposition

For any $m \in M$, the set of isomorphism classes of duality structures is in bijection with the character variety:

$$\mathfrak{R}(\pi_1(M, m), \mathrm{Sp}(2n, \mathbb{R})) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(\pi_1(M, m), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R}) \, ,$$

where $\mathrm{Sp}(2n, \mathbb{R})$ acts through its adjoint representation.

Remark

Duality structures can be viewed as "local systems" of finite-dimensional symplectic vector spaces (i.e. functors $\Pi_1(M) \to \mathrm{Symp}$) by considering their parallel transport functor.

Trivial duality structures

Definition

A duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is called:

1. *topologically trivial* if $\mathcal{S}$ is trivial, i.e. admits a global frame.
2. *symplectically trivial* if $(\mathcal{S}, \omega)$ is isomorphic with the trivial symplectic vector bundle, i.e. admits a global *symplectic* frame.
3. *trivial* (or *holonomy trivial*) if Δ admits a global *flat* symplectic frame, i.e. if the holonomy group of $\mathcal{D}$ is trivial (this amounts to $(\mathcal{S}, \mathcal{D})$ being trivial).

We have: holonomy trivial $\Longrightarrow$ symplectically trivial $\Longrightarrow$ topologically trivial.
If $\pi_1(M) = 0$ then every duality structure is holonomy trivial.

A global flat symplectic frame of a holonomy-trivial duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ of rank $2n$ has the form:

$$\mathcal{E} = (e_1, \ldots, e_n, f_1, \ldots, f_n) \quad ,$$

where e_i, f_j are $\mathcal{D}$-flat sections of $\mathcal{S}$ such that:

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad , \quad \omega(e_i, f_j) = -\omega(f_i, e_j) = -\delta_{ij} \quad \forall i, j = 1, \ldots, n \quad .$$

Any choice of such a frame induces a trivialization isomorphism $\tau_{\mathcal{E}} : \Delta \xrightarrow{\sim} \Delta_n$ between Δ and the *canonical trivial duality structure* $\Delta_n \stackrel{\mathrm{def.}}{=} (\underline{\mathbb{R}}^{2n}_M, \underline{\omega}_{2n}, \mathrm{d})$.

Let $\mathbb{SH}_n$ be the n-th Siegel upper half space (space of symmetric $n \times n$ complex matrices with positive-definite imaginary part).

Definition

A *taming map* of size $2n$ on M is a smooth map $\mathcal{J} \in \mathcal{C}^\infty(M, \mathrm{GL}(2n, \mathbb{R}))$ s.t.:

1. $\mathcal{J}(m)^2 = -I_{2n}$ for all $m \in M$.
2. $\mathcal{J}(m)$ is a taming of ω_{2n} of $\mathbb{R}^{2n}$ for all $m \in M$.

A *period matrix map* of size n on M is a smooth function $\mathcal{N} \in \mathcal{C}^\infty(M, \mathbb{SH}_n)$.

Let $\mathfrak{J}_n(M)$ and $\mathrm{Per}_n(M)$ be the sets of such maps defined on M.

Proposition

A map $\mathcal{J} \in \mathcal{C}^\infty(M, \mathrm{GL}(2n, \mathbb{R}))$ is a taming map iff it can be written as:

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix}$$

where $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary parts of some $\mathcal{N} \in \mathrm{Per}_n(M)$ (which is uniquely determined by $\mathcal{J}$).

This gives a natural bijection $\mathfrak{J}_n(M) \simeq \mathrm{Per}_n(M)$.

Definition

An *electromagnetic structure* defined on M is a pair $\Xi = (\Delta, \mathcal{J})$, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is a duality structure on M and $\mathcal{J}$ is a taming of the symplectic vector bundle $(\mathcal{S}, \omega)$. The *rank* of Ξ is the rank of Δ.

The taming is *not* required to be flat. The *fundamental form* of Ξ:

$$\Psi_\Xi \stackrel{\text{def.}}{=} \mathcal{D}^{\mathrm{ad}}(\mathcal{J}) = \mathcal{D} \circ \mathcal{J} - (\mathrm{id}_{T^*M} \otimes \mathcal{J}) \circ \mathcal{D} \in \Omega^1(M, \mathrm{End}(\mathcal{S}))$$

measures the failure of $\mathcal{D}$ to preserve $\mathcal{J}$. The electromagnetic structure is called *unitary* if $\Psi_\Xi = 0$.

Definition

Let $\Xi = (\mathcal{S}, \mathcal{J})$ and $\Xi' = (\mathcal{S}', \mathcal{J}')$ be two electromagnetic structures defined on M. A *based isomorphism of electromagnetic structures* from Ξ to Ξ' is a based isomorphism of duality structures $f : \Delta \xrightarrow{\sim} \Delta'$ such that $\mathcal{J}' = f \circ \mathcal{J} \circ f^{-1}$.

Proposition

Let Δ be a holonomy-trivial duality structure of rank $2n$ defined on M. Any choice of global flat symplectic frame identifies the set of tamings of Δ with $\mathfrak{J}_n(M) \simeq \mathrm{Per}_n(M)$.

Let (M, g) be an oriented Lorentzian four-manifold and $\Xi = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$ be an electromagnetic structure of rank $2n$ defined on M. Let $\Delta \stackrel{\text{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$.

Definition

The *$\mathcal{J}$-polarized Hodge operator* $\star_{g,\mathcal{J}} \in \mathrm{Aut}_b(\wedge^*(M, \mathcal{S}))$ is defined through:

$$\star_{g,\mathcal{J}} \stackrel{\text{def.}}{=} *_g \otimes \mathcal{J} = \mathcal{J} \otimes *_g \,.$$

The polarized Hodge operator restricts to an involution of $\wedge^2(M, \mathcal{S})$.

Definition

The space of *field strength configurations* of $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is:

$$\mathrm{Conf}(M, \Delta) \stackrel{\text{def.}}{=} \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S}) = \left\{ \mathcal{V} \in \Omega^2(M, \mathcal{S}) | \mathrm{d}_{\mathcal{D}} \mathcal{V} = 0 \right\} \,.$$

The *Abelian equation of motion* defined by $\Xi = (\Delta, \mathcal{J})$ is:

$$\star_{g,\mathcal{J}} \mathcal{V} = \mathcal{V} \qquad (\text{where } \; \mathcal{V} \in \mathrm{Conf}(M, \Delta)) \;\;.$$

The solutions are called *classical field strengths* and form the vector space:

$$\mathrm{Sol}(M, g, \Xi) = \mathrm{Sol}(M, g, \Delta, \mathcal{J}) \stackrel{\text{def.}}{=} \{ \mathcal{V} \in \mathrm{Conf}(M, \Delta) \,|\, \star_{g,\mathcal{J}} \mathcal{V} = \mathcal{V} \} \,.$$

Definition

An *integral symplectic space* is a triple (V, ω, Λ) such that:

- (V, ω) is a finite-dimensional symplectic vector space over $\mathbb{R}$.
- $\Lambda \subset V$ is full lattice in V, i.e. a lattice in V such that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.
- ω is integral with respect to Λ, i.e. we have $\omega(\Lambda, \Lambda) \subset \mathbb{Z}$.

Define:

$$\mathrm{Div}^n \stackrel{\text{def.}}{=} \{(t_1, \ldots, t_n) \in \mathbb{Z}^n_{>0} \mid t_1 | t_2 | \ldots | t_n\} \ .$$

Definition

An *integral symplectic basis* of a $2n$-dimensional integral symplectic space (V, ω, Λ) is a basis $\mathcal{E} = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ of Λ such that:

$$\omega(\xi_i, \xi_j) = \omega(\zeta_i, \zeta_j) = 0 \,, \qquad \omega(\xi_i, \zeta_j) = -t_i \delta_{ij} \ \ , \ \ \forall \ i, j = 1, \ldots, n \,,$$

where $t \in \mathrm{Div}^n$.

Every integral symplectic space admits integral symplectic bases. The *type* $\mathfrak{t}(V, \omega, \Lambda)$ does not depend on the choice of such a basis. (V, ω, Λ) is called *principal* if:

$$\mathfrak{t}(V, \omega, \Lambda) = \delta(n) \stackrel{\text{def.}}{=} (1, \ldots, 1) \in \mathrm{Div}^n \ \ .$$

Modified Siegel modular groups

Proposition

The type gives a bijection between the set of isomorphism classes of integral symplectic spaces and the set Div^n.

For any $\mathfrak{t} \in \mathrm{Div}^n$, let $\Lambda_\mathfrak{t} \subseteq \mathbb{R}^{2n}$ be the full lattice:

$$\Lambda_\mathfrak{t} \stackrel{\text{def.}}{=} \mathbb{Z}^n \oplus (\oplus_{i=1}^n t_i \mathbb{Z})$$

Then $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_\mathfrak{t})$ is the *standard integral symplectic space of type* $\mathfrak{t}$. Let:

$$\mathrm{Sp}(V, \omega, \Lambda) \stackrel{\text{def.}}{=} \{T \in \mathrm{Sp}(V, \omega) \mid T(\Lambda) = \Lambda\} \ .$$

Definition

The *modified Siegel modular group* of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \stackrel{\text{def.}}{=} \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{2n}, \Lambda) \subset \mathrm{Sp}(2n, \mathbb{R})$$

Div^n is a lattice with bottom $\delta(n)$ under the partial order:

$$(t_1, \ldots, t_n) \leq (t'_1, \ldots, t'_n) \ \text{ iff } \ t_i | t'_i \ \ \forall i = 1, \ldots, n \ .$$

We have $\mathrm{Sp}_{\delta(n)}(2n, \mathbb{Z}) = \mathrm{Sp}(2n, \mathbb{Z})$ and $\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \subseteq \mathrm{Sp}_{\mathfrak{t}'}(2n, \mathbb{Z})$ when $\mathfrak{t} \leq \mathfrak{t}'$. Hence $(\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}))_{t \in \mathrm{Div}^n}$ is a direct system of overgroups of $\mathrm{Sp}(2n, \mathbb{Z})$.

Definition

An *affine torus* is a principal homogeneous space $\mathfrak{A}$ for a torus group A. The *standard affine d-torus* is the affine torus $\mathfrak{A}_d$ given by the right action of $\mathrm{U}(1)^d$ on itself. The *toral affine group* Aff_d is the group $\mathrm{Aff}(\mathfrak{A}_d)$.

Definition

A *special affine symplectic torus* is a pair $\mathbb{A} = (\mathfrak{A}, \Omega)$, where $\mathfrak{A}$ is an even-dimensional affine torus and Ω is a translation-invariant symplectic form on $\mathfrak{A}$ such that $(H_1(\mathfrak{A}, \mathbb{R}), H_1(\mathfrak{A}, \mathbb{Z}), \omega)$ is an integral symplectic space, where $\omega = [\Omega] \in H^2(\mathfrak{A}, \mathbb{R}) \simeq \wedge^2 H_1(\mathfrak{A}, \mathbb{R})^\vee$.

Let $\Omega_\mathfrak{t}$ be the symplectic form induced by ω_{2n} on the torus group $\mathbb{R}^{2n}/\Lambda_\mathfrak{t}$.

Definition

The *standard special symplectic torus group* of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathbf{A}_\mathfrak{t} \stackrel{\text{def.}}{=} \left(\mathbb{R}^{2n}/\Lambda_\mathfrak{t}, \Omega_\mathfrak{t}\right) \quad .$$

The *standard special affine symplectic torus* $\mathbb{A}_\mathfrak{t}$ is obtained from $\mathbf{A}_\mathfrak{t}$ by forgetting the origin.

Proposition

Any special affine symplectic torus $\mathbb{A} = (\mathfrak{A}, \Omega)$ *is affinely symplectomorphic to a unique standard special affine symplectic torus* $\mathbb{A}_{\mathfrak{t}}$*, whose type* $\mathfrak{t}$ *is called the* type *of* $\mathbb{A}$*.*

Definition

The *standard special toral affine group* $\mathrm{Aff}_{\mathfrak{t}}$ of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathrm{Aff}_{\mathfrak{t}} \stackrel{\mathrm{def.}}{=} \mathrm{Aut}(\mathbb{A}_{\mathfrak{t}}) \quad .$$

For any $\mathfrak{t} \in \mathrm{Div}^n$, we have $\mathrm{Aff}_{\mathfrak{t}} = A_n \rtimes \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, where $A_n \simeq \mathrm{U}(1)^{2n}$ is the underlying torus group of $\mathbb{A}_{\mathfrak{t}}$.

Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be a duality structure defined on M.

Definition

A *Dirac system* for Δ is a smooth fiber sub-bundle $\mathcal{L} \subset \mathcal{S}$ of full symplectic lattices in $(\mathcal{S}, \omega)$ which is preserved by the parallel transport of $\mathcal{D}$. A pair $\mathbf{\Delta} \stackrel{\text{def.}}{=} (\Delta, \mathcal{L})$ consisting of a duality structure Δ and a choice of Dirac system $\mathcal{L}$ for Δ is called an *integral duality structure*. A duality structure Δ of rank $2n$ is called *semiclassical* if it admits a Dirac system, i.e. if its holonomy group can be conjugated to lie inside $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ for some $\mathfrak{t} \in \mathrm{Div}^n$.

Definition

The *type* $\mathfrak{t}_{\mathbf{\Delta}} \in \mathrm{Div}^n$ of an integral duality structure $\mathbf{\Delta} = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ is the common type of the integral symplectic spaces $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$, where $m \in M$.

Proposition

For any $m \in M$, the set of isomorphism classes of integral duality structures of type $\mathfrak{t}$ defined on M is in bijection with the character variety:

$$\mathfrak{R}_{\mathfrak{t}}(\pi_1(M, m), \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})) \stackrel{\text{def.}}{=} \mathrm{Hom}(\pi_1(M, m), \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}))/\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \quad .$$

Definition

A *Siegel system* of rank $2n$ is a local system of free Abelian groups of rank $2n$ defined on M whose structure group reduces to a subgroup of $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ for some $\mathfrak{t} \in \mathrm{Div}^n$. The smallest $\mathfrak{t}$ with this property is called the *type* $\mathfrak{t}_Z$ of Z.

Proposition

Let Z be a bundle of free Abelian groups of rank $2n$ defined on M. T.a.e.:

- (a) *Z is a Siegel system of type $\mathfrak{t}$ defined on M.*
- (b) *The vector bundle $\mathcal{S} \stackrel{\mathrm{def.}}{=} Z \otimes_{\mathbb{Z}} \mathbb{R}$ admits a symplectic pairing ω which makes $(\mathcal{S}, \omega, \mathcal{D}, Z)$ into an integral duality structure of type $\mathfrak{t}$, where $\mathcal{D}$ is the flat connection induced from Z.*

Let $\ell_{\mathfrak{t}} : \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{2n})$ be the left action of $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ on $\mathbb{Z}^{2n}$. For any principal $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$-bundle Q defined on M, let $Z(Q) \stackrel{\mathrm{def.}}{=} Q \times_{\ell_{\mathfrak{t}}} \mathbb{Z}^{2n}$. For any Siegel system Z, let $\mathrm{Fr}(Z)$ be its principal bundle of frames.

Proposition

The correspondences $Q \mapsto Z(Q)$ and $Z \mapsto \mathrm{Fr}(Z)$ extend to quasi-inverse equivalences between the groupoids of principal $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$-bundles and Siegel systems of type $\mathfrak{t}$ defined on M.

The lattice of charges and the Dirac integrality condition

Let $\mathbf{\Delta} = (\Delta, Z)$ be an integral duality structure on M, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. Consider the flat bundle of torus groups $\mathcal{A} \stackrel{\mathrm{def.}}{=} \mathcal{S}/Z$. The exact sequence of sheaves of Abelian groups:

$$1 \to \mathcal{C}(Z) \xrightarrow{j} \mathcal{C}^\infty_{\mathrm{flat}}(\mathcal{S}) \xrightarrow{\exp} \mathcal{C}^\infty_{\mathrm{flat}}(\mathcal{A}) \to 1$$

induces a long exact sequence in sheaf cohomology:

$$\ldots \to H^1(M, \mathcal{A}_{\mathrm{disc}}) \xrightarrow{\delta_1} H^2(M, Z) \xrightarrow{j_*} H^2_{\mathcal{D}}(M, \mathcal{S}) \xrightarrow{\exp_*} H^2(M, \mathcal{A}_{\mathrm{disc}}) \to \ldots$$

where δ_1 is the connecting morphism.

Definition

The *lattice of charges* of the integral duality structure $\mathbf{\Delta}$ is:

$$L_{\mathbf{\Delta}} \stackrel{\mathrm{def.}}{=} j_*(H^2(M, Z)) \subset H^2_{\mathcal{D}}(M, \mathcal{S})$$

A field strength configuration $\mathcal{V} \in \mathrm{Conf}(M, \Delta) = \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-}\mathrm{cl}}(M, \mathcal{S})$ is called *integral* if $[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}}$.

The condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}}$ is the Dirac integrality condition.

Definition

A *Siegel bundle* P of type $\mathfrak{t} \in \mathrm{Div}^n$ is a principal bundle on M with structure group $\mathrm{Aff}_\mathfrak{t}$. An *isomorphism of Siegel bundles* is a based isomorphism of principal bundles.

Notice that $G \stackrel{\mathrm{def.}}{=} \mathrm{Aff}_\mathfrak{t}$ is a split weakly Abelian Lie group:

$$G \simeq A \rtimes_{\rho_\mathfrak{t}} \Gamma_\mathfrak{t} \quad \text{where} \quad A = \mathbb{R}^{2n}/\Lambda_\mathfrak{t} \simeq \mathrm{U}(1)^{2n} \quad \text{and} \quad \Gamma_\mathfrak{t} = \mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \quad ,$$

with fundamental lattice $\Lambda_\mathfrak{t} \simeq \pi_1(G) \simeq \mathbb{Z}^{2n}$. Notice that $\rho_{\mathfrak{t},0} = \ell_\mathfrak{t}$. Hence Siegel bundles of type $\mathfrak{t}$ are classified up to isomorphism by their remnant bundle $\Gamma_\mathfrak{t}(P)$ and their twisted Chern class $c(P) \in H^2(M, \Lambda_\mathfrak{t}(P))$. In this case:

- The local system $\Lambda_\mathfrak{t}(P) \stackrel{\mathrm{def.}}{=} P \times_{\mathrm{Ad}_0} \Lambda_\mathfrak{t}$ is a Siegel system of type $\mathfrak{t}$, which we denote by $Z(P)$.
- $\mathfrak{g} = \mathbb{R}^{2n}$ and $\mathrm{ad}(P) = Z(P) \otimes_\mathbb{Z} \underline{\mathbb{R}} \stackrel{\mathrm{def.}}{=} \mathcal{S}$ supports the duality structure $(\mathcal{S}, \omega, \mathcal{D})$, induced by $Z(P)$.
- $A(P) = P \rtimes_{\mathrm{Ad}_G^A} A$ coincides with the bundle of symplectic torus groups $\mathcal{A}(P) \stackrel{\mathrm{def.}}{=} \mathcal{S}/Z(P)$.
- The characteristic lattice $L(P) = L_0(P) \in H^2(M, \mathrm{ad}(P)) = H^2(M, \mathcal{S})$ of P coincides with the lattice of charges $L_{\mathbf{\Delta}_P}$ of $\mathbf{\Delta}_P \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D}, Z(P))$.

Let $\Delta_P \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$.

Siegel bundles

Since $\mathrm{Aff}_{\mathfrak{t}}$ is a split weakly Abelian Lie group, any Siegel bundle P has an *integral twisted Chern class* $c(P) \in H^2(M, Z(P))$.

Theorem

Consider the set:

$$\Sigma(M) \stackrel{\mathrm{def.}}{=} \Big\{(Z, c) \,|\, Z \text{ is a Siegel system on } M \ \& \ c \in H^2(M, Z)\Big\} /_\sim ,$$

where $(Z, c) \sim (Z', c')$ iff there exists an isomorphism $\varphi : Z \xrightarrow{\sim} Z'$ s.t. $\varphi_*(c) = c'$. Then the map $P \mapsto (Z(P), c(P))$ induces a bijection between the set of isomorphism classes of Siegel bundles defined on M and the set $\Sigma(M)$.

Let $\mathfrak{c}(P) \in H^2_{\mathcal{D}}(M, \mathrm{ad}(P)) = H^2_{\mathcal{D}}(M, \mathcal{S})$ be the twisted *real* Chern class of P.

Proposition

We have $\mathfrak{c}(P) = j_*(c(P))$, where $j_* : H^*(M, Z) \to H^*(M, \mathcal{S})$ is induced by the inclusion $Z = \Lambda(P) \hookrightarrow \mathcal{S} = \mathrm{ad}(P)$.

Since $\mathfrak{c}(P) \in L(P) = L_{\boldsymbol{\Delta}}$, it follows that the adjoint curvature $\mathcal{V}_{\mathcal{A}}$ of any principal connection $\mathcal{A} \in \mathrm{Conn}(P)$ satisfies the Dirac integrality condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\boldsymbol{\Delta}}$ of the integral duality structure $\boldsymbol{\Delta}_P$ determined by P.

Theorem

Let Z be a Siegel system on M and $\boldsymbol{\Delta} = (\Delta, Z)$ be its integral duality structure, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. For any $\mathfrak{c} \in L_{\boldsymbol{\Delta}}$, there exists a Siegel bundle P on M s.t. $\boldsymbol{\Delta}_P = \boldsymbol{\Delta}$ and $\mathfrak{c} = \mathfrak{c}(P)$. Thus any $\mathcal{V} \in \mathrm{Conf}(M, \Delta) = \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S})$ which satisfies the Dirac integrality condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\boldsymbol{\Delta}}$ coincides with the adjoint curvature $\mathcal{V}_{\mathcal{A}}$ of some principal connection $\mathcal{A} \in \mathrm{Conn}(P)$.

Definition

A *polarized Siegel bundle* is a pair $\mathbf{P} = (P, \mathcal{J})$, where P is a Siegel bundle and $\mathcal{J}$ is a taming of the duality structure Δ_P defined by P.

Definition

Let $\mathbf{P} = (P, \mathcal{J})$ be a polarized Siegel bundle on M. The space of *gauge configurations* defined by P on M is the affine space:

$$\mathfrak{Conf}(M, P) \stackrel{\mathrm{def.}}{=} \mathrm{Conn}(P) \quad .$$

The *Abelian gauge theory* defined by $\mathbf{P}$ on (M, g) has e.o.m:

$$\star_{g,\mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \quad \text{where} \quad \mathcal{A} \in \mathrm{Conn}(P) \, .$$

Let (M, g) be a *simply-connected* (hence orientable) Lorentzian four-manifold and fix an orientation of M.

Any duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ of rank $2n$ on M is holonomy-trivial. Hence any choice of flat global symplectic frame:

- identifies Δ with the standard trivial duality structure $\Delta_n = (\underline{\mathbb{R}}^{2n}, \omega_{2n}, \mathrm{d})$;
- identifies any taming $\mathbb{J}$ of Δ with a taming map $\mathcal{J} \in \mathfrak{J}_n(M)$ of size $2n$ and hence with a matrix period map $\mathcal{N} \in \mathrm{Per}_n(M)$ of size n.

The electromagnetic structure $\Xi = (\Delta, \mathbb{J})$ is unitary iff $\mathbb{J}$ is $\mathcal{D}$-flat, i.e. iff the taming map $\mathcal{J}$ (equivalently, the period map $\mathcal{N}$) is constant on M.

Classical electrodynamics on (M, g) has $n = 1$ and corresponds to a *unitary* electromagnetic structure $\Xi = (\Delta_2, \mathbb{J})$ of rank two defined on M. Thus $\mathbb{J}$ corresponds to a *constant* taming map $\mathcal{J} \in \mathfrak{J}_2(M)$ of size two, i.e. a taming of the standard symplectic plane $(\mathbb{R}^2, \omega_2)$. In turn, this corresponds to a *constant* period map $\mathcal{N} \in \mathrm{Per}_1(M) = \mathbb{SH}_1$ of size one. The latter is simply an element of the upper half-plane $\mathbb{SH}_1 = \mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ which is traditionally denoted by τ and written in the form:

$$\tau = \mathcal{N} = \frac{4\pi}{g^2} + \mathbf{i}\frac{\theta}{2\pi} \quad , \quad \text{i.e. } \mathcal{R} = \frac{4\pi}{g^2} \text{ and } \mathcal{I} = \frac{\theta}{2\pi} \quad .$$

The real constants g and θ are called the *coupling constant* and *theta angle*.

Electrodynamics on simply-connected Lorentzian four-manifolds

The corresponding constant taming is:

$$\mathcal{J} = \begin{pmatrix} -\frac{g^2\theta}{8\pi^2} & \frac{g^2}{4\pi} \\ -\frac{4\pi}{g^2} - \frac{g^2\theta^2}{16\pi^3} & \frac{g^2\theta}{8\pi^2} \end{pmatrix} \quad .$$

The space $\mathrm{Conf}(M, \Delta_2)$ identifies with $\Omega^2_{\mathrm{cl}}(M, \mathbb{R}^2) = \Omega^2_{\mathrm{cl}}(M)^{\oplus 2}$.

Proposition

A size two vector-valued 2-form $\mathcal{V} \in \Omega^2(M, \mathbb{R}^2)$ satisfies $*_{g,\mathcal{J}}\mathcal{V} = \mathcal{V}$ iff it can be written as:

$$\mathcal{V} = \begin{pmatrix} F \\ \mathcal{R}\, F - \mathcal{I} *_g F \end{pmatrix} \tag{1}$$

for some real-valued 2-form $F \in \Omega^2(M)$ (which is uniquely determined by $\mathcal{V}$). Moreover, $\mathcal{V}$ is closed iff F satisfies the source-free Maxwell equations*:*

$$\mathrm{d}F = \mathrm{d} *_g F = 0 \quad . \tag{2}$$

Thus $\mathrm{Sol}(M, g, \Xi)$ identifies with the space of field strength solutions of (source-free) classical electrodynamics.

- Any principal bundle with discrete structure group defined on M is trivial since $\pi_1(M) = 0$. In particular, any Siegel system Z of rank 2 defined on M is isomorphic with the constant local system $\underline{\mathbb{Z}}^2$.
- The modified Siegel modular group of dimension 2 and type $t \in \mathrm{Div}^1 = \mathbb{Z}_{>0}$ is $\mathrm{Sp}_t(2,\mathbb{Z}) = \mathrm{Sp}(\mathbb{R}^2, \mathbb{Z} \oplus t\mathbb{Z}, \omega_2)$. This is an overgroup of the usual modular group $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{Sp}(\mathbb{R}^2, \mathbb{Z}^2, \omega_2) = \mathrm{SL}(2,\mathbb{Z})$, since $\mathbb{Z} \oplus t\mathbb{Z} \subset \mathbb{Z}^2$.
- Siegel bundles P of type $t \in \mathbb{Z}_{>0}$ defined on M are classified up to isomorphism by their integral twisted Chern class $c(P) \in H^2(M, \underline{\mathbb{Z}}^2) = H^2(M,\mathbb{Z})^{\oplus 2}$. They have trivial remnant and adjoint bundles:

$$\Gamma(P) \simeq \underline{\Gamma} = \underline{\mathrm{Sp}_t(2,\mathbb{Z})} \ , \quad \mathrm{ad}(P) \simeq \underline{\mathbb{R}}^2 \ .$$

The adjoint curvature $\mathcal{V}_\mathcal{A} \in H^2(M, \mathbb{R}^2)$ of a principal connection $\mathcal{A} \in \mathrm{Conn}(P)$ is polarized self-dual iff it has the form (1) for some solution $F \in \Omega^2(M)$ of the source-free Maxwell equations (2). Notice that:

Remark

- The source-free Maxwell equations (2) amount to the Bianchi identity $\mathrm{d}\mathcal{V}_\mathcal{A} = 0$, which holds automatically.
- Relation (1) (which reduces $\mathcal{V}_\mathcal{A}$ to F) amounts to the twisted self-duality condition $*_{g,J}\mathcal{V}_\mathcal{A} = \mathcal{V}_\mathcal{A}$, which is the equation of motion for $\mathcal{A}$.

The lattice $\Lambda_t = \mathbb{Z} \oplus t\mathbb{Z}$ is the usual Dirac lattice. The traditional choice in electromagnetism is $t = 1$.