Connections on principal bundles with weakly Abelian structure group

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Outline

1. Recap on principal connections
2. The curvature maps and twisted Chern class for weakly Abelian structure group
3. The gauge group of a principal bundle with weakly Abelian structure group
4. Universal Chern-Weyl theory for weakly Abelian Lie groups

Let:

- G be a Lie group with Lie algebra $\mathfrak{g}$
- $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ be the adjoint representation of G.
- $p : P \to M$ a principal G-bundle on a manifold M
- $VP \subset TP$ be the vertical bundle of P.
- $\mathrm{ad}(P) \stackrel{\text{def.}}{=} P \times_{\mathrm{Ad}} \mathfrak{g}$ be the adjoint bundle of P.

The space of equivariant $\mathfrak{g}$-valued forms defined on P:

$$\Omega^*(P, \mathfrak{g})^G \stackrel{\text{def.}}{=} \{\eta \in \Omega^*(P, \mathfrak{g}) \,|\, r_g^*(\eta) = \mathrm{Ad}(g)^{-1}\eta\}$$

contains the subspace of horizontal forms:

$$\Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \stackrel{\text{def.}}{=} \{\eta \in \Omega^*(P, \mathfrak{g})^G \,|\; \iota_X \eta = 0 \;\; \forall X \in \mathcal{C}^\infty(P, VP)\} \quad .$$

We have inverse isomorphisms of graded vector spaces:

$$\Omega^*(M, \mathrm{ad}(P)) \underset{\varphi_P}{\overset{p^*}{\rightleftarrows}} \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \quad .$$

Principal connections on P form an affine space modeled on $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$:

$$\mathrm{Conn}(P) \stackrel{\text{def.}}{=} \left\{\mathcal{A} \in \Omega^1(P, \mathfrak{g})^G \,|\; \iota_{X^v}\mathcal{A} = v \;\; \forall p \in P \;\; \forall v \in \mathfrak{g}\right\} \quad ,$$

where $X^v \in \mathcal{C}^\infty(P, VP)$ is the vertical vector field defined by $v \in \mathfrak{g}$.

Let $\mathrm{d}_{\mathcal{A}} : \Omega^*(P, \mathfrak{g}) \to \Omega^*(P, \mathfrak{g})[1]$ be the covariant differential of $\mathcal{A} \in \mathrm{Conn}(P)$.

Definition

The *principal curvature* of $\mathcal{A}$ is:

$$\Omega_{\mathcal{A}} \stackrel{\mathrm{def.}}{=} \mathrm{d}_{\mathcal{A}}\mathcal{A} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]_\wedge \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g}) \quad ,$$

The *adjoint curvature of* $\mathcal{A}$ is:

$$\mathcal{V}_{\mathcal{A}} \stackrel{\mathrm{def.}}{=} \varphi_P(\Omega_{\mathcal{A}}) \in \Omega^2(M, \mathrm{ad}(P))$$

The principal curvature satisfies the Bianchi identity:

$$\mathrm{d}_{\mathcal{A}}\Omega_{\mathcal{A}} = 0 \quad .$$

The principal and adjoint *curvature maps* $\Omega : \mathrm{Conn}(P) \to \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$ and $\mathcal{V} : \mathrm{Conn}(P) \to \Omega^2(M, \mathrm{ad}(P))$ are defined through:

$$\Omega(\mathcal{A}) \stackrel{\mathrm{def.}}{=} \Omega_{\mathcal{A}} \quad , \quad \mathcal{V}(\mathcal{A}) \stackrel{\mathrm{def.}}{=} \mathcal{V}_{\mathcal{A}} \quad \forall \mathcal{A} \in \mathrm{Conn}(P) \quad .$$

We have a commutative diagram:

$$\begin{tikzcd} \mathrm{Conn}(P) \arrow[r, "\Omega"] \arrow[dr, "\mathcal{V}"] & \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g}) \arrow[d, "\varphi_P"] \\ & \Omega^2(M, \mathrm{ad}(P)) \end{tikzcd}$$

Let $\mathcal{D}_{\mathcal{A}} : \Gamma(M, \mathrm{ad}(P)) \to \Omega^1(M, \mathrm{ad}(P))$ be the connection induced by $\mathcal{A}$ on $\mathrm{ad}(P)$ and $\mathrm{d}_{\mathcal{D}} : \Omega^*(M, \mathrm{ad}(P)) \to \Omega^*(M, \mathrm{ad}(P))[1]$ its differential. We have a commutative diagram:

$$\begin{tikzcd} \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \arrow[r, "\mathrm{d}_{\mathcal{A}}"] \arrow[d, "\varphi_P"] & \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \arrow[d, "\varphi_P"] \\ \Omega^*(M, \mathrm{ad}(P)) \arrow[r, "\mathrm{d}_{\mathcal{D}_A}"] & \Omega^*(M, \mathrm{ad}(P)) \end{tikzcd}$$

Suppose that G is a weakly Abelian Lie group.

Proposition

The following statements hold:

1. *For any* $\mathcal{A} \in \mathrm{Conn}(P)$, *we have* $\Omega_{\mathcal{A}} = \mathrm{d}\mathcal{A}$ *and the Bianchi identity reduces to* $\mathrm{d}\Omega_{\mathcal{A}} = 0$. *Thus* Ω *is an affine map with linear part:*

$$\mathrm{d}|_{\mathrm{Conn}(P)} : \mathrm{Conn}(P) \to \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$$

2. *We have:*

$$\mathrm{d}_{\mathcal{A}}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})} = \mathrm{d}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})} : \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \to \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g})[1] \qquad (1)$$

3. *All principal connections* $\mathcal{A} \in \mathrm{Conn}(P)$ *induce the same linear connection* $\mathcal{D}_{\mathcal{A}}$ *on the adjoint bundle* $\mathrm{ad}(P)$ *(which we denote by* $\mathcal{D}$*) and this induced connection is flat. Moreover, the adjoint curvature satisfies:*

$$\mathrm{d}_{\mathcal{D}}\mathcal{V}_{\mathcal{A}} = 0 \quad \forall \mathcal{A} \in \mathrm{Conn}(P)$$

   and $\varphi_P : (\Omega_{\mathrm{Ad}}(P, \mathfrak{g}), \mathrm{d}) \to (\Omega(M, \mathrm{ad}(P)), \mathrm{d}_{\mathcal{D}})$ *is an isomorphism of complexes.*

4. $\mathcal{D}$ *coincides with the flat connection induced on* $\mathrm{ad}(P) = \Gamma(P) \times_{\overline{\rho}} \mathfrak{g}$ *by the flat connection of the discrete remnant bundle* $\Gamma(P) \stackrel{\mathrm{def.}}{=} P \times_q \Gamma$.

Proposition

The twisted cohomology class of $\mathcal{V}_\mathcal{A}$:

$$\mathfrak{c}(P) \stackrel{\text{def.}}{=} [\mathcal{V}_\mathcal{A}]_\mathcal{D} = \mathcal{V}_\mathcal{A} + \Omega^2_{\mathrm{d}_\mathcal{D}\text{-ex}}(M, \mathrm{ad}(P)) \in H^2_\mathcal{D}(M, \mathrm{ad}(P))$$

does not depend on the choice of principal connection $\mathcal{A} \in \mathrm{Conn}(P)$. Viewing $\mathfrak{c}(P)$ as an affine space modeled on the vector space $\Omega^2_{\mathrm{d}_\mathcal{D}\text{-ex}}(M, \mathrm{ad}(P))$, the corestricted adjoint curvature map $\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ is a surjective affine map with linear part given by:

$$\mathrm{d}_\mathcal{D} \circ \varphi_P|_{\Omega^1(P,\mathfrak{g})} = \varphi_P \circ \mathrm{d}|_{\Omega^1(P,\mathfrak{g})} : \Omega^1(P, \mathfrak{g}) \to \Omega^2_{\mathrm{d}_\mathcal{D}\text{-ex}}(M, \mathrm{ad}(P)) \quad .$$

Corollary

$\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ is an affine fibration with fiber at $\omega \in \mathfrak{c}(P)$ given by:

$$\mathrm{Conn}_\omega(P) \stackrel{\text{def.}}{=} \{\mathcal{A} \in \mathrm{Conn}(P) \,|\, \mathcal{V}_\mathcal{A} = \omega\} \quad , \tag{2}$$

which is an affine space modeled on the vector space:

$$\Omega^1_{\mathrm{Ad,cl}}(P, \mathfrak{g}) \stackrel{\text{def.}}{=} \ker(\mathrm{d} : \Omega^1_{\mathrm{Ad}}(M, \mathfrak{g}) \to \Omega^2_{\mathrm{Ad}}(M, \mathfrak{g})) \stackrel{\varphi_P}{\simeq} \Omega^1_{\mathrm{d}_\mathcal{D}\text{-cl}}(M, \mathrm{ad}(P)) \quad .$$

Definition

The *gauge group* of P is the group $\mathrm{Aut}_b(P)$ of based automorphisms of P, whose elements are called *(global) gauge transformations* of P.

Let $\mathrm{Aut}_b(\mathrm{ad}(P))$ be the group of based automorphisms of $\mathrm{ad}(P)$.

Definition

The *adjoint representation* of $\mathrm{Aut}_b(P)$ is the linear representation induced on global sections of $\mathrm{ad}(P)$ by the morphism of groups $\mathrm{ad}_P : \mathrm{Aut}_b(P) \to \mathrm{Aut}_b(\mathrm{ad}(P))$ defined through:

$$\mathrm{ad}_P(\psi)([p, v]) \stackrel{\text{def.}}{=} [\psi(p), v] \quad \forall \psi \in \mathrm{Aut}_b(P) \quad \forall p \in P \quad \forall v \in \mathfrak{g} \quad .$$

The *pullback representation* of $\mathrm{Aut}_b(P)$ is the linear representation $\mathfrak{R} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^*(P, \mathfrak{g}))$ defined through:

$$\mathfrak{R}(\psi)(\omega) \stackrel{\text{def.}}{=} (\psi^{-1})^*(\omega) \quad \forall \psi \in \mathrm{Aut}_b(P) \quad \forall \omega \in \Omega^*(P, \mathfrak{g}) \quad .$$

Remark. Suppose that M is compact. Then $\mathrm{Aut}_b(P)$ is an infinite-dimensional Fréchet-Lie groups whose Lie algebra identifies with $\mathcal{C}^\infty(M, \mathrm{ad}(P))$. In this case, the linear action induced by ad_P on $\mathcal{C}^\infty(M, \mathrm{Ad}_G(P))$ identifies with the adjoint representation of $\mathrm{Aut}_b(P)$ as a Lie group (hence our terminology).

The pullback representation preserves $\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})$, on which it restricts to a representation $\mathfrak{R}_{\mathrm{Ad}} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}))$.

Proposition

The following diagram commutes:

$$\begin{tikzcd} \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) \arrow[r, "\mathcal{R}_{\mathrm{Ad}}(\psi)"] \arrow[d, "\varphi_P"] & \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) \arrow[d, "\varphi_P"] \\ \Omega^*(M,\mathrm{ad}(P)) \arrow[r, "\mathrm{ad}_P(\psi)"] & \Omega^*(M,\mathrm{ad}(P)) \end{tikzcd}$$

Proposition

For any $\psi \in \mathrm{Aut}_b(P)$, we have:

$$\mathrm{d}\circ\mathfrak{R}_{\mathrm{Ad}}(\psi) = \mathfrak{R}_{\mathrm{Ad}}(\psi)\circ \mathrm{d}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})}\quad,\quad \mathrm{d}_{\mathcal{D}}\circ\mathrm{ad}_P(\psi) = \mathrm{ad}_P(\psi)\circ\mathrm{d}_{\mathcal{D}}\quad.$$

Thus $\mathfrak{R}_{\mathrm{Ad}}$ and ad_P induce linear representations of the gauge group on the spaces $H^*_{\mathrm{d}}(\Omega_{\mathrm{Ad}}(P,\mathfrak{g}))$ and $H^*_{\mathrm{d}_{\mathcal{D}}}(M,\mathrm{ad}(P))$, which are equivalent through the isomorphism $\varphi_{P*} : H^*_{\mathrm{d}}(\Omega_{\mathrm{Ad}}(P,\mathfrak{g})) \xrightarrow{\sim} H^*_{\mathrm{d}_{\mathcal{D}}}(M,\mathrm{ad}(P))$ induced by φ_P.

The pull-back action preserves the affine space $\mathrm{Conn}(P) \subset \Omega^1(P,\mathfrak{g})^G$, on which it restricts to an affine action $\mathfrak{R}_c : \mathrm{Aut}_b(P) \to \mathrm{Aff}(\mathrm{Conn}(P))$ with linear part:

$$\mathfrak{R}^1_{\mathrm{Ad}} \stackrel{\mathrm{def.}}{=} \mathfrak{R}_{\mathrm{Ad}}|_{\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})) \quad .$$

Proposition

The principal and adjoint curvature maps of P are gauge-equivariant:

$$\Omega \circ \mathfrak{R}_c(\psi) = \mathfrak{R}_{\mathrm{Ad}}(\psi) \circ \Omega \quad \text{and} \quad \mathcal{V} \circ \mathfrak{R}_c(\psi) = \mathrm{ad}_P(\psi) \circ \mathcal{V} \quad \forall \psi \in \mathrm{Aut}_b(P) \quad .$$

Moreover, ad_P preserves the affine subspace $\mathfrak{c}(P) \subset \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M,\mathrm{ad}(P))$, on which it acts through affine transformations with linear part:

$$\mathrm{ad}_P(\psi)|_{\Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M,\mathrm{ad}(P))} : \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M,\mathrm{ad}(P)) \to \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M,\mathrm{ad}(P)) \;\; \forall \psi \in \mathrm{Aut}_b(P) \; .$$

In particular, the affine fibration $\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ is equivariant with respect to the affine actions of $\mathrm{Aut}_b(P)$ on $\mathrm{Conn}(P)$ and $\mathfrak{c}(P)$.

Discrete gauge transformations

Recall that the *discrete remnant* of P is the principal Γ-bundle $\Gamma(P) \stackrel{\text{def.}}{=} P \times_q \Gamma$. This comes with a (G, q)-lift of structure group $\Phi_P : P \to \Gamma(P)$.

Definition

The group $\mathrm{Aut}_b(\Gamma(P))$ is called the *discrete gauge group* of P and its elements are called *discrete gauge transformations* of P.

Any $\psi \in \mathrm{Aut}_b(P)$ induces an automorphism $Q_P(\psi) \stackrel{\text{def.}}{=} \bar{\psi} \in \mathrm{Aut}_b(\Gamma(P))$ by:

$$\bar{\psi}([p, \gamma]) = [\psi(p), \gamma] \quad \forall [p, \gamma] \in \Gamma(P) \ \ .$$

This fits into a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\Phi_P"] \arrow[d, "\psi"] & \Gamma(P) \arrow[d, "\bar{\psi}"] \\ P \arrow[r, "\Phi_P"] & \Gamma(P) \end{tikzcd}$$

The map $Q_P : \mathrm{Aut}_b(P) \to \mathrm{Aut}_b(\Gamma(P))$ is a morphism of groups.

Definition

The discrete gauge transformation $Q_P(\psi) = \bar{\psi} \in \mathrm{Aut}_b(\Gamma(P))$ is called the *discrete remnant* of the gauge transformation $\psi \in \mathrm{Aut}_b(P)$.

Let $\mathrm{ad}_{\Gamma(P)} : \mathrm{Aut}_b(\Gamma(P)) \to \mathrm{Aut}_b(\mathrm{ad}(P))$ be the morphism of groups given by:

$$\mathrm{ad}_{\Gamma(P)}(\chi)([p,v]_{\mathrm{Ad}}) \stackrel{\mathrm{def.}}{=} [\chi(\Phi_P(p)),v]_{\overline{\rho}} = [p,\overline{\rho}(h_\chi(p))(v)]_{\mathrm{Ad}} \ \forall p \in P \ \forall v \in \mathfrak{g}$$

with $\chi \in \mathrm{Aut}_b(\Gamma(P))$, where $\overline{\rho} : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is the reduced adjoint representation of G and we used the presentation $\mathrm{ad}(P) = \Gamma(P) \times_{\overline{\rho}} \Gamma$.

Proposition

We have $\mathrm{ad}_P = \mathrm{ad}_{\Gamma(P)} \circ Q_P$, *i.e.:*

$$\mathrm{ad}_P(\psi) = \mathrm{ad}_{\Gamma(P)}(\bar{\psi}) \quad \forall \psi \in \mathrm{Aut}_b(P) \quad .$$

Hence $\mathrm{ad}_P(\psi)$ *depends only on the discrete remnant of* ψ.

Let NG be the nerve of G (the nerve of the one-object groupoid defined by G):

- $N_n G = G^{\times n} \quad \forall n \geq 1 \quad , \quad N_0 G = \{1_G\}$
- face maps $\epsilon_i := \epsilon_i^n : N_n G \to N_{n-1} G$ $(n \geq 1)$ and degeneracy maps $\eta^i := \eta_i^n : N_n G \to N_{n+1} G$ $(n \geq 0)$ given by:

$$\epsilon_0^1(g) = \epsilon_1^1(g) = 1_G \quad , \quad \eta_0^0(1_G) = 1_G$$

$$\epsilon_i^n(g_1, \ldots, g_n) \overset{\text{def.}}{=} \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

$$\eta_i^n(g_1, \ldots, g_n) \overset{\text{def.}}{=} \begin{cases} (1_G, g_1, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_{i-1}, 1_G, g_i, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_n, 1_G) & i = n \end{cases}$$

for all $n \geq 1$.

Let $\| \ \| : \mathrm{sTop} \to \mathrm{Top}$ be the fat realization functor, where sTop is the category of simplicial spaces and maps thereof. Then $\|NG\|$ is homotopy-equivalent with $\mathrm{B}G$. Notice that the fat model $\|NG\|$ of $\mathrm{B}G$ differs up to homotopy from the Segal model (which uses the thin realization functor $| \ |$) and from the Milnor model (which uses the join construction).

Definition

The *simplicial de Rham bicomplex* $\Omega(NG)$ has components $\Omega^{p,q}(NG) \stackrel{\text{def.}}{=} \Omega^q(N_pG)$ and differentials:

$$\delta' = \sum_{i=0}^{p+1}(-1)^i(\epsilon_i^{p+1})^* : \Omega^{p,q}(NG) \to \Omega^{p+1,q}(NG)$$

$$\delta'' = (-1)^p \mathrm{d}_{N_pG} : \Omega^{p,q}(NG) \to \Omega^{p,q+1}(NG) \quad .$$

Let $H^*(\Omega(NG))$ be the total cohomology of this bicomplex, which is a graded ring under the obvious operation:

$$\mathring{\wedge} : \Omega^{k_1}(N_{q_1}G) \times \Omega^{k_2}(N_{q_2}G) \to \Omega^{k_1+k_2}(N_{q_1+q_2}G) \quad .$$

$H^*(\Omega(NG))$ is computed by the spectral sequence of the vertical filtration $\Omega(NG)_q \stackrel{\text{def.}}{=} \oplus_{j\geq q} \oplus_{i\geq 0} \Omega^{i,j}(NG)$. The first page is $E_1^{p,q} = H^q_{\delta'}(\Omega^{*,p}(NG)) = H^q_{\delta'}(\Omega^p(N_*G))$ with differential $\delta_1 = \delta'' : E_1^{p,q} \to E_1^{p+1,q}$, while the second page is $E_2^{p,q} = H^p_{\delta''}(H^q_{\delta'}(\Omega(NG)))$.

Theorem (Bott, Shulman, Stasheff)

There exists an isomorphism of graded rings $\zeta : H^*(\Omega(NG)) \xrightarrow{\sim} H^*(BG, \mathbb{R})$ *and isomorphisms of vector spaces:*

$$\beta_{p-q,q} : H^{p-q}(G, S^q(\mathfrak{g}^*)) \xrightarrow{\sim} H^p_{\delta'}(\Omega^q(NG)) = E_1^{q,p} \quad \forall p \geq q \quad .$$

Moreover, we have $E_1^{q,p} = 0$ *for* $p < q$.

Since $\delta_1|_{E_1^{q,q}} = 0$, we have epimorphisms $E_1^{q,q} \to E_2^{q,q} \to E_3^{q,q} \to \ldots$ and an edge morphism $e_q : E_1^{q,q} \to E_\infty^{q,q} \subset H^{2q}(\Omega(NG))$. Since $H^0(G, S^q(\mathfrak{g}^*)) = S^q(\mathfrak{g}^*)^G = S^q(\mathfrak{g}^*)^\Gamma$, we have $\beta_{0,q} : S^q(\mathfrak{g}^*)^\Gamma \xrightarrow{\sim} E_1^{q,q}$.

Definition

The *simplicial* and *universal* Chern-Weil morphisms of G are the morphism of graded rings:

$$\beta \stackrel{\text{def.}}{=} \oplus_{q\geq 0} e_q \circ \beta_{0,q} : S^*(\mathfrak{g}[2]^\vee)^\Gamma \xrightarrow{\sim} H^{\mathrm{even}}(\Omega(NG)) \quad .$$

and:

$$\psi \stackrel{\text{def.}}{=} \zeta \circ \beta : S^*(\mathfrak{g}[2]^\vee)^\Gamma \to H^{\mathrm{even}}(BG) \quad .$$

Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the left Maurer-Cartan form of G, which is closed by the MC equation. For any $q \geq 1$ and $i = 1, \ldots, q$, let $\theta_i^{(q)} \stackrel{\text{def.}}{=} (\pi_i^q)^*(\theta) \in \Omega^1(N_q G)$, where $\pi_i^q : N_q G = G^{\times q} \to G$ is the i-th projection.

Proposition

For any $T \in S^q(\mathfrak{g}^*)^\Gamma$, the form $T(\theta_1^{(q)} \mathring{\wedge} \ldots \mathring{\wedge} \theta_q^{(q)}) \in \Omega^q(N_q G)$ is δ-closed and we have:

$$\beta(T) = [T(\theta_1^{(q)} \mathring{\wedge} \ldots \mathring{\wedge} \theta_q^{(q)})]_\delta \in H^{\mathrm{even}}(\Omega(NG)) \quad .$$

Theorem (Cartan, Bott)

Suppose that G is compact. Then the following statements hold:

- $H^p(G, S^q(\mathfrak{g}^*)) = 0$ *for* $p > 0$
- *The spectral sequence E_* collapses at the first page, giving isomorphisms* $e_q : E_1^{q,q} \xrightarrow{\sim} H^{2q}(\Omega(NG))$.
- *The simplicial and universal Chern-Weil morphisms are isomorphisms of graded rings.*

Let $\bar{N}G \to NG$ be the simplicial universal bundle and $\bar{\mathcal{D}}$ be the simplicial flat connection on the simplicial vector bundle $\mathrm{ad}(\bar{N}G)$.

Definition

The *twisted simplicial de Rham bicomplex* $\Omega(NG, \mathrm{ad}(\bar{N}G))$ has components $\Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \stackrel{\mathrm{def.}}{=} \Omega^q(N_pG, \mathrm{ad}(\bar{N}_pG))$ and differentials:

$$\delta'_{\mathrm{ad}} = \sum_{i=0}^{p+1}(-1)^i(\epsilon_i^{p+1})^* : \Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \to \Omega^{p+1,q}(NG, \mathrm{ad}(\bar{N}G))$$

$$\delta''_{\mathrm{ad}} = (-1)^p \mathrm{d}_{\bar{\mathcal{D}}} : \Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \to \Omega^{p,q+1}(NG, \mathrm{ad}(\bar{N}G)) \quad .$$

Let $\delta_{\mathrm{ad}} \stackrel{\mathrm{def.}}{=} \delta'_{\mathrm{ad}} + \delta''_{\mathrm{ad}}$ and $H^*(\Omega(NG, \mathrm{ad}(\bar{N}G)))$ be the total differential and total cohomology of this bicomplex, which is a ring under the obvious operation $\mathring{\wedge} : \Omega^{k_1}(N_qG, \mathfrak{g}^{\otimes l_1}) \times \Omega^{k_2}(N_qG, \mathfrak{g}^{\otimes l_2}) \to \Omega^{k_1+k_2}(N_qG, \mathfrak{g}^{\otimes (l_1+l_2)})$.

Theorem

There exists a natural isomorphism of vector spaces:

$$\zeta_{\mathrm{ad}} : H^*(\Omega(NG, \mathrm{ad}(\bar{N}G))) \xrightarrow{\sim} H^*(BG, \mathrm{ad}(EG)_{\mathrm{disc}}) \quad .$$

which respects the cup product.

Definition

The *universal real twisted Chern class* of G is the real twisted Chern class of EG:

$$\mathfrak{c}(G) \stackrel{\mathrm{def.}}{=} \mathfrak{c}(EG) \in H^2(BG, \mathrm{ad}(EG)_{\mathrm{disc}})$$

We have:

$$\mathfrak{c}(G) = \zeta_{\mathrm{ad}}([\mathcal{V}(G)]_{\delta_{\mathrm{ad}}})$$

where the *universal simplicial adjoint curvature* $\mathcal{V}(G) \in \Omega^2_{\delta_{\mathrm{ad}}-\mathrm{cl}}(NG, \mathrm{ad}(\bar{N}G))$ is induced by Dupont's universal simplicial connection.

Proposition

We have:

$$\mathcal{V}(G) = \theta \in \Omega^{1,1}_{\delta_{\mathrm{ad}}-\mathrm{cl}}(NG, \mathrm{ad}(PG)) = \Omega^1_{\mathrm{cl}}(G, \mathfrak{g}) \quad .$$

Moreover, for any $T \in S^q(\mathfrak{g}^*)^{\Gamma}$, we have:

$$\psi(T) = T(\mathfrak{c}(G) \cup \ldots \cup \mathfrak{c}(G)) \in H^{2q}(BG, \mathbb{R}) \quad .$$

Let P be a principal G-bundle on a manifold M. Recall that the Chern-Weil morphism $\psi_P : S^q(\mathfrak{g}[2]^*) \to H^{\mathrm{even}}(M, \mathbb{R})$ of P is defined through:

$$\psi_P(T) \stackrel{\mathrm{def.}}{=} [T(\mathcal{V}_{\mathcal{A}} \wedge \ldots \wedge \mathcal{V}_{\mathcal{A}})]_{\mathrm{d}} = T(\mathfrak{c}(P) \cup \ldots \cup \mathfrak{c}(P)) \quad ,$$

where $\mathcal{A} \in \mathrm{Conn}(P)$ is an arbitrary principal connection on P and the cup product includes tensorization along $\mathrm{ad}(P)$ (it is the cup product for the sheaf cohomology of $\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{ad}(P))$).

Proposition

Let $f : M \to BG$ be a classifying map for P. Then:

$$\mathfrak{c}(P) = f^\sharp(\mathfrak{c}(G)) \in H^2(M, \mathrm{ad}(P)_{\mathrm{disc}}) = H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$$

and:

$$\psi_P = f^* \circ \psi \quad .$$

Relation between $\mathrm{ad}(EG)$ and $\mathfrak{g}(E\Gamma)$

Applying the universal bundle functor E to the projection morphism $q : G \to \Gamma$ gives a q-morphism of principal bundles $Eq : EG \to E\Gamma$ which covers the map $Bq : BG \to B\Gamma$. This is equivalent with a based isomorphism of principal Γ-bundles $\phi : \Gamma(EG) \xrightarrow{\sim} (Bq)^*(E\Gamma)$, i.e. a (G, q)-lift of structure group of $(Bq)^*(E\Gamma)$.

$$\begin{tikzcd} EG \arrow[r, "Eq"] \arrow[d] & E\Gamma \arrow[d] \\ BG \arrow[r, "Bq"] & B\Gamma \end{tikzcd}$$

Since $\mathrm{ad}(EG) \stackrel{\mathrm{def.}}{=} EG \times_{\mathrm{Ad}} \mathfrak{g} = \Gamma(EG) \times_{\overline{\rho}} \mathfrak{g}$, this gives:

$$\mathrm{ad}(EG) \simeq (Bq)^*(E\Gamma) \times_{\overline{\rho}} \mathfrak{g} = (Bq)^*(E\Gamma \times_{\overline{\rho}} \mathfrak{g}) = (Bq)^*(\mathfrak{g}(E\Gamma)) \quad .$$