The classification of principal bundles with weakly Abelian structure group

Calin Lazaroiu

DFT, IFIN-HH

Outline

Definition

A Lie group G is called *weakly Abelian* if its Lie algebra $\mathfrak{g}$ is Abelian.

Proposition

A Lie group G is weakly Abelian iff its connected component of the identity G_0 is an Abelian Lie group, which we denote by A.

Let G be weakly Abelian and $\Gamma \stackrel{\mathrm{def.}}{=} \pi_0(G)$ be its group of components. The exact sequence $1 \to G_0 \to G \to \pi_0(G)$ presents G as an Abelian extension of Γ:

$$1 \to A \stackrel{i}{\to} G \stackrel{q}{\to} \Gamma \to 1 \quad , \tag{1}$$

which need not be split or central. The adjoint (a.k.a. conjugation) action $\mathrm{Ad}_G : G \to \mathrm{Aut}(G)$ of G preserves the normal subgroup A, on which it induces the *restricted adjoint action* $\mathrm{Ad}_G^A : G \to \mathrm{Aut}(A)$. In turn, this factors through q to the *characteristic morphism* $\rho : \Gamma \to \mathrm{Aut}(A)$:

$$\mathrm{Ad}_G^A = \rho \circ q \quad ,$$

which depends only on the equivalence class of the extension. Let $\mathrm{Ext}_\rho(\Gamma, A)$ be the group of equivalence classes of extensions (1) with characteristic morphism ρ, with addition given by the Baer sum. This is isomorphic with $H^2(\Gamma, A_\rho) = \mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A_\rho)$, where A_ρ is the Γ-module defined by ρ.

Definition

The *extension class* of G is the group cohomology class $e(G) \in H^2(\Gamma, A_\rho)$ defined by the extension sequence $1 \to A \stackrel{i}{\to} G \stackrel{q}{\to} \Gamma \to 1$.

The Lie group extension (1) gives a Lyndon-Hochschild-Serre spectral sequence in (Segal-Mitchison) cohomology of continuous groups, which in turn produces a five-term *inflation-restriction exact sequence*:

$$0 \to H^1(\Gamma, A_\rho) \stackrel{q^*}{\longrightarrow} H^1(G, A_{\mathrm{Ad}_G^A}) \stackrel{i^*}{\longrightarrow} H^1(A, A)^\Gamma \stackrel{\lambda_G}{\longrightarrow} H^2(\Gamma, A) \stackrel{q^*}{\longrightarrow} H^2(G, A_{\mathrm{Ad}_G^A}) \ , \tag{2}$$

where λ_G is the transgression morphism.

Proposition

We have:

$$e(G) = -\lambda_G(\mathrm{id}_A) \ , \tag{3}$$

where $\mathrm{id}_A \in \mathrm{Hom}(A, A) = H^1(A, A)^\Gamma$ *is the identity morphism of* A*. In particular, we have* $q^*(e(G)) = 0$*.*

Since A is Abelian, the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ of G factors through q to the *reduced adjoint representation* $\overline{\rho} : \Gamma \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$:

$$\mathrm{Ad} = \overline{\rho} \circ q \quad . \tag{4}$$

Proposition

The exponential map $\exp_G : (\mathfrak{g}, +) \to A$ *of* G *is a surjective morphism of Lie groups. The Abelian group:*

$$\Lambda \stackrel{\mathrm{def.}}{=} \ker(\exp_G) = \{\lambda \in \mathfrak{g} \mid \exp_G(\lambda) = 1_G\}$$

is a (generally non-full) lattice in $\mathfrak{g}$ *which is stable under* G *and* Γ*. The map* $C_G : \Lambda \to \pi_1(G) \stackrel{\mathrm{def.}}{=} \pi_1(G_0, 1_G)$ *which sends* $\lambda \in \Lambda$ *to the homotopy class of the curve* $c_\lambda : [0,1] \to G_0 = A$ *defined through:*

$$c_\lambda(t) \stackrel{\mathrm{def.}}{=} \exp_G(t\lambda) \quad \forall t \in [0,1] \quad . \tag{5}$$

is an isomorphism of groups, whose inverse embeds $\pi_1(G)$ *as the lattice* $\Lambda \subset \mathfrak{g}$*.*

Definition

The lattice $\Lambda \subset \mathfrak{g}$ is called the *fundamental lattice* of G. The morphism of groups $\mathrm{Ad}_0 : G \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ obtained by restricting Ad to Λ is called the *restricted adjoint representation* of G. The morphism of groups $\rho_0 : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ obtained by restricting $\overline{\rho}$ to Λ is called the *coefficient morphism* of G, while the Γ-module Λ_{ρ_0} is called the *coefficient module*.

We have:

$$\mathrm{Ad}_0 = \rho_0 \circ q \quad .$$

The *coefficient crossed module* $\mathcal{X}_0(G) \stackrel{\mathrm{def.}}{=} (\Lambda, \Gamma, \mathbb{1}_\Gamma, \rho_0)$ is *algebraically* weakly-equivalent with the *exponential crossed module* $\mathcal{X}_1(G) \stackrel{\mathrm{def.}}{=} (\mathfrak{g}, G, \exp_G, \mathrm{Ad})$.

Proposition

The crossed module defined by $\Pi_1(G)$ is isomorphic with the exponential crossed module $\mathcal{X}_1(G)$ and hence the fundamental 2-group $\Pi_1(G)$ is isomorphic with the 2-group $\mathrm{X}_1(G) = G /\!\!/_{\exp_G} \mathfrak{g}$ defined by $\mathcal{X}_1(G)$.

Let $\xi(G) \in H^3(\Gamma, \Lambda_{\rho_0})$ be the *Taylor obstruction class* of G, which vanishes iff G admits a proper universal covering *group*. Given a topological group H and a morphism of topological groups $\alpha : H \to \Gamma$, the exponential sequence

$1 \to \Lambda \xrightarrow{j} \mathfrak{g} \xrightarrow{\exp} A \to 1$ induces a long exact sequence in group cohomology:

$$\ldots \to H^k(H, \Lambda_{\rho_0 \circ \alpha}) \xrightarrow{j_*} H^k(H, \mathfrak{g}_{\bar{\rho} \circ \alpha}) \xrightarrow{\exp_*} H^k(H, A_{\rho \circ \alpha}) \xrightarrow{\Delta_k^H} H^{k+1}(H, \Lambda_{\rho_0 \circ \alpha}) \to \ldots \quad ,$$

where Δ_k^H are the connecting morphisms. The inflation-restriction sequences of the extension (1) for group cohomology with coefficients in A and Λ fit into a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(\Gamma, A_\rho) & \longrightarrow & H^1(G, A_{\mathrm{Ad}_G^A}) & \xrightarrow{i^*} & H^1(A, A)^\Gamma & \xrightarrow{\lambda_G} & H^2(\Gamma, A_\rho) & \xrightarrow{q^*} & H^2(G, A_{\mathrm{Ad}_G^A}) \\
 & & \downarrow{\scriptstyle \Delta_1^\Gamma} & & \downarrow{\scriptstyle \Delta_1^G} & & \downarrow{\scriptstyle \Delta_1^A} & & \downarrow{\scriptstyle \Delta_2^\Gamma} & & \downarrow{\scriptstyle \Delta_2^G} \\
0 & \longrightarrow & H^2(\Gamma, \Lambda_{\rho_0}) & \longrightarrow & H^2(G, \Lambda_{\mathrm{Ad}_0}) & \xrightarrow{i^*} & H^2(A, \Lambda_{\rho_0}) & \xrightarrow{\mu_G} & H^3(\Gamma, \Lambda_{\rho_0}) & \xrightarrow{q^*} & H^3(G, \Lambda_{\mathrm{Ad}_0})
\end{array} \quad (6)$$

Let $\epsilon(G) \stackrel{\text{def.}}{=} \Delta_1^A(\mathrm{id}_A) \in H^2(A, \Lambda_{\rho_0})$ be the *fundamental class* of A.

Proposition

We have:

$$\xi(G) = \Delta_2^\Gamma(e(G)) = -\mu_G(\epsilon(G)) \quad .$$

In particular, we have $q^*(\xi(G)) = 0$

Relation between $\xi(G)$ and the k-invariant of BG

To any principal Γ-bundle Q on a topological space X we associate the local coefficient system $\Lambda_{\rho_0}(Q) \stackrel{\text{def.}}{=} Q \times_{\rho_0} \Lambda$.

Proposition (Segal-Mitchison)

For any topological group morphism $H \stackrel{\alpha}{\to} \Gamma$, we have a natural isomorphism:

$$H^*(H, \Lambda_{\rho_0 \circ \alpha}) \simeq H^*(BH, \Lambda_{\rho_0}(E_\alpha \Gamma)) \quad , \tag{7}$$

where $E_\alpha \Gamma \to BH$ is the $B\alpha$-pull-back to BH of the universal bundle $E\Gamma \to B\Gamma$.

In particular, the fundamental class $\epsilon(G) \in H^2(A, \Lambda)$ of A identifies with the fundamental class $\iota \in H^2(K(\Lambda, 2), \Lambda) \simeq [K(\Lambda, 2), K(\Lambda, 2)]$ of $K(\Lambda, 2)$.
The extension sequence (1) implies that the classifying space of G is an Eilenberg-MacLane fibration with fiber $BA \simeq K(\Lambda, 2)$ over the classifying space $B\Gamma \simeq K(\Gamma, 1)$ of Γ:

$$* \to BA \to BG \to B\Gamma \to * \quad . \tag{8}$$

Such fibrations are classified by an element $\kappa \in H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma))$, which is the single k-invariant of BG.

Proposition

The extension class $\xi(G)$ identifies with κ under the isomorphism of groups (7).

Since $H^1(K(\Lambda, 2), \Lambda) = 0$, the Leray-Serre spectral sequence for Λ-valued cohomology of this fibration gives a five term exact sequence:

$$0 \to H^2(B\Gamma, \Lambda_{\rho_0}(E\Gamma)) \to H^2(BG, \Lambda_{\mathrm{Ad}_0}(E\Gamma)) \to H^2(BA, \Lambda) \xrightarrow{\delta} H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma)) \to H^3(BG, \Lambda_{\mathrm{Ad}_0}(E\Gamma)) \ . \quad (9)$$

which identifies with the Λ-valued inflation-restriction sequence on the bottom row of (6). Hence the Leray-Serre spectral sequence for Λ-valued cohomology of (8) identifies with the Λ-valued Lyndon-Hochschild-Serre spectral sequence of (1) and the inflation-restriction sequence in the bottom row of diagram (6) identifies with the five-term sequence induced by the Leray-Serre spectral sequence.

Proposition

We have:

$$\kappa = \delta(\iota)$$

where $\delta : H^2(BA, \Lambda) \to H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma))$ *is the connecting morphism of* (9).

Let M be a d-manifold. To any principal Γ-bundle Q defined on M we associate two bundles of Ablian groups and a vector bundle, namely:

- The *coefficient system* $\Lambda(Q) \stackrel{\text{def.}}{=} Q \times_{\rho_0} \Lambda$ of Γ of Q relative to G, where $\rho_0 : \Gamma \to \mathrm{Aut}(G)$ is the coefficient morphism of G. This is a bundle of discrete Abelian groups with fiber given by the exponential lattice of G, which can also be viewed as a local system of discrete Abelian groups defined on M.
- The *characteristic bundle* $A(Q) \stackrel{\text{def.}}{=} Q \times_{\rho} A$ of Q relative to G, where $\rho : \Gamma \to \mathrm{Aut}(A)$ is the reduced adjoint action of G. This is a bundle of Abelian *Lie* groups whose fiber is given by the connected component of the identity in G.
- The *reduced adjoint bundle* $\mathfrak{g}(Q) = Q \times_{\overline{\rho}} \mathfrak{g}$ of Q relative to G, where $\overline{\rho} : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is the reduced adjoint representation of G. This is a smooth vector bundle defined on M, whose fiber is the Lie algebra of G.

Notice that Q carries a unique flat connection since it is a principal bundle with discrete structure group. This induces a flat Ehresmann connection on $A(Q)$ (whose parallel transport acts through isomorphisms of groups) and a flat connection $\mathcal{D}$ on the vector bundle $\mathfrak{g}(Q)$. Notice that $\Lambda(Q)$ is a fiber sub-bundle of $\mathfrak{g}(Q)$ which is preserved by the parallel transport of $\mathcal{D}$.

Definition

The k-th $\mathfrak{g}(Q)$*-valued twisted de Rham cohomology space* $H^k_{\mathcal{D}}(M, \mathfrak{g}(Q))$ is the k-th cohomology space of the *twisted de Rham complex*:

$$0 \to \Omega^0(M, \mathfrak{g}(Q)) \xrightarrow{\mathrm{d}_{\mathcal{D}}} \Omega^1(M, \mathfrak{g}(Q)) \xrightarrow{\mathrm{d}_{\mathcal{D}}} \dots \xrightarrow{\mathrm{d}_{\mathcal{D}}} \Omega^d(M, \mathfrak{g}(Q)) \to 0 \quad .$$

Proposition

There exists a natural isomorphism of graded vector spaces:

$$H^*_{\mathcal{D}}(M, \mathfrak{g}(Q)) \simeq H^*(M, \mathcal{C}^\infty_{\mathrm{flat}}(\mathfrak{g}(Q))) = H^*(M, \mathfrak{g}(Q)_{\mathrm{disc}}) \quad .$$

The exponential sequence of A induces a commutative diagram with exact rows, where δ_0 and δ are the connecting morphisms:

$$\begin{array}{ccccccccccccccccc}
\dots & \longrightarrow & H^1(M, \Lambda(Q)) & \xrightarrow{j_{0,*}} & H^1_{\mathcal{D}}(M, \mathfrak{g}(Q)) & \xrightarrow{\exp_{0,*}} & H^1(M, A(Q)_{\mathrm{disc}}) & \xrightarrow{\delta_0} & H^2(M, \Lambda(Q)) & \xrightarrow{j_{0,*}} & H^2_{\mathcal{D}}(M, \mathfrak{g}(Q)) & \xrightarrow{\exp_{0,*}} & H^2(M, A(Q)_{\mathrm{disc}}) & \xrightarrow{\delta_0} & H^3(M, \Lambda(Q)) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\dots & \longrightarrow & H^1(M, \Lambda(Q)) & \xrightarrow{j_*} & 0 & \xrightarrow{\exp_*} & H^1(M, \mathcal{C}^\infty(A(Q))) & \xrightarrow{\delta} & H^2(M, \Lambda(Q)) & \xrightarrow{j_*} & 0 & \xrightarrow{\exp_*} & H^2(M, \mathcal{C}^\infty(A(Q))) & \xrightarrow{\delta} & H^3(M, \Lambda(Q)) & \longrightarrow & \dots
\end{array} \tag{10}$$

The sheaf $\mathcal{C}^\infty(\mathfrak{g}(Q))$ is acyclic. Hence $\delta : H^k(M, \mathcal{C}^\infty(A(Q))) \xrightarrow{\sim} H^{k+1}(M, \Lambda(Q))$ are isomorphisms for all $k \geq 1$ and we have:

$$\delta_0 = \delta \circ \iota_* \quad , \quad \kappa_* \circ j_{0,*} = 0 \quad . \tag{11}$$

Let Q be a principal Γ-bundle on a manifold M.

Definition

The *G-extension class* and *G-obstruction class* of Q are defined through:

$$e_G(Q) \stackrel{\mathrm{def.}}{=} f^\sharp(e(G)) \in H^2(M, A(Q)_{\mathrm{disc}}) \ , \ \ \xi_G(Q) \stackrel{\mathrm{def.}}{=} f^\sharp(\xi(G)) \in H^3(M, \Lambda(Q)) \ ,$$

where $f : M \to B\Gamma$ is a classifying map for Q. The *smooth image* of $e(Q)$ is defined through:

$$e^s_G(Q) \stackrel{\mathrm{def.}}{=} \iota_*(e_G(Q)) \in H^2(M, \mathcal{C}^\infty(A(Q)) \ .$$

where $\iota_* : H^2(M, A(Q)_{\mathrm{disc}}) = H^*(M, \mathcal{C}^\infty_{\mathrm{flat}}(A(Q))) \to H^2(M, \mathcal{C}^\infty(A(Q)))$ is the morphism induced by the sheaf inclusion $\mathcal{C}^\infty_{\mathrm{flat}}(A(Q)) \hookrightarrow \mathcal{C}^\infty(A(Q))$.

We have:

$$\delta_0(e_G(Q)) = \delta(e^s_G(Q)) = \xi_G(Q) \ . \qquad (12)$$

Definition

A (G, q)-*lift of structure group* of Q is a pair (P, φ), where P is principal G-bundle defined on M and $\varphi : P \to Q$ is a based morphism of principal bundles above $q : G \to \Gamma$, i.e. a based isomorphism of principal Γ-bundles $\varphi : \Gamma(P) \xrightarrow{\sim} Q$, where $\Gamma(P) \stackrel{\text{def.}}{=} P \times_q \Gamma$.

Isomorphisms of (G, q)-lifts of structure group are defined obviously. Let $T_{G,q}(Q)$ be the set of isomorphism classes of (G, q)-lifts of Q.

Theorem

Q admits a smooth (G, q)-lift of structure group iff $\xi_G(Q) = 0$, i.e. $e^s_G(Q) = 0$. In this case, $T_{G,q}(Q)$ is a torsor over $H^1(M, \mathcal{C}^\infty(A(Q))) = H^2(M, \Lambda(Q))$.

Definition

Suppose that Q admits a (G, q)-lift of structure group, thus $e_G(Q) \in \ker \delta_0 = \mathrm{im}(\exp_{0,*}(H^2_{\mathcal{D}}(M, \mathfrak{g}(Q)))$. The *linear and affine characteristic lattices* of Q are the lattices in $H^2_{\mathcal{D}}(M, \mathfrak{g}(Q))$ defined through:

$$L_0(Q) \stackrel{\text{def.}}{=} j_{0,*}(H^2(M, \Lambda(Q))) = \exp_{0,*}^{-1}(\{0\}) \quad , \quad L(Q) \stackrel{\text{def.}}{=} \exp_{0,*}^{-1}(\{e_G(Q)\}) \quad .$$

Define:

$$Prin^0_\Gamma(M) \stackrel{\text{def.}}{=} \{Q \in Prin_\Gamma(M) \mid \xi_G(Q) = 0\} \ , \ \ T^{G,q}_\Gamma(M) \stackrel{\text{def.}}{=} \sqcup_{Q \in Prin^0_\Gamma(M)} T_{G,q}(Q)$$

The groupoid $Prin^0_\Gamma(M)$ acts from the left on $T^{G,q}_\Gamma(M)$. The set of orbits $T^{G,q}_\Gamma(M)/Prin^0_\Gamma(M)$ fibers over $\mathrm{Prin}^0_\Gamma(M)$.

Theorem

There exists a natural bijection:

$$\mathrm{Prin}_G(M) \xrightarrow{\sim} T^{G,q}_\Gamma(M)/Prin^0_\Gamma(M) \ \ .$$

The extension class and affine characteristic lattice of P

Let P be a principal G-bundle defined on M.

Definition

The *discrete remnant* of P is the principal Γ-bundle $\Gamma(P) \stackrel{\mathrm{def.}}{=} P \times_q \Gamma$.

We have $\mathrm{ad}(P) = \mathfrak{g}(\Gamma(P))$. Define:

$$A(P) \stackrel{\mathrm{def.}}{=} A(\Gamma(P)) = P \times_{\mathrm{Ad}_G^A} A \quad , \quad \Lambda(P) \stackrel{\mathrm{def.}}{=} \Lambda(\Gamma(P)) = P \times_{\mathrm{Ad}_0} \Lambda$$

Notice that $\xi(\Gamma(P)) = 0$, hence $e_G^s(\Gamma(P)) = 0$.

Definition

The *G-extension class* of P is defined through:

$$e_G(P) \stackrel{\mathrm{def.}}{=} e_G(\Gamma(P)) \in H^2(M, \mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{ad}(P))) = H^2(M, \mathrm{ad}(P)_{\mathrm{disc}})$$

The linear and affine *characteristic lattices* of P are defined to be the corresponding lattices of $\Gamma(P)$:

$$L_0(P) \stackrel{\mathrm{def.}}{=} L_0(\Gamma(P)) = j_{0,*}(H^2(M, \Lambda(P))) = \exp_{0,*}^{-1}(0) \subset H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$$
$$L(P) \stackrel{\mathrm{def.}}{=} L(\Gamma(P)) = \exp_{0,*}^{-1}(\{e_G(P)\}) \subset H^2_{\mathcal{D}}(M, \mathrm{ad}(P)) \quad .$$

Let P be a principal G-bundle defined on M.

Proposition

All principal connections defined on P induce the same adjoint connection, which coincides with the distinguished flat connection $\mathcal{D}$ of $\mathrm{ad}(P) = \mathfrak{g}(\Gamma(P))$.

Proposition

The adjoint curvature $\mathcal{V}_\mathcal{A} \in \Omega^2(M, \mathrm{ad}(P))$ of any principal connection $\mathcal{A} \in \mathrm{Conn}(P)$ satisfies:

$$\mathrm{d}_\mathcal{D} \mathcal{V}_\mathcal{A} = 0 \quad .$$

Moreover, the $\mathrm{d}_\mathcal{D}$-cohomology class $\mathfrak{c} \stackrel{\mathrm{def.}}{=} [\mathcal{V}_\mathcal{A}]_{\mathrm{d}_\mathcal{D}} \in H^2_\mathcal{D}(M, \mathrm{ad}(P))$ does not depend on the choice of $\mathcal{A}$ in $\mathrm{Conn}(P)$.

Definition

The twisted de Rham cohomology class $\mathfrak{c}(P) \in H^2_\mathcal{D}(M, \mathrm{ad}(P))$ is called the *real twisted Chern class* of P.

Theorem

For any principal G-bundle P defined on M, we have $\mathfrak{c}(P) \in L(P)$. Given a principal Γ-bundle Q defined on M which admits (G, q)-lifts of structure group, the map:

$$T_{G,q}(Q) \ni P \to \mathfrak{c}(P) \in L(Q)$$

is a morphism of torsors above the surjective morphism of groups $j_{0,*} : H^2(M, \Lambda(Q)) \to L_0(Q)$.

Notice that $j_{0,*}$ kills torsion, so it need not be injective.

Remark. Suppose that $e_G(Q) = 0$. In this case, $T_{G,q}(Q)$ identifies with the Abelian group $H^2(M, \Lambda(P))$ and (G, q)-extensions of Q are classified by their *integral twisted Chern class* $c(P) \in H^2(M, \Lambda(P)) = H^2(M, \Lambda(Q))$, which satisfies $j_{0,*}(c(P)) = \mathfrak{c}(P)$.

This occurs for example when G is a split extension of Γ by A (i.e. when $G \simeq A \rtimes_\rho \Gamma$). Then $e(G) = 0$, hence $e_G(Q) = 0$ for any principal Γ-bundle Q. In this case, any principal Γ-bundle admits (G, q)-extensions of structure group and principal G-bundles P are classified by pairs $(\Gamma(P), c(P))$.