Open-closed B-type LG modes with Stein manifold targets

Calin Lazaroiu

(with Mirela Babalic, Dmitry Doryn, Mehdi Tavakol)

IBS Center for Geometry and Physics, Pohang, Korea

Mar 28, 2019

IMAR, Bucharest

Outline

1. Introduction
2. Classical LG models
3. TFT data
4. B-type LG models with Stein manifold target
5. Topologically non-trivial elementary holomorphic factorizations
6. Examples
   - Domains of holomorphy
   - Non-compact Riemann surfaces
   - Analytic complete intersections
   - Complements of affine hyperplane arrangements
   - Complements of anticanonical divisors
   - Total spaces of holomorphic vector bundles

We study the quantization of general B-type Landau-Ginzburg models (including their coupling to topological D-branes).

Definition

A Landau-Ginzburg pair is a pair (X, W), where:

- X is a non-compact Kählerian manifold (called target space), whose canonical line bundle is holomorphically trivial.
- $W : X \to \mathbb{C}$ is a non-constant holomorphic function, called superpotential.

The critical set of (X, W) is the set of critical points of the superpotential:

$$Z_W \stackrel{\text{def.}}{=} \{p \in X | (\partial W)(p) = 0\} \ .$$

The signature of (X, W) is the mod 2 reduction of the complex dimension of X:

$$\mu(X, W) \stackrel{\text{def.}}{=} d \mod 2 \in \mathbb{Z}_2$$

Classical oriented open-closed B-type topological Landau-Ginzburg models are classical field theories defined on compact oriented surfaces with corners and parameterized by Landau-Ginzburg pairs (X, W). It is expected that such models admit a non-anomalous quantization.

The bulk action is:

$$\tilde{S}_{bulk} = S_B + S_W + s \quad ,$$

where:

$$\begin{aligned} S_B &= \int_\Sigma d^2\sigma\sqrt{g}\Big[G_{i\bar{j}}\Big(g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \mathrm{i}\varepsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \frac{1}{2}g^{\alpha\beta}\rho^i_\alpha D_\beta\eta^{\bar{j}} \\ &\quad -\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha D_\beta\theta^{\bar{j}} - \tilde{F}^i\tilde{F}^{\bar{j}}\Big) + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}R_{i\bar{l}k\bar{j}}\rho^i_\alpha\bar{\chi}^{\bar{l}}\rho^k_\beta\chi^{\bar{j}}\Big] \end{aligned}$$

is the action of the B-twisted sigma model and $S_W = S_0 + S_1$ is the potential-dependent term, with:

$$\begin{aligned} S_0 &= -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\bar{\chi}^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}}\Big] \\ S_1 &= -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[(\partial_i W)\tilde{F}^i + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}D_i\partial_j W\rho^i_\alpha\rho^j_\beta\Big] \quad . \end{aligned}$$

Here:

$$s := \mathrm{i}\int_\Sigma d^2\sigma\sqrt{g}\varepsilon^{\alpha\beta}\partial_\alpha(G_{\bar{i}j}\chi^{\bar{i}}\rho^j_\beta) = \mathrm{i}\int_\Sigma d(G_{\bar{i}j}\chi^{\bar{i}}\rho^j) \quad .$$

is a correction needed to solve the so-called “Warner problem”.

The fields involved are:

- Grassmann even fields:
  - the scalar field $\phi : \Sigma \to X$
  - the Riemannian metric g on Σ, which is treated as a background field.
  - the auxiliary fields $\tilde{F} \in \Gamma_\infty(\phi^*(\mathcal{T}_\mathbb{C} X))$
- Grassmann odd fields:
  - $\eta, \chi, \bar{\chi} \in \Gamma_\infty(\phi^*(\bar{T}X))$, $\theta \in \Gamma_\infty(\phi^*(T^*X))$, $\rho \in \Gamma_\infty(\phi^*(TX) \otimes \mathcal{T}^*\Sigma)$

Here $\mathcal{T}X$ is the real tangent bundle of X and $\mathcal{T}_\mathbb{C}X = \mathcal{T}X \otimes \mathbb{C} = TX \oplus \bar{T}X$ its complexification, while TX and $\bar{T}X$ are the holomorphic and antiholomorphic tangent bundles of X. $\mathcal{T}\Sigma$ is the real tangent bundle of Σ. We wrote the Lagrange density in local holomorphic coordinates $z = (z_1, \ldots, z_d)$ defined on $U \subset X$.

Consider a complex superbundle $E = E^{\hat{0}} \oplus E^{\hat{1}}$ on X and a superconnection $\mathcal{B}$ on E. The bundle $End(E)$ is $\mathbb{Z}_2$-graded:

$$
\begin{aligned}
End^{\hat{0}}(E) &:= \quad End(E^{\hat{0}}) \oplus End(E^{\hat{1}}) \\
End^{\hat{1}}(E) &:= \quad Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \quad .
\end{aligned}
$$

In a local frame of E compatible with the grading, $\mathcal{B}$ corresponds to a matrix:

$$
\mathcal{B} = \begin{bmatrix} A^{(+)} & v \\ u & A^{(-)} \end{bmatrix}
$$

where v and u are smooth sections of $Hom(E^{\hat{1}}, E^{\hat{0}})$ and $Hom(E^{\hat{0}}, E^{\hat{1}})$, while the diagonal entries $A^{(+)}$ and $A^{(-)}$ are connection one-forms on $E^{\hat{0}}$ and $E^{\hat{1}}$, such that $A^{(+)} \in \Omega^{(0,1)}(End(E^{\hat{0}}))$ and $A^{(-)} \in \Omega^{(0,1)}(End(E^{\hat{1}}))$.

We ‘define’ the *naive partition function* on an oriented Riemann surface (Σ, g) with corners by:

$$
\mathcal{Z} := \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta] e^{-\tilde{S}_{bulk}} \mathcal{U}_1 \ldots \mathcal{U}_h \quad ,
$$

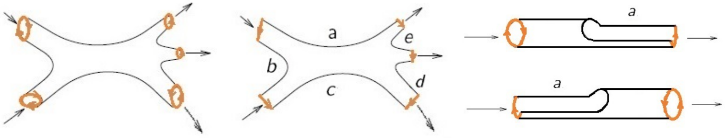
where h is the number of holes and the factors $\mathcal{U}_h$ have complicated expressions depending on the superconnection $\mathcal{B}$ and the fields as well as on “boundary condition changing operators” inserted at the corners of each hole.
$(\mathcal{U}_1 \ldots \mathcal{U}_h = e^{-\tilde{S}_{boundary}})$

The monoidal functor formalism

A non-anomalous quantum oriented 2-dimensional open-closed TFT can be described axiomatically (CIL 2001) as a strongly monoidal functor from the symmetric monoidal category Cob_2 of oriented cobordisms with corners to the symmetric monoidal category $\mathrm{vect}^s_{\mathbb{C}}$ of finite-dimensional $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$:

$$Z : (\mathrm{Cob}_2, \sqcup, \emptyset) \longrightarrow (\mathrm{vect}^s_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$$

The objects of Cob_2 are finite disjoint unions of oriented circles and oriented segments. The morphisms are compact oriented smooth surfaces with corners (corresponding to the worldsheets of open and closed strings). The corners coincide with the boundary points of the segments.



The labels associated to the ends of the intervals indicate the corresponding boundary conditions (or the corresponding D-branes).

TFT data

Theorem (CIL 2001)

Any (non-anomalous) oriented 2-dimensional open-closed TFT can be described equivalently by an algebraic structure known as a TFT datum*.*

Definition

A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

- $\mathcal{H}$ = bulk algebra, a finite-dimensional supercommutative $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra with unit $1_\mathcal{H}$ (the space of 'on-shell states' of the closed topological string)
- $\mathcal{T}$ = category of topological D-branes, a Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category, with composition of morphisms denoted by $\circ$ and units:

  $$1_a \in \mathrm{Hom}_\mathcal{T}(a, a) \ , \ \forall a \in \mathrm{Ob}\mathcal{T}$$

  The objects of $\mathcal{T}$ are the topological D-branes while $\mathrm{Hom}_\mathcal{T}(a, b)$ is the space of 'on-shell states' of the open topological string stretching from a to b.
- $e = (e_a)_{a\in \mathrm{Ob}\mathcal{T}}$, a family of $\mathbb{C}$-linear bulk-boundary maps $e_a : \mathcal{H} \to \mathrm{Hom}_\mathcal{T}(a, a)$ such that the following conditions are satisfied:
  - For any $a \in \mathrm{Ob}\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{Z}2$-graded $\mathbb{C}$-algebras from $\mathcal{H}$ to $(\mathrm{End}_\mathcal{T}(a), \circ)$, where $\mathrm{End}_\mathcal{T}(a) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_\mathcal{T}(a, a)$.
  - For any $a, b \in \mathrm{Ob}\mathcal{T}$, any $\mathbb{Z}_2$-homogeneous $h \in \mathcal{H}$ and any $\mathbb{Z}_2$-homogeneous $t \in \mathrm{Hom}_\mathcal{T}(a, b)$, we have:

    $$e_b(h) \circ t = (-1)^{\mathrm{deg} h \, \mathrm{deg} t} t \circ e_a(h).$$

Definition

A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in Ob\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-degree μ

$$\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$$

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$, the $\mathbb{C}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a, b) \times \mathrm{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$$

  defined through:

$$\langle t_1, t_2 \rangle_{a,b} = \mathrm{tr}_b(t_1 \circ t_2),\ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b),\ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$$

  is non-degenerate.

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$, we have:

$$\langle t_1, t_2 \rangle_{a,b} = (-1)^{\deg t_1 \deg t_2} \langle t_2, t_1 \rangle_{b,a}$$

Definition

A TFT datum of parity $\mu \in \mathbb{Z}_2$ is a system $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a pre-TFT datum
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the bulk trace)
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called boundary traces)

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra, meaning that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate (i.e. the condition $\mathrm{Tr}(hh') = 0$ for all $h' \in \mathcal{H}$ implies $h = 0$)
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi-Yau supercategory of parity μ.
- The so-called *topological Cardy constraint* holds for all $a, b \in \mathrm{Ob}\mathcal{T}$.

The topological Cardy constraint has the form:

$$\boxed{\mathrm{Tr}(f_a(t_a)f_b(t_b)) = \mathrm{str}(\Phi_{ab}(t_a,t_b)) \;\;,\;\; \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \;,\; \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)}$$

where:

- "str" is the supertrace on the $\mathbb{Z}_2$-graded vector space $End_{\mathbb{C}}(Hom_{\mathcal{T}}(a,b))$
- $f_a : \mathrm{Hom}_{\mathcal{T}}(a,a) \to \mathcal{H}$ is the boundary-bulk map of a, defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to Hom_{\mathcal{T}}(a,a)$ with respect to Tr and tr:

$$\mathrm{Tr}(hf_a(t_a)) = \mathrm{tr}_a(e_a(h) \circ t_a), \;\forall h \in \mathcal{H}, \;\forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a)$$

- $\Phi_{ab}(t_a,t_b) : \mathrm{Hom}_{\mathcal{T}}(a,b) \to \mathrm{Hom}_{\mathcal{T}}(a,b)$ is the $\mathbb{C}$-linear map defined through:

$$\Phi_{ab}(t_a,t_b)(t) = t_b \circ t \circ t_a \;,\; \forall t \in \mathrm{Hom}_{\mathcal{T}}(a,b) \;,\; \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \;,\; \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b) \;.$$

In our work, we made a mathematically rigorous proposal for the TFT datum associated to a general Landau-Ginzburg pair; this proposal is inspired by path integral arguments.

For any Landau-Ginzburg pair (X, W), we proved that our proposal satisfies all axioms of a TFT datum.

This amounts to a rigorous construction of the non-anomalous quantum oriented open-closed B-type Landau-Ginzburg topological field theory in such extreme generality.

Remark

According to our proposal, the TFT datum of the quantum B-type LG model associated to an LG pair (X, W) is not only $\mathbb{C}$-linear but also partially $\mathrm{O}(X)$-linear, *where $\mathrm{O}(X)$ is the unital commutative ring of complex-valued holomorphic functions defined on X. Namely:*

- *$\mathcal{H}$ is an $\mathrm{O}(X)$-module (which is finite-dimensional over $\mathbb{C} \subset \mathrm{O}(X)$).*
- *$\mathcal{T}$ is an $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded category (which is Hom-finite over $\mathbb{C}$).*
- *The maps e_a and f_a are $\mathrm{O}(X)$-linear.*
- *However, the bulk and boundary traces Tr and tr_a* are only $\mathbb{C}$-linear.

Stein manifolds

Definition

Let X be a complex manifold of dimension $d > 0$. We say that X is Stein if it admits a holomorphic embedding as a *closed* complex submanifold of $\mathbb{C}^N$ for some $N > 1$.

Remarks

- *There exist numerous equivalent definitions of Stein manifolds.*
- *Any Stein manifold is Kählerian.*
- *The analyticization of any non-singular complex affine variety is Stein, but the vast majority of Stein manifolds are not of that type.*

Example

- $\mathbb{C}^d$ is a Stein manifold.
- Every domain of holomorphy in $\mathbb{C}^d$ is a Stein manifold.
- Every closed complex submanifold of a Stein manifold is a Stein manifold.
- Any non-singular analytic complete intersection in $\mathbb{C}^N$ is a Stein manifold.
- Any (non-singular) connected open Riemann surface without border is a Stein manifold.

Theorem (Cartan's theorem B)

For every coherent analytic sheaf $\mathcal{F}$ on a Stein manifold X, the sheaf cohomology $\mathrm{H}^i(X, \mathcal{F})$ vanishes for all $i > 0$.

Since Stein manifolds are Kählerian, we can consider B-type Landau-Ginzburg models with Stein manifold target X, provided that $K_X \simeq \mathcal{O}_X$. In this case, our model for the TFT datum simplifies.

Let (X, W) be a Landau-Ginzburg pair and:

- $\mathcal{O}_X$ be the sheaf of complex-valued holomorphic functions on X.
- $\mathrm{O}(X) \stackrel{\text{def.}}{=} \Gamma(X, \mathcal{O}_X)$ be the ring of complex-valued holomorphic functions which are globally-defined on X.
- $\boldsymbol{\iota}_W \stackrel{\text{def.}}{=} -\mathrm{i}(\partial W)\lrcorner : TX \to \mathcal{O}_X$ the morphism of sheaves of $\mathcal{O}_X$-modules given by left contraction with the holomorphic 1-form $-\mathrm{i}\partial W$.

Definition

- The critical sheaf $\mathcal{J}_W \stackrel{\text{def.}}{=} \mathrm{im}(\boldsymbol{\iota}_W : TX \to \mathcal{O}_X)$.
- The Jacobi sheaf $Jac_W \stackrel{\text{def.}}{=} \mathcal{O}_X/\mathcal{J}_W$.
- The Jacobi algebra $\mathrm{Jac}(X, W) \stackrel{\text{def.}}{=} \Gamma(X, Jac_W)$, which is a unital and commutative $\mathrm{O}(X)$-algebra.
- The critical ideal $\mathrm{J}(X, W) \stackrel{\text{def.}}{=} \mathcal{J}_W(X) = \boldsymbol{\iota}_W(\Gamma(X, TX)) \subset \mathrm{O}(X)$.

Theorem

Let (X, W) be a Landau-Ginzburg pair such that X is a Stein manifold of complex dimension d and suppose that Z_W is compact. Then the following statements hold:

1. *The critical locus Z_W is necessarily finite.*
2. *The bulk algebra is concentrated in even degree and can be identified with the Jacobi algebra:*
$$\mathcal{H} \equiv_{\mathrm{O}(X)} \mathrm{Jac}(X, W) \quad .$$
Moreover, we have an isomorphism of $\mathrm{O}(X)$-algebras:
$$\mathrm{Jac}(X, W) \simeq \mathrm{O}(X)/\mathrm{J}(X, W) \quad .$$
3. *If X is holomorphically parallelizable (i.e. if TX is holomorphically trivial), then:*
$$\mathrm{J}(X, W) = \langle u_1(W), \ldots, u_d(W) \rangle \quad ,$$
where $u_1, \ldots, u_d$ is any global holomorphic frame of TX.

Let (X, W) be a Stein LG pair with finite critical set Z_W. For any $p \in Z_W$, let:

$$\mathrm{M}(\hat{W}_p) \stackrel{\mathrm{def.}}{=} \frac{\mathcal{O}_{X,p}}{\langle \partial_1 \hat{W}_p, \ldots, \partial_d \hat{W}_p \rangle}$$

denote the **analytic** Milnor algebra of the analytic function germ $\hat{W}_p$ of W at p. The map $\mathrm{germ}_p : \mathrm{O}(X) \to \mathcal{O}_{X,p}$ induces a morphism of $\mathrm{O}(X)$-algebras:

$$\Lambda_p : \mathrm{Jac}(X, W) \to \mathrm{M}(\hat{W}_p) \quad .$$

Proposition

The map:

$$\Lambda \stackrel{\mathrm{def.}}{=} \oplus_{p \in Z_W} \Lambda_p : \mathrm{Jac}(X, W) \to \oplus_{p \in Z_W} \mathrm{M}(\hat{W}_p)$$

is an isomorphism of $\mathrm{O}(X)$*-algebras.*

This allows us to identify the bulk algebra with the direct sum of analytic Milnor algebras of W at its critical points.

Let (X, W) be a Landau-Ginzburg pair.

Definition

The holomorphic dg category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg category $\mathrm{F}(X, W)$ defined as follows:

- The objects are the holomorphic factorizations of W.
- $\mathrm{Hom}_{\mathrm{F}(X,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \Gamma(X, Hom(E_1, E_2))$, endowed with the $\mathbb{Z}_2$-grading:

$$\mathrm{Hom}^{\kappa}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom^{\kappa}(E_1, E_2)) \ , \ \forall \kappa \in \mathbb{Z}_2$$

  and with the differentials $\mathfrak{d}_{a_1,a_2}$ determined uniquely by the condition:

$$\mathfrak{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^{\kappa} f \circ D_1 \ , \ \forall f \in \Gamma(X, Hom^{\kappa}(E_1, E_2)) \ , \ \forall \kappa \in \mathbb{Z}_2 \quad .$$

- The composition of morphisms is the obvious one.

Let $\mathrm{HF}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{F}(X, W))$ be the total cohomology category of the dg category $\mathrm{F}(X, W)$.

The topological D-brane category in the Stein case

Definition

A projective analytic factorization of W is a pair (P, D), where P is a finitely-generated projective $\mathrm{O}(X)$-supermodule and $D \in \mathrm{End}^1_{\mathrm{O}(X)}(P)$ is an odd endomorphism of P such that $D^2 = W\mathrm{id}_P$.

Definition

The dg category $\mathrm{PF}(X, W)$ of projective analytic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg category defined as follows:

- The objects are the projective analytic factorizations of W.
- $\mathrm{Hom}_{\mathrm{PF}(X,W)}((P_1, D_1), (P_2, D_2)) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$, endowed with the obvious $\mathbb{Z}_2$-grading and with the $\mathrm{O}(X)$-linear odd differential $\mathfrak{d} := \mathfrak{d}_{(P_1,D_1),(P_2,D_2)}$ determined uniquely by the condition:

$$\mathfrak{d}(f) = D_2 \circ f - (-1)^{\mathrm{deg} f} f \circ D_1 f$$

  for all $\mathbb{Z}_2$-homogeneous module morphisms $f \in \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$.
- The composition of morphisms is the obvious one.

Let $\mathrm{HPF}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{PF}(X, W))$ be the total cohomology category of the dg category $\mathrm{PF}(X, W)$.

For any unital commutative ring R, let $\mathrm{MF}(R, W)$ denote category of finite rank matrix factorizations of W over R and $\mathrm{HMF}(R, W)$ denote its total cohomology category (which is $\mathbb{Z}_2$-graded and R-linear).

Theorem

Let (X, W) be a Landau-Ginzburg pair such that X is a Stein manifold. Then:

1. *There exists a natural equivalence of $\mathrm{O}(X)$-linear and $\mathbb{Z}_2$-graded dg categories:*
$$\mathrm{F}(X, W) \simeq_{\mathrm{O}(X)} \mathrm{PF}(X, W) \quad .$$
2. *If the critical locus Z_W is finite, then the topological D-brane category $\mathcal{T}$ is given by:*
$$\mathcal{T} \equiv \mathrm{HF}(X, W) \simeq_{\mathrm{O}(X)} \mathrm{HPF}(X, W) \quad .$$
3. *Multiplication with elements of the critical ideal $\mathrm{J}(X, W)$ acts trivially on $\mathrm{HF}(X, W)$, so $\mathcal{T}$ can be viewed as a $\mathbb{Z}_2$-graded $\mathrm{Jac}(X, W)$-linear category.*
4. *The even subcategory $\mathcal{T}^{\hat{0}}$ has a natural triangulated structure.*
5. *There exists a natural $\mathrm{O}(X)$-linear dg functor $\Xi : \mathrm{F}(X, W) \to \oplus_{p \in Z_W} \mathrm{MF}(\mathcal{O}_{X,p}, \hat{W}_p)$ which induces a full and faithful $\mathrm{Jac}(X, W)$-linear functor $\Xi_* : \mathrm{HF}(X, W) \to \oplus_{p \in Z_W} \mathrm{HMF}(\mathcal{O}_{X,p}, \hat{W}_p)$.*

The remaining objects of the TFT datum are as follows, where $d = \dim_{\mathbb{C}} X$:

- The bulk trace is given by:

$$\mathrm{Tr}(f) = \sum_{p \in Z_W} A_p \,\mathrm{Res}_p \left[\frac{\hat{f}_p \hat{\Omega}_p}{\det_{\hat{\Omega}_p}(\partial W)} \right] ,$$

- The boundary trace of the D-brane (holomorphic factorization) $a = (E, D)$ is given by the sum of *generalized Kapustin-Li traces*:

$$\mathrm{tr}_a(s) = \frac{(-1)^{\frac{d(d-1)}{2}}}{d!} \sum_{p \in Z_W} A_p \,\mathrm{Res}_p \left[\frac{\mathrm{str}\left(\det_{\hat{\Omega}_p}(\partial \hat{D}_p) \hat{s}_p \right) \hat{\Omega}_p}{\det_{\hat{\Omega}_p}(\partial \hat{W}_p)} \right] .$$

Here Ω is a holomorphic volume form on X, A_p are normalization constants and Res_p denotes the Grothendieck residue on $\mathcal{O}_{X,p}$.

- The bulk-boundary and boundary-bulk maps of $a = (E, D)$ are given by:

$$e_a(f) \equiv \mathrm{i}^d (-1)^{\frac{d(d-1)}{2}} \bigoplus_{p \in Z_W} \hat{f}_p \mathrm{id}_{E_p} \quad , \forall f \in \mathcal{H} \equiv \mathrm{Jac}(X, W)$$

$$f_a(s) \equiv \frac{\mathrm{i}^d}{d!} \bigoplus_{p \in Z_W} \mathrm{str}\left(\det_{\hat{\Omega}_p}(\partial \hat{D}_p)\, \hat{s}_p \right) \quad , \quad \forall s \in \mathrm{End}_{\mathcal{T}}(a) \equiv \Gamma(X, End(E)) \;.$$

By the Oka-Grauert principle, a holomorphic vector bundle V on a Stein manifold X is holomorphically trivial iff it is topologically trivial.

Definition

Let (X, W) be an LG pair where X is a Stein manifold. A holomorphic factorization (E, D) of W is called elementary if $\mathrm{rk}\, E^{\hat{0}} = \mathrm{rk}\, E^{\hat{1}} = 1$. It is called topologically trivial if E is trivial as a $\mathbb{Z}_2$-graded holomorphic vector bundle, i.e. if both $E^{\hat{0}}$ and $E^{\hat{1}}$ are trivial.

Construction. Let X be a CY Stein manifold with $\mathrm{H}^2(X, \mathbb{Z}) \neq 0$. Given a non-trivial holomorphic line bundle L on X and non-trivial holomorphic sections $v \in \Gamma(X, L)$ and $u \in \Gamma(X, L^{-1})$, the $\mathbb{Z}_2$-graded holomorphic vector bundle $E \stackrel{\mathrm{def.}}{=} \mathcal{O}_X \oplus L$ admits the odd global holomorphic section $D \stackrel{\mathrm{def.}}{=} \left[\begin{array}{cc} 0 & v \\ u & 0 \end{array}\right]$.

Then $u \otimes v \in \mathrm{H}^0(L \otimes L^{-1})$ identifies with a non-trivial holomorphic function W through any isomorphism $L \otimes L^{-1} \simeq \mathcal{O}_X$ and (E, D) is a topologically non-trivial elementary holomorphic factorization of W.

Remark

The assignment $L \to c_1(L)$ *gives an isomorphism* $\mathrm{Pic}_{\mathrm{an}}(X) \simeq \mathrm{H}^2(X, \mathbb{Z})$. *When* X *is the analyticization of a complex algebraic variety* X_{alg}, *the natural map* $\mathrm{Pic}_{\mathrm{alg}}(X_{\mathrm{alg}}) \to \mathrm{Pic}_{\mathrm{an}}(X)$ *need not be isomorphism.*

Theorem (Cartan-Thullen)

The following statements are equivalent for a domain $U \subset \mathbb{C}^d$:

- *U is Stein*
- *U is a domain of holomorphy.*

Let $X = U \subseteq \mathbb{C}^d$ be a domain of holomorphy. Then U is holomorphically parallelizable (and hence Calabi-Yau). Moreover, TU admits global holomorphic frames given by $u_i = \partial_i$, where $\partial_i := \frac{\partial}{\partial z^i}$ and $\{z^1, \ldots, z^d\}$ is a system of globally-defined complex coordinates on U. Assume that $W \in \mathrm{O}(U)$ has isolated critical points. Then:

$$\mathcal{H} = \mathcal{H}^{\hat{0}} \simeq_{\mathrm{O}(U)} \mathrm{Jac}(U, W) = \mathrm{O}(U)/\langle \partial_1 W, \ldots, \partial_d W \rangle \ .$$

Let us further assume that U is contractible. Then any finitely-generated projective $\mathrm{O}(U)$-module is free and projective analytic factorizations coincide with analytic matrix factorizations. Thus $\mathcal{T} \simeq \mathrm{HMF}(U, W)$.

Let X be a smooth, connected and non-compact Riemann surface without boundary. Every such surface is Stein by a result of Behnke and Stein. Moreover, any holomorphic vector bundle on X is holomorphically trivial (Grauert and Röhrl). In particular, X is holomorphically parallelizable and hence Calabi-Yau.

Proposition

Let $W \in \mathrm{O}(X)$ be a non-constant holomorphic function. Then:

$$\mathcal{H} = \mathcal{H}^{\hat{0}}(X, W) \simeq_{\mathrm{O}(X)} \mathrm{Jac}(X, W) = \mathrm{O}(X)/\langle v(W) \rangle \quad ,$$

where v is any non-trivial globally-defined holomorphic vector field on X. Moreover, there exist equivalences of $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded categories:

$$\mathcal{T} \equiv \mathrm{HF}(X, W) \simeq \mathrm{HMF}(X, W) \quad .$$

Notice that X need not be affine algebraic; in particular, it can have infinite genus and an infinite number of ends.

Proposition

Let $X \subset \mathbb{C}^N$ be an analytic complete intersection of dimension d, defined by the regular sequence of holomorphic functions $f_1, \ldots, f_{N-d} \in \mathrm{O}(\mathbb{C}^N)$. Then X is Stein and holomorphically parallelizable (in particular, Calabi-Yau)

Since $\mathrm{O}(X) \simeq \mathrm{O}(\mathbb{C}^N)/\langle f_1, \ldots, f_{N-d}\rangle$, any $W \in \mathrm{O}(X)$ is the restriction of some $\mathcal{W} \in \mathrm{O}(\mathbb{C}^N)$. We have $Z_{\mathcal{W}} \cap X \subseteq Z_W$, but the inclusion can be strict.

Proposition

Let $X \subset \mathbb{C}^N$ be a non-singular analytic hypersurface defined by the equation $f = 0$. Then the holomorphic vector fields $(v_{ij})_{1 \leq i < j \leq N}$ defined on X through:

$$v_{ij}^k = (\partial_j f)\delta_{ik} - (\partial_i f)\delta_{jk} \quad (k = 1, \ldots, N)$$

generate each fiber of TX. For any $\mathcal{W} \in \mathrm{O}(\mathbb{C}^N)$, the the critical locus Z_W of $W \stackrel{\mathrm{def.}}{=} \mathcal{W}|_X$ is defined by:

$$f = 0\,, \quad \partial_i \mathcal{W} \partial_j f - \partial_j \mathcal{W} \partial_i f = 0 \quad (1 \leq i < j \leq N) \quad .$$

If W has isolated critical points on X, then we have $\mathrm{Jac}(X, W) = \mathrm{O}(\mathbb{C}^N)/I$, where the ideal $I \subset \mathrm{O}(\mathbb{C}^N)$ is generated by f and by the holomorphic functions $\partial_i \mathcal{W} \partial_j f - \partial_j \mathcal{W} \partial_i f$ with $1 \leq i < j \leq N$.

Consider the non-singular hypersurface $X \subset \mathbb{C}^3$ defined by the equation $f(x_1, x_2, x_3) = x_1 e^{x_2} + x_2 e^{x_3} + x_3 e^{x_1} = 0$. Let $\mathcal{W} \in \mathrm{O}(\mathbb{C}^3)$ be the holomorphic function given by $\mathcal{W}(x_1, x_2, x_3) = x_1^{n+1} + x_2 x_3$ (where $n \geq 1$) and let $W \stackrel{\text{def.}}{=} \mathcal{W}|_X \in \mathrm{O}(X)$. The critical locus of $\mathcal{W}$ coincides with the origin of $\mathbb{C}^3$, which lies on X. Thus Z_W contains the point $(0,0,0) \in X$. We have:

$$\partial_1 f = e^{x_2} + x_3 e^{x_1} \ , \ \partial_2 f = e^{x_3} + x_1 e^{x_2} \ , \ \partial_3 f = e^{x_1} + x_2 e^{x_3}$$

and the vector fields of the proposition are:

$$\begin{aligned}
v_{23} &= (0, \partial_3 f, -\partial_2 f) = \left(0, e^{x_1} + x_2 e^{x_3}, -e^{x_3} - x_1 e^{x_2}\right) \\
v_{13} &= (\partial_3 f, 0, -\partial_1 f) = \left(e^{x_1} + x_2 e^{x_3}, 0, -e^{x_2} - x_3 e^{x_1}\right) \\
v_{12} &= (\partial_2 f, -\partial_1 f, 0) = \left(e^{x_3} + x_1 e^{x_2}, -e^{x_2} - x_3 e^{x_1}, 0\right)
\end{aligned}$$

The defining equations of Z_W take the form:

$$\begin{gathered}
x_1 e^{x_2} + x_2 e^{x_3} + x_3 e^{x_1} = 0 \\
(n+1) x_1^n (e^{x_1} + x_2 e^{x_3}) - x_2 (e^{x_2} + x_3 e^{x_1}) = 0 \\
x_3 (e^{x_1} + x_2 e^{x_3}) - x_2 (e^{x_3} + x_1 e^{x_2}) = 0 \\
(n+1) x_1^n (e^{x_3} + x_1 e^{x_2}) - x_3 (e^{x_2} + x_3 e^{x_1}) = 0 \quad .
\end{gathered}$$

The bulk algebra $\mathcal{H}$ is isomorphic with $\mathrm{Jac}(X, W) = \mathrm{O}(\mathbb{C}^3)/I$, where the ideal $I \subset \mathrm{O}(\mathbb{C}^3)$ is generated by f and the four holomorphic functions appearing in the left hand side of the previous system. For generic $n \geq 1$, the transcendental system above admits solutions different from $x_1 = x_2 = x_3 = 0$, so $W = \mathcal{W}|_X$ has critical points on X which differ from the origin.

n	x
1	$(0.512, -0.505, 1.957)$, $(2.048, -2.114, 2.017)$, $(0.450 + 0.985\,\mathbf{i}, -0.241 - 0.613\,\mathbf{i}, -0.848 + 0.747\,\mathbf{i})$, ...
2	$(-0.435, -0.109, 2.314)$, $(-0.385, 0.315, 0.207)$, $(0.604, -0.553, 1.960)$, $(-0.338 - 0.599\,\mathbf{i},\ 0.056 + 0.370\,\mathbf{i},\ 0.050 + 0.678\,\mathbf{i})$, ...
3	$(0.658, -0.583, 1.963)$, $(0.112 - 0.298\,\mathbf{i}, -0.075 + 0.122\,\mathbf{i}, -0.089 + 0.121\,\mathbf{i})$, ...

Figure: Some non-zero critical points of W on X for small n (4 significant digits).

For any $k \in \{0, \ldots, n+1\}$, an example of holomorphic (in fact, algebraic) factorization (E, D_k) of $\mathcal{W}$ on $\mathbb{C}^3$ is given by $E^{\hat{0}} = E^{\hat{1}} = \mathcal{O}_{\mathbb{C}^3}^{\oplus 2}$ with:

$$D_k = \begin{bmatrix} 0 & b_k \\ a_k & 0 \end{bmatrix}, \text{ where } a_k = \begin{bmatrix} x_2 & x_1^{n+1-k} \\ x_1^k & -x_3 \end{bmatrix} \text{ and } b_k = \begin{bmatrix} x_3 & x_1^{n+1-k} \\ x_1^k & -x_2 \end{bmatrix}$$

This induces a holomorphic factorization $(E|_X, D_k|_X)$ of W.

Let $\mathcal{A}$ be a d-dimensional central complex affine hyperplane arrangement. Let $\alpha_H : \mathbb{C}^d \to \mathbb{C}$ be defining linear functional of the hyperplane $H \in \mathcal{A}$. Let $X \stackrel{\text{def.}}{=} \mathbb{C}^d \setminus (\cup_{H\in\mathcal{A}} H)$. Then X is a holomorphically parallelizable (and hence Calabi-Yau) Stein manifold. Moreover, X is the analytic space associated to a non-singular complex affine variety which can be realized as the hypersurface in $\mathbb{C}^{d+1}$ given by the equation:

$$x_{d+1} \prod_{H\in\mathcal{A}} \alpha_H(x_1, \ldots, x_d) = 1 \quad .$$

The Orlik-Solomon algebra. Fix a total ordering of $\mathcal{A}$ and let E be the exterior $\mathbb{Z}$-algebra on the degree one generators $(e_H)_{H\in\mathcal{A}}$ and ∂ be the degree -1 differential on E which is determined uniquely by the condition $\partial(e_H) = 1$ for all $H \in \mathcal{A}$. For any non-empty subset S of $\mathcal{A}$, let e_S be the corresponding Grassmann monomial with respect to the total order chosen on $\mathcal{A}$. We say that S is *dependent* if $\cap_{H\in S} H \neq \emptyset$ and the defining linear functionals α_H of the hyperplanes $H \in S$ are linearly dependent. Let I be the homogeneous ideal of E generated by the elements $\partial(e_S)$ with dependent S. The Orlik-Solomon algebra of $\mathcal{A}$ is the $\mathbb{Z}$-graded quotient algebra $A \stackrel{\text{def.}}{=} E/I$.

Theorem (Brieskorn)

The cohomology ring $\mathrm{H}(X, \mathbb{Z})$ *is isomorphic with* A *as a graded* $\mathbb{Z}$*-algebra. In particular, each cohomology group* $\mathrm{H}^k(X, \mathbb{Z})$ *is a free* $\mathbb{Z}$*-module of finite rank.*

The *intersection poset* L of $\mathcal{A}$ is the bounded lattice formed by those subspaces F of $\mathbb{C}^d$ (called *flats*) which arise as finite intersections of hyperplanes from $\mathcal{A}$, ordered by reverse inclusion. Let $P_X(t) \stackrel{\text{def.}}{=} \sum_{j=0}^{d} \mathrm{rk}\mathrm{H}^j(X, \mathbb{Z})\, t^j$ be the Poincaré polynomial of X.

Proposition (Dimca)

Let μ_L *be the Möbius function of the locally-finite poset* L*. Then* $\mu_L(\mathbb{C}^d, F) \neq 0$ *for all flats* $F \in L$ *and the sign of* $\mu_L(\mathbb{C}^d, F)$ *equals* $(-1)^{\mathrm{codim}_{\mathbb{C}} F}$*. Moreover, we have:*

$$P_X(t) = \sum_{F \in L} |\mu_L(\mathbb{C}^d, F)| t^{\mathrm{codim}_{\mathbb{C}} F} = \sum_{F \in L} \mu_L(\mathbb{C}^d, F)(-t)^{\mathrm{codim}_{\mathbb{C}} F} .$$

In particular, $\mathrm{H}^2(X, \mathbb{Z})$ *is non-trivial iff* $\mathcal{A}$ *contains two distinct hyperplanes.*

One has $\mathrm{rk}\mathrm{H}^j(X, \mathbb{Z}) > 0$ for all $j = 0, \ldots, \mathrm{rk}\mathcal{A}$ and $\mathrm{rk}\mathrm{H}^j(X, \mathbb{Z}) = 0$ for $j > \mathrm{rk}\mathcal{A}$, where $\mathrm{rk}\mathcal{A} \stackrel{\text{def.}}{=} \mathrm{codim}_{\mathbb{C}} [\cap_{H \in \mathcal{A}} H]$ is the *rank* of $\mathcal{A}$. Hence $\mathrm{rk}\mathcal{A} \geq 2$ implies $\mathrm{H}^2(X, \mathbb{Z}) \neq 0$.

Example

Let $\mathcal{A}$ be the hyperplane arrangement defined in $\mathbb{C}^3$ by the six linear functionals $x, y, z, x-y, x-z$ and $y-z$ and let $X = \mathbb{C}^3 \setminus (\cup_{H\in\mathcal{A}} H)$. Then $P_X(t) = 1 + 6t + 11t^2 + 6t^3$. In particular, we have $\mathrm{H}^2(X, \mathbb{Z}) = \mathbb{Z}^{11}$.

Example

Let $\mathcal{A}$ be the Boolean arrangement, defined in $\mathbb{C}^d$ by $x_1 \cdot \ldots \cdot x_d = 0$. Then $X = (\mathbb{C}^*)^d$, $I = 0$ and A is the Grassmann $\mathbb{Z}$-algebra E on d generators. Thus $P_X(t) = (t+1)^d$ and $\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{\frac{d(d-1)}{2}}$. Write $x_k = r_k e^{2\pi \mathrm{i} \theta_k}$ with $r_k > 0$ and real θ_k and let $T^d \stackrel{\mathrm{def.}}{=} \{x \in (\mathbb{C}^*)^d \,\big|\, |x_1| = \ldots = |x_d| = 1\} \simeq (\mathrm{S}^1)^d$. Then the map $\pi : X \to T^d$ given by $\pi(x_1, \ldots, x_d) = \left(\frac{x_1}{|x_1|}, \ldots, \frac{x_d}{|x_d|}\right)$ is a homotopy retraction of X onto T^d. Thus $\mathrm{H}^2(T^d, \mathbb{Z}) \simeq \mathrm{H}^2(X, \mathbb{Z})$. Moreover, X is the analytic space associated to the affine variety:

$$X_{\mathrm{alg}} \stackrel{\mathrm{def.}}{=} \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}] = \mathrm{Spec}\Big(\mathbb{C}[x_1, \ldots, x_{d+1}]/\langle x_1 \ldots x_{d+1} - 1\rangle\Big) .$$

We have $\mathrm{Pic}_{\mathrm{alg}}(X) = 0$ (since $\mathbb{C}[x_1, \ldots, x_d, x_1^{-1}, \ldots x_d^{-1}]$ is a UFD) while $\mathrm{Pic}_{\mathrm{an}}(X) \simeq \mathbb{Z}^{\frac{d(d-1)}{2}}$.

Let $X \simeq (\mathbb{C}^*)^2$ be the complement of the 2-dimensional Boolean hyperplane arrangement, which embeds in $\mathbb{C}^3$ as the affine hypersurface $x_1 x_2 x_3 = 1$. We have $\mathrm{Pic}_{\mathrm{alg}}(X) = 0$ but $\mathrm{Pic}_{\mathrm{an}}(X) \simeq \mathrm{H}^2(X, \mathbb{Z}) \simeq \mathbb{Z}$. Write $X = \mathbb{C}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by:

$$(n_1, n_2) \cdot (z_1, z_2) \stackrel{\mathrm{def.}}{=} (z_1 + n_1, z_2 + n_2) \ \ , \ \ \forall (z_1, z_2) \in \mathbb{C}^2 \ , \ \forall (n_1, n_2) \in \mathbb{Z}^2 \ \ .$$

Consider the lattice $\Lambda := \mathbb{Z} \oplus \mathrm{i}\mathbb{Z} \subset \mathbb{C}$ and the elliptic curve $X_0 \stackrel{\mathrm{def.}}{=} \mathbb{C}/\Lambda$ of modulus $\tau = \mathrm{i}$. The maps $\varphi^{\pm} : \mathbb{C}^2 \to \mathbb{C}$ given by $\varphi^{\pm}(z_1, z_2) = z_1 \pm \mathrm{i} z_2$ induce homotopy retractions $\boldsymbol{\varphi}^{\pm} : X \to X_0$ and isomorphisms $\boldsymbol{\varphi}^{\pm} : \mathrm{NS}(X_0) \to \mathrm{H}^2(X, \mathbb{Z})$ which differ by sign, where $\mathrm{NS}(X_0) \stackrel{\mathrm{def.}}{=} \mathrm{Pic}_{\mathrm{an}}(X_0)/\mathrm{Pic}^0_{\mathrm{an}}(X_0) = \mathrm{Pic}_{\mathrm{alg}}(X_0)/\mathrm{Pic}^0_{\mathrm{alg}}(X_0) \simeq \mathrm{H}^2(X_0, \mathbb{Z})$ is the Neron-Severi group. Let $p_0 \in X_0$ be the $\mathrm{mod}\,\Lambda$ image of the point $z_0 = \frac{1+\mathrm{i}}{2} \in \mathbb{C}$ and consider the holomorphic line bundle $\mathcal{L} = \mathcal{O}_{X_0}(p_0)$ on X_0. The class of $\mathcal{L}$ modulo $\mathrm{Pic}^0_{\mathrm{an}}(X_0)$ generates $\mathrm{NS}(X_0)$ and each of the holomorphic line bundles $L_{\pm} \stackrel{\mathrm{def.}}{=} (\boldsymbol{\varphi}^{\pm})^*(\mathcal{L})$ generates $\mathrm{Pic}_{\mathrm{an}}(X)$. We have $L_- = L_+^{-1}$. Up to multiplication by a non-zero complex number, there exists a unique holomorphic section of $\mathcal{O}_{X_0}(p_0)$ which vanishes at p_0. A convenient choice $s_0 \in \Gamma(\mathcal{O}_{X_0}(p_0))$ is described by the Riemann-Jacobi theta function (traditionally denoted by ϑ_{00} or ϑ_3) at modulus $\tau = \mathrm{i}$:

$$\vartheta(z) = \vartheta_{00}(z)|_{\tau=\mathrm{i}} = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi \mathrm{i} n z} \quad .$$

This satisfies:

$$\vartheta(z+1)=\vartheta(z) \quad , \quad \vartheta(z\pm \mathrm{i})=e^{\pi\mp 2\pi \mathrm{i} z}\vartheta(z)$$

and vanishes on the lattice $\frac{1+\mathrm{i}}{2}+\Lambda$. The $\varphi^\pm$-pullbacks of s_0 give global holomorphic sections $s_\pm\in \mathrm{H}^0(L_\pm)$, which are described by the $\mathbb{Z}^2$-quasiperiodic holomorphic functions $f_\pm \in \mathrm{O}(\mathbb{C}^2)$ defined through $f_\pm(z_1,z_2)\stackrel{\mathrm{def.}}{=}\vartheta(z_1\pm \mathrm{i} z_2)=\sum_{n\in \mathbb{Z}} e^{-\pi n^2+2\pi \mathrm{i} n(z_1\pm \mathrm{i} z_2)}$. The tensor product $s_+\otimes s_-\in \mathrm{H}^0(L_+\otimes L_-)\simeq \mathrm{O}(X)$ corresponds to the holomorphic function $f\stackrel{\mathrm{def.}}{=} f_+f_-\in \mathrm{O}(\mathbb{C}^2)$, which satisfies:

$$f(z_1+1,z_2)=f(z_1,z_2) \quad , \quad f(z_1,z_2+1)=e^{2\pi+4\pi z_2}f(z_1,z_2) \quad .$$

The isomorphism $L_+\otimes L_-\simeq \mathcal{O}_X$ is realized on $\mathbb{Z}^2$-factors of automorphy by the holomorphic function $S:\mathbb{C}^2\rightarrow \mathbb{C}^\ast$ given by $S(z_1,z_2)\stackrel{\mathrm{def.}}{=} e^{-2\pi z_2^2}$, which satisfies $\frac{S(z_1,z_2)}{S(z_1,z_2+1)}=e^{2\pi+4\pi z_2}$. The section $s_+\otimes s_-\in \mathrm{H}^0(L_+\otimes L_-)$ corresponds through this isomorphism to a holomorphic function $W\in \mathrm{H}^0(\mathcal{O}_X)=\mathrm{O}(X)$ whose lift to $\mathbb{C}^2$ is the $\mathbb{Z}^2$-periodic function:

$$\widetilde{W}(z_1,z_2)\stackrel{\mathrm{def.}}{=} S(z_1,z_2)f(z_1,z_2)=e^{-2\pi z_2^2}\vartheta(z_1+\mathrm{i} z_2)\vartheta(z_1-\mathrm{i} z_2) \quad .$$

This gives a topologically non-trivial elementary holomorphic factorization (E,D) of W, where $E=\mathcal{O}_X\oplus L_+$ and $D=\left[\begin{array}{cc} 0 & s_- \\ s_+ & 0\end{array}\right]$.

Proposition

Let Y be a non-singular complex projective Fano variety and D be a smooth anticanonical divisor on Y. Then the complement $X := Y \setminus D$ is a non-compact Stein Calabi-Yau manifold.

The Stein manifolds produced by this proposition are analytifications of non-singular affine varieties.

Example

Let $\mathbb{P}^d$ be the d-dimensional projective space with $d \geq 2$. In this case, we have $K_{\mathbb{P}^d} = \mathrm{O}_{\mathbb{P}^d}(-d-1)$. Let D be an irreducible smooth hypersurface of degree $d+1$ in $\mathbb{P}^d$. Notice that D defines an anticanonical divisor in $\mathbb{P}^d$ (which is a Fano manifold). The complement $X = \mathbb{P}^d \setminus D$ is a Stein Calabi-Yau manifold. In this example, we have $\mathrm{H}_1(X, \mathbb{Z}) = \mathbb{Z}_{d+1}$. By the universal coefficient theorem, the torsion part of $\mathrm{H}^2(X, \mathbb{Z})$ is isomorphic with $\mathrm{Ext}^1(\mathrm{H}_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Ext}^1(\mathbb{Z}_{d+1}, \mathbb{Z}) \simeq \mathbb{Z}_{d+1}$. Thus X admits non-trivial holomorphic line bundles and hence topologically non-trivial elementary holomorphic factorizations.

Let Y be a Stein manifold and $\pi : E \to Y$ be a holomorphic vector bundle. Then the total space X of E is Stein . The complex tangent bundle $\mathcal{T}X$ of X is an extension:

$$0 \to \pi^* E \to \mathcal{T}X \to \pi^* \mathcal{T}Y \to 0 \quad ,$$

hence the Chern polynomial $c_t(\mathcal{T}X)$ equals $\pi^* \left(c_t(E)\right) \cdot \pi^*(c_t(\mathcal{T}Y))$. The projection π is a homotopy equivalence since each fiber of E is contractible to a point; hence $\pi^* : \mathrm{H}(Y, \mathbb{Z}) \to \mathrm{H}(X, \mathbb{Z})$ is an isomorphism of groups. In particular, the first Chern class of X vanishes iff $c_1(E) + c_1(Y) = 0$ (i.e. $c_1(E) = c_1(K_Y)$), in which case X is a Stein Calabi-Yau manifold.

A particular case of this construction is obtained by taking E to be the canonical line bundle K_Y of Y. In this case, the first Chern class of X vanishes automatically. For example, let $Y \subset \mathbb{P}^n$ be the complement of an algebraic hypersurface $Z \subset \mathbb{P}^n$. Then the total space X of K_Y is Stein and Calabi-Yau.

Consider the (non-anticanonical) surface $S = \{[x, y, z] \in \mathbb{P}^2 \,|\, x^2 + y^2 + z^2 = 0\}$.

Proposition (Forster)

The complement $Y \stackrel{\text{def.}}{=} \mathbb{P}^2 \setminus S$ is Stein but not Calabi-Yau. Moreover, we have $\mathrm{H}^2(Y, \mathbb{Z}) \simeq \mathbb{Z}_2$ and $c_1(Y) = \gamma$, where γ is the generator of $\mathrm{H}^2(Y, \mathbb{Z})$.

Thus K_Y is non-trivial with $c_1(K_Y) = -\gamma$. Let X be the total space of K_Y. Then X is Stein and Calabi-Yau. Moreover, the pull-back L of K_Y to X has non-trivial first Chern class. Hence the $\mathbb{Z}_2$-graded holomorphic vector bundle $E = \mathcal{O}_X \oplus L$ supports topologically non-trivial holomorphic factorizations of some non-zero function $W \in \mathrm{O}(X)$.