Non-degeneracy of cohomological traces for general Landau-Ginzburg models

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- C. I. Lazaroiu, M. Babalic, D. Doryn, M. Tavakol, *Differential models for B-type open-closed topological Landau-Ginzburg theories*, Comun. Math. Phys (2018)
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1. Axiomatics of 2-dimensional oriented and non-anomalous open-closed TFTs
2. A framework for B-type topological Landau-Ginzburg models
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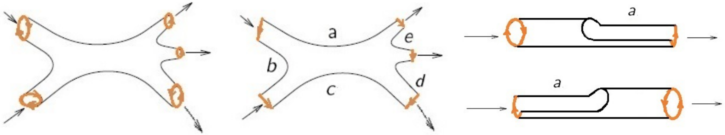
A non-anomalous oriented quantum 2-dimensional open-closed TFT (with finite-dimensional open and closed state spaces) can be described axiomatically as a symmetric monoidal functor from a symmetric monoidal category Cob_2 of oriented and labeled cobordisms with corners to the symmetric monoidal category $\mathrm{vect}^s_{\mathbb{C}}$ of finite-dimensional $\mathbb{Z}_2$-graded $\mathbb{C}$-vector spaces:

$$Z : (\mathrm{Cob}_2, \sqcup, \emptyset) \longrightarrow (\mathrm{vect}^s_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$$

The objects of Cob_2 are finite disjoint unions of oriented circles and oriented closed segments. The morphisms are compact oriented 2-manifolds with corners (corresponding to worldsheets of open and closed strings). The monoidal product is the disjoint union, while the composition is the sewing of cobordisms.



The labels associated to the ends of the intervals indicate the boundary conditions/D-branes.

The TFT datum of a non-anomalous and oriented open-closed 2d TFT

Theorem (C.I.L. 2000)

A non-anomalous oriented 2-dimensional open-closed TFT is equivalent to an algebraic structure known as a **TFT datum***.*

To define a TFT datum, we first define the notions of **pre-TFT datum** and **Calabi-Yau supercategory**.

Pre-TFT datum

Definition

A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

- The **bulk algebra** $\mathcal{H}$, which is a finite-dimensional supercommutative $\mathbb{C}$-superalgebra with unit $1_{\mathcal{H}}$. This describes the algebra of on-shell states of the closed oriented topological string.
- The **category of topological D-branes**, which is a Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category, with composition of morphisms denoted by $\circ$ and units:

  $$1_a \in \mathrm{Hom}_{\mathcal{T}}(a, a) \ , \ \forall a \in \mathrm{Ob}\mathcal{T}$$

  The objects of $\mathcal{T}$ describe the topological D-branes, while $\mathrm{Hom}_{\mathcal{T}}(a, b)$ describes the space of on-shell boundary states of the *open* oriented topological string stretching from the D-brane a to the D-brane b.
- A family $e = (e_a)_{a \in \mathrm{Ob}\mathcal{T}}$ of $\mathbb{C}$-linear **bulk-boundary maps** $e_a : \mathcal{H} \to \mathrm{Hom}_{\mathcal{T}}(a, a)$ such that the following conditions are satisfied:
  - For any $a \in \mathrm{Ob}\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the algebra $(\mathrm{End}_{\mathcal{T}}(a), \circ)$, where $\mathrm{End}_{\mathcal{T}}(a) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathcal{T}}(a, a)$.
  - For any $a, b \in \mathrm{Ob}\mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous bulk state $h \in \mathcal{H}$ and any $\mathbb{Z}_2$-homogeneous elements $t \in \mathrm{Hom}_{\mathcal{T}}(a, b)$, we have:

    $$e_b(h) \circ t = (-1)^{\deg h \deg t} t \circ e_a(h) \ \ .$$

Definition

A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category,
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in Ob\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-**degree** μ:

$$\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C} \quad ,$$

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$, the $\mathbb{C}$-bilinear pairing:

$$\langle \cdot, \cdot \rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a, b) \times \mathrm{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$$

  defined through:

$$\langle t_1, t_2 \rangle_{a,b} = \mathrm{tr}_b(t_1 \circ t_2), \ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b), \ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$$

  is non-degenerate.
- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$, we have:

$$\langle t_1, t_2 \rangle_{a,b} = (-1)^{\deg t_1 \deg t_2} \langle t_2, t_1 \rangle_{b,a} \quad .$$

Definition

A TFT datum of parity $\mu \in \mathbb{Z}_2$ is a quintuple $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a **pre-TFT datum**
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the **bulk trace**),
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called **boundary traces**),

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra, meaning that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate (i.e. the condition $\mathrm{Tr}(hh') = 0$ for all $h' \in \mathcal{H}$ implies $h = 0$)
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi-Yau supercategory of parity μ.
- The so-called *topological Cardy constraint* holds for all $a, b \in \mathrm{Ob}\mathcal{T}$.

The topological Cardy constraint has the form:

$$\boxed{\mathrm{Tr}(f_a(t_a)f_b(t_b)) = \mathrm{str}(\Phi_{ab}(t_a,t_b)) \ \ , \ \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)} \ ,$$

where:

- "str" is the supertrace on the $\mathbb{Z}_2$-graded vector space $End_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{T}}(a,b))$.
- $f_a : \mathrm{Hom}_{\mathcal{T}}(a,a) \to \mathcal{H}$ is the *boundary-bulk map of a*, which is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to Hom_{\mathcal{T}}(a,a)$ with respect to Tr and tr:

$$\mathrm{Tr}(hf_a(t_a)) = \mathrm{tr}_a(e_a(h)\circ t_a), \ \forall h \in \mathcal{H}, \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a)$$

- $\Phi_{ab}(t_a,t_b) : \mathrm{Hom}_{\mathcal{T}}(a,b) \to \mathrm{Hom}_{\mathcal{T}}(a,b)$ is the $\mathbb{C}$-linear map defined through:

$$\Phi_{ab}(t_a,t_b)(t) = t_b\circ t\circ t_a \ , \ \forall t \in \mathrm{Hom}_{\mathcal{T}}(a,b) \ , \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b) \ .$$

Definition

A *Landau-Ginzburg (LG) pair* is a pair (X, W) such that:

- X is a *non-compact* Kählerian manifold (called **target space**), whose canonical line bundle K_X is holomorphically trivial.
- $W : X \to \mathbb{C}$ is a *non-constant* holomorphic function (called **superpotential**).

Let $d \stackrel{\text{def.}}{=} \dim_{\mathbb{C}} X$.

Non-anomalous two-dimensional open-closed topological B-type Landau-Ginzburg models with D-branes can be associated to any LG pair. In general, such models are *not* scale invariant and hence are B-type twists of two-dimensional $\mathcal{N} = 2$ supersymmetric field theories which are *not* conformally invariant (the latter have non-anomalous axial $\mathrm{U}(1)$ R-symmetry but no vector $\mathrm{U}(1)$ R-symmetry).

Definition

The *critical set of W* is the set of its critical points:

$$Z_W \stackrel{\text{def.}}{=} \{p \in X | (\partial W)(p) = 0\} .$$

Definition

The *signature* of a Landau-Ginzburg pair (X, W) is defined as the mod 2 reduction of the complex dimension of X:

$$\mu(X, W) \stackrel{\mathrm{def.}}{=} \hat{d} \in \mathbb{Z}_2$$

Technical Assumption

We will assume that the critical set Z_W is compact. This insures finite-dimensionality of the (on-shell) closed and open topological string state spaces.

Remarks

- *The class of all non-compact Kählerian manifolds is extremely large.*
- *Stein manifolds form a* **very special** *sub-class of Kählerian manifolds.*
- *Analyticizations of non-singular complex affine varieties form a* **very special** *subclass of Stein manifolds.*

B-type topological Landau-Ginzburg models

In our previous work, we made a mathematically rigorous proposal for the TFT datum associated to a general Landau-Ginzburg pair; this proposal is inspired by path integral arguments.

For any Landau-Ginzburg pair (X, W), we proved that our proposal satisfies all axioms of a TFT datum (including the non-degeneracy of bulk and boundary traces), except for the topological Cardy constraint (whose proof in full generality is work in progress).

Modulo the proof of the topological Cardy constraint, our proposal amounts to a rigorous construction of the non-anomalous quantum oriented open-closed B-type Landau-Ginzburg 2-dimensional topological field theory in such extreme generality.

Remark

According to our proposal, the TFT datum of the B-type Landau-Ginzburg theory associated to a Landau-Ginzburg pair (X, W) is not only $\mathbb{C}$-linear but also **partially $\mathrm{O}(X)$-linear**, *where $\mathrm{O}(X)$ is the unital commutative ring of complex-valued holomorphic functions defined on X. Namely:*

- *$\mathcal{H}$ is an $\mathrm{O}(X)$-module (which is finite-dimensional over $\mathbb{C} \subset \mathrm{O}(X)$).*
- *$\mathcal{T}$ is an $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded category (which is Hom-finite over $\mathbb{C}$).*
- *The maps e_a and f_a are $\mathrm{O}(X)$-linear.*
- *However, the bulk and boundary traces Tr and tr_a* are only $\mathbb{C}$-linear.

Definition

Let (X, W) be a Landau-Ginzburg pair with $\dim_{\mathbb{C}} X = d$. The space of polyvector fields is defined through:

$$\mathrm{PV}(X) \stackrel{\text{def.}}{=} \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathrm{PV}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathcal{A}^j(X, \wedge^{|i|} TX)$$

where $\mathcal{A}^j(X, \wedge^{|i|} TX) \equiv \Omega^{0,j}(X, \wedge^{|i|} TX)$.

The *twisted Dolbeault differential* determined by W on $\mathrm{PV}(X)$ is the operator

$$\delta_W : \mathrm{PV}(X) \to \mathrm{PV}(X)$$

defined through $\delta_W = \boldsymbol{\overline{\partial}} + \boldsymbol{\iota}_W$, where:

- $\boldsymbol{\iota}_W = -\mathrm{i}(\partial W) \lrcorner$, which satisfies $\boldsymbol{\iota}_W(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i+1,j}(X)$
- $\boldsymbol{\overline{\partial}}$ is the (antiholomorphic) Dolbeault operator of $\wedge TX$, which satisfies $\boldsymbol{\overline{\partial}}(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i,j+1}(X)$

Notice that $(\mathrm{PV}(X), \boldsymbol{\overline{\partial}}, \boldsymbol{\iota}_W)$ is a bicomplex of $\mathrm{O}(X)$-modules

$$\delta_W^2 = \boldsymbol{\overline{\partial}}^2 = \boldsymbol{\iota}_W^2 = \boldsymbol{\overline{\partial}}\boldsymbol{\iota}_W + \boldsymbol{\iota}_W\boldsymbol{\overline{\partial}} = 0$$

Definition

The twisted Dolbeault algebra of polyvectors of the Landau-Ginzburg pair (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear dG algebra $(\mathrm{PV}(X), \delta_W)$, where $\mathrm{PV}(X)$ is endowed with its total $\mathbb{Z}$-grading.

Definition

The cohomological twisted Dolbeault algebra of (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear algebra given by the total cohomology of the dGA $(\mathrm{PV}(X), \delta_W)$:

$$\mathrm{HPV}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{PV}(X), \delta_W) \quad .$$

Definition

The *sheaf Koszul complex* of W is the following complex of locally-free sheaves of $\mathcal{O}_X$-modules:

$$(\mathcal{K}_W): \quad 0 \to \wedge^d TX \stackrel{\iota_W}{\to} \wedge^{d-1} TX \stackrel{\iota_W}{\to} \cdots \stackrel{\iota_W}{\to} \mathcal{O}_X \to 0$$

where $\mathcal{O}_X$ sits in degree zero and we identify the exterior power $\wedge^k TX$ with its locally-free sheaf of holomorphic sections.

An analytic model for the off-shell bulk algebra

Proposition

Let $\mathbb{H}(\mathcal{K}_W)$ denote the hypercohomology of the Koszul complex $\mathcal{K}_W$. There exists a natural isomorphism of $\mathbb{Z}$-graded $\mathrm{O}(X)$-modules:

$$\mathrm{HPV}(X, W) \simeq_{\mathrm{O}(X)} \mathbb{H}(\mathcal{K}_W) \quad .$$

Moreover, we have:

$$\mathbb{H}^k(\mathcal{K}_W) = \bigoplus_{i+j=k} \mathbf{E}_\infty^{i,j}$$

where $(\mathbf{E}_r^{i,j}, d_r)_{r\geq 0}$ is a spectral sequence which starts with:

$$\mathbf{E}_0^{i,j} := \mathrm{PV}^{i,j}(X) = \mathcal{A}^j(X, \wedge^{|i|} TX), \;\; \mathrm{d}_0 = \boldsymbol{\overline{\partial}} \;\; , \; (i = -d, \ldots, 0, \; j = 0, \ldots, d)$$

Spectral sequence

$$
\begin{array}{ccccccc}
\mathbf{E}_0^{-d,d} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+1,d} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+2,d} & \cdots\cdots & \mathbf{E}_0^{0,d} \\
\vdots & & \vdots & & \vdots & \iddots & \vdots \\
\mathbf{E}_0^{-d,2} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+1,2} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+2,2} & \cdots\cdots & \mathbf{E}_0^{0,2} \\
\uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} \\
\mathbf{E}_0^{-d,1} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+1,1} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+2,1} & \cdots\cdots & \mathbf{E}_0^{0,1} \\
\uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} \\
\mathbf{E}_0^{-d,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+1,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_0^{-d+2,0} & \cdots\cdots & \mathbf{E}_0^{0,0}
\end{array}
$$

The zeroth page of the spectral sequence.

The category of topological D-branes

Definition

A holomorphic vector superbundle on X is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on X, i.e. a complex holomorphic vector bundle E endowed with a direct sum decomposition $E = E^{\hat{0}} \oplus E^{\hat{1}}$, where $E^{\hat{0}}$ and $E^{\hat{1}}$ are holomorphic sub-bundles of E.

Definition

A holomorphic factorization of W is a pair $a = (E, D)$, where $E = E^{\hat{0}} \oplus E^{\hat{1}}$ is a holomorphic vector superbundle on X and $D \in \Gamma_{\mathrm{sm}}(X, End^{\hat{1}}(E))$ is a holomorphic section of the bundle $End^{\hat{1}}(E) = Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \subset End(E)$ which satisfies the condition $D^2 = W\mathrm{id}_E$.

Let $a = (E, D)$ be a holomorphic factorization of W. Decomposing $E = E^{\hat{0}} \oplus E^{\hat{1}}$, the condition that D is odd implies:

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$$

where $u \in \Gamma_{\mathrm{sm}}(X, Hom(E^{\hat{0}}, E^{\hat{1}}))$ and $v \in \Gamma_{\mathrm{sm}}(X, Hom(E^{\hat{1}}, E^{\hat{0}}))$. The condition $D^2 = W\mathrm{id}_E$ amounts to:

$$v \circ u = W\mathrm{id}_{E^{\hat{0}}} \ , \quad u \circ v = W\mathrm{id}_{E^{\hat{1}}}$$

Definition

The twisted Dolbeault category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category $\mathrm{DF}(X, W)$ defined as follows:

- The objects of $\mathrm{DF}(X, W)$ are the holomorphic factorizations of W.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$:

$$\mathrm{Hom}_{\mathrm{DF}(X,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \mathcal{A}(X, Hom(E_1, E_2))$$

  endowed with the total $\mathbb{Z}_2$-grading and with the twisted differentials δ_{a_1,a_2}:

$$\delta_{a_1,a_2} \stackrel{\mathrm{def.}}{=} \boldsymbol{\overline{\partial}}_{a_1,a_2} + \mathfrak{d}_{a_1,a_2} \quad , \quad \textit{where}$$
$$\boldsymbol{\overline{\partial}}_{a_1,a_2} := \boldsymbol{\overline{\partial}}_{Hom(E_1,E_2)}$$
$$\mathfrak{d}_{a_1,a_2}(\rho \otimes f) = (-1)^{\mathrm{rk}\rho} \rho \otimes (D_2 \circ f) - (-1)^{\mathrm{rk}\rho + \sigma(f)} \rho \otimes (f \circ D_1)$$

- The composition of morphisms $\circ : \mathcal{A}(X, Hom(E_2, E_3)) \times \mathcal{A}(X, Hom(E_1, E_2)) \to \mathcal{A}(X, Hom(E_1, E_3))$ is determined uniquely through the condition:

$$(\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f)\,\mathrm{rk}\eta} (\rho \wedge \eta) \otimes (f \circ g)$$

  for all pure rank forms $\rho, \eta \in \mathcal{A}(X)$ and all pure $\mathbb{Z}_2$-degree elements $f \in \Gamma_{\mathrm{sm}}(X, Hom(E_2, E_3))$ and $g \in \Gamma_{\mathrm{sm}}(X, Hom(E_1, E_2))$.

We have $\boldsymbol{\overline{\partial}}^2 = \mathfrak{d}^2 = \boldsymbol{\overline{\partial}} \circ \mathfrak{d} + \mathfrak{d} \circ \boldsymbol{\overline{\partial}} = 0$.

Let Ω be a holomorphic volume form on X and $\mathcal{A}_c(X) \stackrel{\rm def.}{=} \mathcal{A}_c(X, \mathcal{O}_X)$ be the space of compactly supported forms of type $(0,*)$ on X.

Definition

The *Serre trace* induced by Ω on $\mathcal{A}_c(X)$ is the $\mathbb{C}$-linear map $\int_\Omega : \mathcal{A}_c(X) \to \mathbb{C}$ defined through:

$$\int_\Omega \rho \stackrel{\rm def.}{=} \int_X \Omega \wedge \rho \quad , \quad \forall \rho \in \mathcal{A}_c(X) \quad .$$

Definition

The *canonical off-shell trace induced by* Ω *on* $\mathrm{PV}_c(X)$ is the $\mathbb{C}$-linear map $\mathrm{Tr}_B := \mathrm{Tr}_B^\Omega : \mathrm{PV}_c(X) \to \mathbb{C}$ defined through:

$$\mathrm{Tr}_B^\Omega(\omega) = \int_X \Omega \wedge (\Omega \lrcorner \omega) = \int_\Omega \Omega \lrcorner \omega \quad , \quad \forall \omega \in \mathrm{PV}_c(X) \quad .$$

Proposition

For any $\eta \in \mathrm{PV}_c(X)$, *we have:*

$$\mathrm{Tr}_B(\delta_W \eta) = \mathrm{Tr}_B(\boldsymbol{\overline{\partial}} \eta) = \mathrm{Tr}_B(\boldsymbol{\iota}_W \eta) = 0 \quad .$$

In particular, Tr_B *descends to* $\mathrm{HPV}_c(X, W)$.

The cohomological bulk trace induced by a holomorphic volume form

Definition

The *cohomological trace induced by* Ω *on* $\mathrm{HPV}_c(X, W)$ is the $\mathbb{C}$-linear map $\mathrm{Tr}_c := \mathrm{Tr}_c^\Omega : \mathrm{HPV}_c(X, W) \to \mathbb{C}$ induced by Tr_B^Ω.

Proposition

Assume that the critical set Z_W *is compact. In this case,* $\mathrm{HPV}(X, W)$ *is finite-dimensional and the inclusion map* $i : \mathrm{PV}_c(X, W) \hookrightarrow \mathrm{PV}(X, W)$ *induces a linear isomorphism* $i_* : \mathrm{HPV}_c(X, W) \xrightarrow{\sim} \mathrm{HPV}(X, W)$.

Definition

The *cohomological bulk trace induced by* Ω is the $\mathbb{C}$-linear map $\mathrm{Tr} := \mathrm{Tr}^\Omega \stackrel{\text{def.}}{=} \mathrm{Tr}_c^\Omega \circ i_*^{-1} : \mathrm{HPV}(X, W) \to \mathbb{C}$.

Let $a = (E, D)$ be a holomorphic factorization of W. Let $\delta_a := \delta_{a,a}$ and $\mathfrak{d}_a := \mathfrak{d}_{a,a}$ denote the twisted Dolbeault and defect differentials on $\mathrm{End}_{\mathrm{DF}(X,W)}(a)$. Let $\boldsymbol{\bar\partial}_a := \boldsymbol{\bar\partial}_{a,a} = \boldsymbol{\bar\partial}_{End(E)}$ denote the Dolbeault operator of $End(E)$. We have:

$$\delta_a = \boldsymbol{\bar\partial}_a + \mathfrak{d}_a \quad , \quad \mathfrak{d}_a = [D, \cdot] \quad ,$$

where $[\cdot, \cdot]$ denotes the graded commutator.

Definition

The *canonical off-shell boundary trace* induced by Ω on $\mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}_a^B := \mathrm{tr}_a^{B,\Omega} : \mathrm{End}_{\mathrm{DF}_c(X,W)}(a) \to \mathbb{C}$ defined through:

$$\mathrm{tr}_a^{B,\Omega}(\alpha) = \int_X \Omega \wedge \mathrm{str}(\alpha) = \int_\Omega \mathrm{str}(\alpha) \quad ,$$

for all $\alpha \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a) = \mathcal{A}_c(X, End(E))$, where str denotes the extended supertrace.

Proposition

For any holomorphic factorizations a_1 *and* a_2 *of* W*, we have:*

$$\mathrm{tr}^B_{a_2}(\alpha\beta) = (-1)^{\deg\alpha\,\deg\beta}\mathrm{tr}^B_{a_1}(\beta\alpha) \quad ,$$

when $\alpha \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_1, a_2)$ *and* $\beta \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_2, a_1)$ *have pure total* $\mathbb{Z}_2$*-degree.*

Proposition

For any $\alpha \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$*, we have:*

$$\mathrm{tr}^B_a(\delta_a\alpha) = \mathrm{tr}^B_a(\boldsymbol{\overline{\partial}}_a\alpha) = \mathrm{tr}^B_a(\mathfrak{d}_a\alpha) = 0 \quad .$$

In particular, tr^B_a *descends to* $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) = \mathrm{H}(\mathcal{A}_c(X, End(E)), \delta_a)$.

Definition

The *cohomological boundary trace induced by* Ω *on* $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}^c_a := \mathrm{tr}^{c,\Omega}_a : \mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) \to \mathbb{C}$ induced by $\mathrm{tr}^{B,\Omega}_a$ on $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a)$.

Proposition

Assume that the critical set Z_W is compact. In this case, $\mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a,b)$ is finite-dimensional for all holomorphic factorizations a, b of W and the inclusion map $j_{a,b} : \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a,b) \hookrightarrow \mathrm{Hom}_{\mathrm{DF}(X,W)}(a,b)$ induces a linear isomorphism $j_{*,a,b} : \mathrm{Hom}_{\mathrm{HDF}_c(X,W)}(a,b) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{HDF}(X,W)}(a,b)$.

Definition

Suppose that the critical locus Z_W is compact. Then the *cohomological boundary trace induced by* Ω *on* $\mathrm{End}_{\mathrm{HDF}(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}_a \stackrel{\mathrm{def.}}{=} \mathrm{tr}_a^c \circ j_{*,a,a}^{-1} : \mathrm{End}_{\mathrm{HDF}(X,W)}(a) \to \mathbb{C}$.

Theorem (A)

Suppose that the critical set Z_W is compact. Then the graded-symmetric pairing $\langle \cdot, \cdot \rangle_\Omega : \mathrm{HPV}(X, W) \times \mathrm{HPV}(X, W) \to \mathbb{C}$ given by:

$$\langle u_1, u_2 \rangle_\Omega \stackrel{\mathrm{def.}}{=} \mathrm{Tr}^\Omega(u_1 u_2) \ , \quad \forall u_1, u_2 \in \mathrm{HPV}(X, W)$$

is non-degenerate. Hence $(\mathrm{HPV}(X, W), \mathrm{Tr}^\Omega)$ is a finite-dimensional, $\mathbb{Z}$-graded supercommutative Frobenius algebra over $\mathbb{C}$.

Let $\mathcal{K}_W$ denote the sheaf Koszul complex defined by $\boldsymbol{\iota}_W$:

$$(\mathcal{K}_W): \qquad 0 \longrightarrow \wedge^d TX \xrightarrow{\boldsymbol{\iota}_W} \wedge^{d-1} TX \xrightarrow{\boldsymbol{\iota}_W} \ldots \xrightarrow{\boldsymbol{\iota}_W} TX \xrightarrow{\boldsymbol{\iota}_W} \mathcal{O}_X \longrightarrow 0 \ .$$

We have $\mathrm{HPV}^k(X, W) \simeq_{\mathbb{C}} \mathbb{H}^k(\mathcal{K}_W)$ and Theorem A. amounts to:

Theorem (A′)

Suppose that the critical set Z_W is compact. Then for every $k \in \{-d, \ldots, d\}$ we have a linear isomorphism:

$$\mathbb{H}^k(\mathcal{K}_W) \simeq_{\mathbb{C}} \mathbb{H}^{-k}(\mathcal{K}_W)^\vee \ ,$$

which depends on the choice of a holomorphic volume form Ω.

Let tr^Ω denote the family $(\mathrm{tr}_a^\Omega)_{a\in \mathrm{Ob}\mathrm{HDF}(X,W)}$.

Theorem (B)

Suppose that the critical set Z_W is compact. Then the bilinear pairing

$$\langle\cdot,\cdot\rangle_{a_1,a_2}^\Omega : \mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_1,a_2)\times \mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_2,a_1)\longrightarrow \mathbb{C}$$

defined through:

$$\langle t_1,t_2\rangle_{a_1,a_2}^\Omega \stackrel{\mathrm{def.}}{=} \mathrm{tr}_{a_2}^\Omega(t_1t_2)~,~\forall t_1\in \mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_1,a_2)~,~\forall t_2\in \mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_2,a_1)$$

is non-degenerate for any two holomorphic factorizations a_1, a_2 of W. Hence $(\mathrm{HDF}(X,W),\mathrm{tr}^\Omega)$ is a Calabi-Yau supercategory of signature $\mu=\hat{d}$.

The $\mathbb{Z}_2$-graded category $\mathrm{HDF}(X,W)$ admits an involutive automorphism Σ and natural isomorphisms:

$$\mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_1,\Sigma(a_2))\simeq \mathrm{Hom}_{\mathrm{HDF}(X,W)}(\Sigma(a_1),a_2)\simeq \Pi\mathrm{Hom}_{\mathrm{HDF}(X,W)}(a_1,a_2)~,$$

where Π is the parity change functor of $\mathrm{vect}^s_{\mathbb{C}}$. Thus $\mathrm{HDF}(X,W)$ can be recovered from its even subcategory $\mathrm{HDF}^{\hat{0}}(X,W)$.

Theorem B. amounts to:

Theorem (B′)

Suppose that the critical set Z_W is compact. Then Σ^d is a Serre functor for the category $\mathrm{HDF}^{\hat{0}}(X, W)$, where:

$$\Sigma^d = \begin{cases} \Sigma & \text{if } d \text{ is odd} \\ \mathrm{id}_{\mathrm{HDF}^{\hat{0}}(X,W)} & \text{if } d \text{ is even} \end{cases} .$$

Notice that Σ^d depends only on the signature $\mu = \hat{d}$.

The differentials $\delta_W = \overline{\partial}_{\wedge TX} + \boldsymbol{\iota}_W$ and $\delta_{a_1,a_2} = \overline{\partial}_{Hom(E_1,E_2)} + \mathfrak{d}_{a_1,a_2}$ deform the the Dolbeault operators $\overline{\partial}_{\wedge TX}$ and $\overline{\partial}_{Hom(E_1,E_2)}$. When $W = 0$, Theorem (A) reduces to Serre duality on the non-compact complex manifold X for the holomorphic vector bundle $\wedge TX$. In this case, a particular class of holomorphic factorizations is given by pairs $(E, 0)$, for which Theorem (B) also reduces to Serre duality. Hence both theorems are "deformed" versions of ordinary Serre duality on non-compact complex manifolds. We shall prove both by reduction to Serre duality using certain spectral sequences.

Notations

- F^* denotes the topological dual of a topological vector space (tvs) F, endowed with the strong topology.
- Given a continuous linear map $f : F_1 \to F_2$ between two tvs, we denote its transpose by $f^t : F_2^* \to F_1^*$; which is continuous wrt the strong topologies.

Definition

A continuous linear map $f : F_1 \to F_2$ is a *topological homomorphism* if the corestriction $f_0 : F_1 \to f(F_1)$ is open. f is a *topological isomorphism* if it is a homeomorphism.

Given two tvs F_1 and F_2, write $F_1 \simeq F_2$ if there exists a topological isomorphism from F_1 to F_2.

FS and DFS spaces. A Fréchet-Schwartz (FS) space is a Fréchet space which is also a Schwartz space. Every such space is *reflexive*, i.e. naturally topologically isomorphic with the strong topological dual of its strong topological dual. A tvs is called a *DFS space* if it is the strong topological dual of an FS space; DFS spaces are also reflexive.

Definition

Let F_1 and F_2 be topological vector spaces. A *topological pairing* between F_1 and F_2 is a bilinear map $\langle\cdot,\cdot\rangle : F_1 \times F_2 \to \mathbb{C}$ which is (jointly) continuous.

Definition

Let $\langle\cdot,\cdot\rangle : F_1 \times F_2 \to \mathbb{C}$ be a topological pairing. The *left Riesz morphism* of $\langle\cdot,\cdot\rangle$ is the linear map $\tau_l : F_1 \to F_2^*$ defined through:

$$\tau_l(u)(v) \stackrel{\text{def.}}{=} \langle u, v\rangle \ , \ \ \forall u \in F_1 \ , \ \forall v \in F_2 \ \ .$$

The *right Riesz morphism* of $\langle\cdot,\cdot\rangle$ is the linear map $\tau_r : F_2 \to F_1^*$ defined through:

$$\tau_r(v)(u) \stackrel{\text{def.}}{=} \langle u, v\rangle \ , \ \ \forall u \in F_1 \ , \ \forall v \in F_2 \ \ .$$

We say that $\langle\cdot,\cdot\rangle$ is a *perfect topological pairing* if both τ_l and τ_r are topological isomorphisms.

Definition

A *topological cochain complex* (tcc) is a sequence:

$$(F^\bullet): \quad \ldots \longrightarrow F^{k-1} \stackrel{\delta_{k-1}}{\longrightarrow} F^k \stackrel{\delta_k}{\longrightarrow} F^{k+1} \longrightarrow \ldots \quad ,$$

where F^k are topological vector spaces and δ_k are continuous linear maps which satisfy $\delta_{k+1} \circ \delta_k = 0$ for all $k \in \mathbb{Z}$. The cohomology of such a complex in degree $k \in \mathbb{Z}$ is the vector space:

$$\mathrm{H}^k(F^\bullet, \delta) \stackrel{\mathrm{def.}}{=} \ker \delta_k / \mathrm{im} \delta_{k-1} \quad ,$$

endowed with the quotient topology. We say that the topological complex $F^\bullet$ is *bounded* if there exist integers $k_1 < k_2$ such that $F^k = 0$ unless $k_1 \leq k \leq k_2$.

Given a bounded tcc $F^\bullet$, set $F \stackrel{\mathrm{def.}}{=} \oplus_{k \in \mathbb{Z}} F^k = F^{k_1} \times \ldots \times F^{k_2}$ and $\delta = \sum_{k \in \mathbb{Z}} \delta_k = \sum_{k=k_1}^{k_2} \delta_k$. Then F is a finitely $\mathbb{Z}$-graded tvs and δ is a continuous differential of degree $+1$. The *total* cohomology:

$$\mathrm{H}(F, \delta) = \ker \delta / \mathrm{im} \delta = \mathrm{H}(F^\bullet, \delta) = \bigoplus_{k \in \mathbb{Z}} \mathrm{H}^k(F^\bullet, \delta) = \mathrm{H}^{k_1}(F, \delta) \times \ldots \times \mathrm{H}^{k_2}(F, \delta)$$

is a finitely $\mathbb{Z}$-graded tvs. When $u \in F^k$, we set $\deg u \stackrel{\mathrm{def.}}{=} k$.

Definition

The *topological dual* of a bounded tcc (F, δ) is the tcc (F^*, δ^t) defined through:

$$(F^*)^k \stackrel{\text{def.}}{=} (F^{-k})^* \quad , \quad (\delta^t)_k \stackrel{\text{def.}}{=} \delta^t_{-k-1} \quad .$$

Definition

Let (F, δ) and $(\widehat{F}, \hat{\delta})$ be two bounded tccs. A $\mathbb{C}$-bilinear map $\langle \cdot, \cdot \rangle : F \times \widehat{F} \to \mathbb{C}$ is a *topological pairing of complexes* if it satisfies the conditions:

1. $\langle \cdot, \cdot \rangle$ is jointly continuous.
2. $\langle \cdot, \cdot \rangle$ is of degree zero: $\langle \cdot, \cdot \rangle\big|_{F^i \times \widehat{F}^j} = 0$ if $i + j \neq 0$.
3. $\langle \delta u, v \rangle + (-1)^{\deg u} \langle u, \hat{\delta} v \rangle = 0$ for all homogeneous elements $u \in F$ and $v \in \widehat{F}$.

In this case, we say that $\langle \cdot, \cdot \rangle$ is a *perfect topological pairing of complexes* if its restriction to $F^k \times \widehat{F}^{-k}$ is perfect for all $k \in \mathbb{Z}$.

A topological pairing of bounded tccs induces a degree zero topological pairing $\langle \cdot, \cdot \rangle^H : \mathrm{H}(F, \delta) \times \mathrm{H}(\widehat{F}, \hat{\delta}) \to \mathbb{C}$.

Definition

A topological pairing of bounded tccs is *cohomologically perfect* if the restriction $\langle\cdot,\cdot\rangle^H|_{\mathrm{H}^k(F,\delta)\times \mathrm{H}^{-k}(\widehat{F},\hat{\delta})}$ is a perfect topological pairing for all $k\in\mathbb{Z}$.

When $\langle\cdot,\cdot\rangle$ is cohomologically perfect, the vector spaces $\mathrm{H}^k(F,\delta)$ and $\mathrm{H}^k(\widehat{F},\hat{\delta})$ are reflexive for all k and $\langle\cdot,\cdot\rangle^H$ induces topological isomorphisms $\mathrm{H}^k(F,\delta)\simeq \mathrm{H}^{-k}(\widehat{F},\hat{\delta})^*$.

Proposition

Let (F,δ) be a bounded tcc **of FS spaces** *such that $\mathrm{H}^k(F,\delta)$ is finite-dimensional for all $k\in\mathbb{Z}$. Then:*

1. *δ_k is a topological homomorphism for all $k\in\mathbb{Z}$.*
2. *The dual complex (F^*,δ^t) is a bounded tcc* **of DFS spaces** *whose differentials are topological homomorphisms and whose cohomology is finite-dimensional in every degree.*
3. *The natural linear map:*

$$\mathrm{H}^{-k}(F^*,\delta^t)\to \mathrm{H}^k(F,\delta)^\vee$$

is bijective for all $k\in\mathbb{Z}$.

Corollary

Let (F,δ) and $(\widehat{F},\hat{\delta})$ be two bounded tccs let $\langle\cdot,\cdot\rangle : F \times \widehat{F} \to \mathbb{C}$ be a perfect topological pairing between them. Suppose that $\mathrm{H}^k(F,\delta)$ is finite-dimensional for all $k \in \mathbb{Z}$. Then $\mathrm{H}^k(\widehat{F},\hat{\delta})$ is finite-dimensional for all $k \in \mathbb{Z}$ and $\langle\cdot,\cdot\rangle$ is cohomologically perfect.

Let $K = \oplus_{p,q\in\mathbb{Z}} K^{p,q}$ be a double complex of $\mathbb{C}$-vector spaces with *vertical* differential $\mathrm{d}_1 : K^{p,q} \to K^{p,q+1}$ and *horizontal* differential $\mathrm{d}_2 : K^{p,q} \to K^{p+1,q}$. We are interested in the following special cases:

- (A) K is concentrated in the first quadrant, i.e. $K^{p,q}$ vanishes unless $p \geq 0$ and $q \geq 0$.
- (B) K is concentrated in a horizontal strip above the horizontal axis, i.e. there exists $N > 0$ such that $K^{p,q}$ vanishes unless $0 \leq q \leq N$.

Let $K = \oplus_{n\in\mathbb{Z}} K^n$ be the decomposition corresponding to the total grading of K, where $K^n \stackrel{\mathrm{def.}}{=} \bigoplus_{p+q=n} K^{p,q}$. Let $\delta \stackrel{\mathrm{def.}}{=} \mathrm{d}_1 + \mathrm{d}$ be the total differential, where $\mathrm{d} \stackrel{\mathrm{def.}}{=} (-1)^p \mathrm{d}_2$. Endow the double complex with the *standard* decreasing filtration F^p given by:

$$F^p K \stackrel{\mathrm{def.}}{=} \bigoplus_{i\geq p} \bigoplus_{q\in\mathbb{Z}} K^{i,q} \quad .$$

The filtration can be unbounded in case B. but the spaces:

$$\mathrm{gr}_F^p(K^n) \stackrel{\mathrm{def.}}{=} \frac{[K^n \cap F^p K]}{[K^n \cap F^{p-1} K]} = \Big[\bigoplus_{\substack{i\geq p\\ q=n-i}} K^{i,q}\Big] \Big/ \Big[\bigoplus_{\substack{i\geq p-1,\\ q=n-i}} K^{i,q}\Big] \simeq_{\mathbb{C}} K^{p,n-p}$$

vanish in both cases (A) and (B) for any $n \in \mathbb{Z}$, except for a finite number of values of p.

Proposition

Assume that the double complex $(K, \mathrm{d}_1, \mathrm{d}_2)$ *satisfies condition (A) or (B) above. Then the standard decreasing filtration defines a spectral sequence* $\mathbf{E} = (\mathbf{E}_r, \mathbf{d}_r)_{r\geq 0}$ *which converges to the total cohomology* $\mathrm{H}_\delta(K) \stackrel{\mathrm{def.}}{=} \oplus_{n\in\mathbb{Z}}\mathrm{H}^n_\delta(K)$. *For each* $r \geq 0$, *the page* $\mathbf{E}_r$ *is endowed with the bigrading given by the decomposition* $\mathbf{E}_r = \oplus_{p,q\in\mathbb{Z}}\mathbf{E}_r^{p,q}$ *and with the differential* $\mathbf{d}_r : \mathbf{E}_r^{p,q} \to \mathbf{E}_r^{p+r,q-r+1}$ *defined recursively by:*

$$\mathbf{E}_r^{p,q} \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathbf{E}_{r-1}^{p,q}, \mathbf{d}_{r-1}) .$$

For the first pages we have $\mathbf{d}_0 = \mathrm{d}_1$ *and* $\mathbf{d}_1 = \mathrm{d} := (-1)^p\mathrm{d}_2$, *hence:*

$$\mathbf{E}_1^{p,q} = \mathrm{H}^q_{\mathrm{d}_1}(\mathrm{gr}^p_F K) , \quad \mathbf{E}_2^{p,q} = \mathrm{H}^p_{\mathrm{d}}(\mathbf{E}_1^{p,q}) \quad .$$

For each $n \in \mathbb{Z}$, *the standard decreasing filtration induces a decreasing filtration* $(\mathbf{F}^p\mathrm{H}^n_\delta(K))_{p\in\mathbb{Z}}$ *of the vector space* $\mathrm{H}^n_\delta(K)$, *whose associated graded pieces* $\mathrm{gr}^p_{\mathbf{F}}\mathrm{H}^n_\delta(K) \stackrel{\mathrm{def.}}{=} \frac{\mathbf{F}^p\mathrm{H}^n_\delta(K)}{\mathbf{F}^{p-1}\mathrm{H}^n_\delta(K)}$ *satisfy:*

$$\mathrm{gr}^p_{\mathbf{F}}\mathrm{H}^n_\delta(K) \simeq_{\mathbb{C}} \mathbf{E}_\infty^{p,n-p} , \quad \forall p \in \mathbb{Z} \quad ,$$

where $\mathbf{E}_\infty = \oplus_{p,q\in\mathbb{Z}}\mathbf{E}_\infty^{p,q}$ *is the limit of* $\mathbf{E}$.

Consider another double complex $(\widehat{K}, \hat{d}_1, \hat{d}_2)$ with the total differential $\hat{\delta} = \hat{d}_1 + \hat{d}$, where $\hat{d} = (-1)^p \hat{d}_2$. Let $\tau : K \to \widehat{K}$ be a morphism of double complexes and $\tau_* : \mathrm{H}_\delta(K) \to \mathrm{H}_{\hat{\delta}}(\widehat{K})$ denote the morphism of graded $\mathbb{C}$-vector spaces induced by τ on total cohomology. Let $F^p\widehat{K}$ denote the analogue of the filtration (32) for $\widehat{K}$ and $(\widehat{\mathbf{F}}^p \mathrm{H}^n_{\hat{\delta}}(\widehat{K}))_{p\in\mathbb{Z}}$ denote the filtration induced by F^p on the homogeneous components of total cohomology.

Theorem

Suppose that τ is injective and induces isomorphisms of vector spaces $\mathrm{H}^{p,q}_{\mathrm{d}_1}(K) \simeq_{\mathbb{C}} \mathrm{H}^{p,q}_{\hat{d}_1}(\widehat{K})$ for all $p, q \in \mathbb{Z}$ in vertical cohomology. Assume that both double complexes $(K, \mathrm{d}_1, \mathrm{d}_2)$ and $(\widehat{K}, \hat{d}_1, \hat{d}_2)$ satisfy condition (A) or that both satisfy condition (B) above. Then τ_* satisfies:

$$\tau_*(\mathbf{F}^p \mathrm{H}^n_\delta(K)) \subset \widehat{\mathbf{F}}^p \mathrm{H}^n_{\hat{\delta}}(\widehat{K}) \ , \ \forall p, n \in \mathbb{Z}$$

and restricts to isomorphisms of vector spaces between the associated gradeds:

$$\tau_* : \mathrm{gr}^p_{\mathbf{F}} \mathrm{H}^n_\delta(K) \xrightarrow{\sim} \mathrm{gr}^p_{\widehat{\mathbf{F}}} \mathrm{H}^n_{\hat{\delta}}(\widehat{K}) \ , \ \forall p, n \in \mathbb{Z} \ .$$

Topological complexes of differential forms valued in a holomorphic vector bundle

For any holomorphic vector bundle V on X, let $\Omega^{p,q}(X,V)$ be the space of V-valued smooth forms of type (p,q) defined on X and $\Omega_c^{p,q}(X,V)$ be the subspace of compactly-supported forms. Then $\Omega^{p,q}(X,V)$ is a FS space when endowed with the topology of uniform convergence of all derivatives on compact subsets and $\Omega_c^{p,q}(X,V)$ is dense in $\Omega^{p,q}(X,V)$. Let:

$$\Omega(X,V) \stackrel{\mathrm{def.}}{=} \bigoplus_{p,q=0}^{d} \Omega^{p,q}(X,V) \quad , \quad \Omega_c(X,V) \stackrel{\mathrm{def.}}{=} \bigoplus_{p,q=0}^{d} \Omega_c^{p,q}(X,V)$$

$$\mathcal{A}(X,V) \stackrel{\mathrm{def.}}{=} \bigoplus_{q=0}^{d} \Omega^{0,q}(X,V) \quad , \quad \mathcal{A}_c(X,V) \stackrel{\mathrm{def.}}{=} \bigoplus_{q=0}^{d} \Omega_c^{0,q}(X,V) \quad .$$

Then $\Omega(X,V)$ and $\mathcal{A}(X,V)$ are FS spaces which contain $\Omega_c(X,V)$ (respectively $\mathcal{A}_c(X,V)$) as dense subspaces. Notice that $\mathcal{A}(X,V)$ is a closed subspace of $\Omega(X,V)$. Moreover, $(\mathcal{A}(X,V),\overline{\partial}_V)$ is a bounded topological complex of FS spaces, where $\overline{\partial}_V$ is the Dolbeault operator on $\Omega(X,V)$.

Topological complexes of bundle-valued currents with compact support and the classical Serre pairing

Let $\hat{\Omega}^{p,q}(X,V)$ denote the space of distributions with compact support valued in the bundle $\wedge^p T^*X \otimes \wedge^q \bar{T}^*X \otimes V$. Consider the bigraded topological vector space:

$$\hat{\Omega}(X,V) \stackrel{\mathrm{def.}}{=} \bigoplus_{p,q=0}^{d} \hat{\Omega}^{p,q}(X,V)$$

of distributions with compact support valued in the vector bundle $\wedge T^*X \otimes \wedge \bar{T}^*X \otimes V$. Let $V^\vee \stackrel{\mathrm{def.}}{=} Hom(V,\mathcal{O}_X)$ denote the dual bundle to V. Then $\hat{\Omega}^{d-p,d-q}(X,V)$ is topologically isomorphic with the topological dual $\Omega^{p,q}(X,V^\vee)^*$, where the latter is endowed with the strong topology. The corresponding perfect duality pairing is known as the *Serre pairing* and is given by:

$$(\omega,T) \to \int_X \omega\wedge T~,~~\forall \omega\in \Omega^{p,q}(X,V^\vee)~,~\forall T\in \hat{\Omega}^{d-p,d-q}(X,V)~~,$$

where $\int_X$ denotes integration of compactly supported currents of type (d,d) on X with respect to the orientation induced by the complex structure of X. Below, we introduce a version of this pairing adapted to the case when V is replaced by a $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded holomorphic vector bundle.

The graded Serre pairing of a $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded holomorphic vector bundle

Let $A \in \{\mathbb{Z}, \mathbb{Z}_2\}$. Let $Q = \oplus_{j\in A} Q^j$ be an A-graded holomorphic vector bundle whose dual we grade by $Q^\vee = \bigoplus_{j\in A}(Q^\vee)^j$, where $(Q^\vee)^j \stackrel{\mathrm{def.}}{=} (Q^{-j})^\vee$. We use the convention $\wedge^k T^*X = \wedge^k \bar{T}^*X = 0$ for both $k<0$ or $k>d$. Define:

$$(\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q)^{p,q,i} \stackrel{\mathrm{def.}}{=} \wedge^p T^*X \otimes \wedge^q \bar{T}^*X \otimes Q^i \quad .$$

The bundles $\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q$ and $\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q^\vee$ are $(\mathbb{Z}^2 \times A)$-graded and we have:

$$(\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q^\vee)^{p,q,j} = \wedge^p T^*X \otimes \wedge^q \bar{T}^*X \otimes (Q^{-j})^\vee \quad .$$

Viewing $\mathcal{O}_X$ as an A-graded holomorphic vector bundle concentrated in degree zero, the bundle $\wedge T^*X \otimes \wedge \bar{T}^*X \simeq \wedge T^*X \otimes \wedge \bar{T}^*X \otimes \mathcal{O}_X$ is $\mathbb{Z}^2 \times A$-graded with third grading concentrated in degree zero, while the spaces $\Omega(X,Q)$ and $\hat{\Omega}(X,Q)$ are trigraded accordingly. Notice that $\Omega(X,Q)$ is an FS space, while $\hat{\Omega}(X,Q)$ is a DFS space.

Definition

The *graded duality morphism of* Q is the morphism $\mathrm{ev}_Q : Q \otimes Q^\vee \to \mathcal{O}_X$ of A-graded holomorphic vector bundles determined uniquely by the condition:

$$\mathrm{ev}_Q(x)(v,w) \stackrel{\mathrm{def.}}{=} (-1)^i \delta_{i+j,0} w(v) \ , \ \ \forall v \in Q^i \ , \ \forall w \in (Q^\vee)^j \ , \ \forall x \in X \quad .$$

The graded Serre pairing of a $\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded holomorphic vector bundle

Together with the wedge product of differential forms, ev_Q induces a morphism of holomorphic vector bundles:

$$\mathcal{S}_Q : (\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q) \otimes (\wedge T^*X \otimes \wedge \bar{T}^*X \otimes Q^\vee) \longrightarrow \wedge T^*X \otimes \wedge \bar{T}^*X$$

$$\mathcal{S}_Q(x)(\omega_1 \otimes v, \omega_2 \otimes w) \stackrel{\mathrm{def.}}{=} (-1)^{i(p_2+q_2)} \mathrm{ev}_Q(x)(v,w)\, \omega_1 \wedge \omega_2$$

for $\omega_1 \in \wedge^{p_1} T_x^*X \otimes \wedge^{q_1} \bar{T}_x^*X$, $\omega_2 \in \wedge^{p_2} T_x^*X \otimes \wedge^{q_2} \bar{T}_x^*X$ and $v \in Q_x^i$, $w \in (Q_x^\vee)^j$ (where $x \in X$). The bundle morphism $\mathcal{S}_Q$ induces a continuous bilinear map:

$$S_Q : \Omega(X,Q) \times \hat{\Omega}(X,Q^\vee) \longrightarrow \hat{\Omega}(X) \quad .$$

Definition

The *graded Serre pairing* of the A-graded holomorphic vector bundle Q is the topological pairing $\mathbb{S}_Q : \Omega(X,Q) \times \hat{\Omega}(X,Q^\vee) \to \mathbb{C}$ defined through $\mathbb{S}_Q(\omega, T) \stackrel{\mathrm{def.}}{=} \int_X S_Q(\omega, T)$, where $\int_X L$ is defined to be zero unless $L \in \hat{\Omega}(X)$ has type (d,d).

Lemma

$\mathbb{S}_Q$ is a perfect pairing between the topological vector spaces $\Omega(X,Q)$ and $\hat{\Omega}(X,Q^\vee)$.

Let (X, W) be a holomorphic Landau-Ginzburg pair. Consider the cochain complex $(\mathrm{PV}(X), \delta_W)$, graded by total degree:

$$0 \longrightarrow \mathrm{PV}^{-d}(X) \xrightarrow{\delta_W} \mathrm{PV}^{-d+1}(X) \xrightarrow{\delta_W} \ldots \xrightarrow{\delta_W} \mathrm{PV}^{d-1}(X) \xrightarrow{\delta_W} \mathrm{PV}^{d}(X) \longrightarrow 0 \quad .$$

Let $\mathrm{PV}_c(X)$ be the subcomplex of compactly-supported forms. Then $\boldsymbol{\iota}_W$ and δ_W are continuous with respect to the Fréchet topology. $(\mathrm{PV}(X), \delta_W)$ is a bounded tcc of FS spaces containing $\mathrm{PV}_c(X)$ as a dense subcomplex. For any $i \in \{-d, \ldots, 0\}$ and $j \in \{0, \ldots, d\}$, let $\widehat{\mathrm{PV}}^{i,j}(X) \stackrel{\text{def.}}{=} \hat{\Omega}^{0,j}(X, \wedge^{|i|} TX)$ and set:

$$\widehat{\mathrm{PV}}(X) \stackrel{\text{def.}}{=} \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \widehat{\mathrm{PV}}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \hat{\Omega}^{0,j}(X, \wedge^{|i|} TX) \quad ,$$

endowed with the total grading $\widehat{\mathrm{PV}}^{k}(X) = \bigoplus_{i+j=k} \widehat{\mathrm{PV}}^{i,j}(X)$, which is concentrated in degrees $k \in \{-d, \ldots, d\}$. Let $\hat{\delta}_W : \widehat{\mathrm{PV}}(X) \to \widehat{\mathrm{PV}}(X)$ be the natural extension of δ_W. Then $(\widehat{\mathrm{PV}}(X), \hat{\delta}_W)$ is a tcc of DFS spaces:

$$0 \longrightarrow \widehat{\mathrm{PV}}^{-d}(X) \xrightarrow{\hat{\delta}_W} \widehat{\mathrm{PV}}^{-d+1}(X) \xrightarrow{\hat{\delta}_W} \ldots \xrightarrow{\hat{\delta}_W} \widehat{\mathrm{PV}}^{d-1}(X) \xrightarrow{\hat{\delta}_W} \widehat{\mathrm{PV}}^{d}(X) \longrightarrow 0 \quad .$$

Let $\widehat{\mathrm{HPV}}^{k}(X, W) \stackrel{\text{def.}}{=} \mathrm{H}^{k}(\widehat{\mathrm{PV}}(X), \hat{\delta}_W)$ denote its cohomology in degree k. Then $(\mathrm{PV}_c(X), \delta_W)$ is a subcomplex. The wedge product induces an associative and jointly continuous multiplication on $\mathrm{PV}(X)$ (denoted by juxtaposition) which makes $(\mathrm{PV}(X), \delta_W)$ into a supercommutative dGA.

Let Ω be a holomorphic volume form on X.

Definition

The *canonical off-shell bulk pairing* determined by Ω is the continuous bilinear map $\langle \cdot, \cdot \rangle_B : \mathrm{PV}(X) \times \mathrm{PV}_c(X) \to \mathbb{C}$ defined through:

$$\langle \omega, \eta \rangle_B \stackrel{\text{def.}}{=} \mathrm{Tr}_B(\omega\eta) = \int_X \Omega \wedge [\Omega \lrcorner (\omega\eta)] \ , \quad \forall \omega \in \mathrm{PV}(X) \ , \ \forall \eta \in \mathrm{PV}_c(X) \quad .$$

Definition

The *extended canonical off-shell bulk pairing* is the continuous bilinear map $\langle \cdot, \cdot \rangle : \mathrm{PV}(X) \times \widehat{\mathrm{PV}}(X) \to \mathbb{C}$ defined through:

$$\langle \omega, T \rangle \stackrel{\text{def.}}{=} \int_X \Omega \wedge [\Omega \lrcorner (\omega T)] \quad , \quad \forall \omega \in \mathrm{PV}(X) \ , \ \forall T \in \widehat{\mathrm{PV}}(X) \quad .$$

Proposition

The canonical off-shell bulk pairing $\langle \cdot, \cdot \rangle_B$ *is a topological pairing of bounded* $\mathbb{Z}$*-graded tccs between* $(\mathrm{PV}(X), \delta_W)$ *and* $(\mathrm{PV}_c(X), \delta_W)$*, while the extended canonical off-shell bulk pairing* $\langle \cdot, \cdot \rangle$ *is a topological pairing of bounded* $\mathbb{Z}$*-graded tccs between* $(\mathrm{PV}(X), \delta_W)$ *and* $(\widehat{\mathrm{PV}}(X), \hat{\delta}_W)$.

View $\wedge T^*X$ and $\wedge TX$ as a $\mathbb{Z}$-graded bundles with $\wedge^k T^*X$ and $\wedge^k TX$ in degrees $+k$ and $-k$ respectively. Let $\Omega_\lrcorner, \Omega_{\lrcorner 0} : \wedge TX \to \wedge T^*X$ be the degree d linear bundle maps given by contraction/reduced contraction with Ω and $\mathrm{ev} := \mathrm{ev}_{\wedge TX} : \wedge TX \otimes \wedge T^*X \to \mathcal{O}_X$ be the graded duality morphism.

Proposition

For any $x \in X$ and $v_1, v_2 \in \wedge T_x X$, we have $\Omega_{x \lrcorner 0}(v_1 \wedge v_2) = (-1)^{k_1 d} \mathrm{ev}_{\wedge TX}(x)(v_1, \Omega_{x \lrcorner} v_2)$.

Contraction and wedge product with Ω induce topological isomorphisms:

$$\Omega_\lrcorner : \widehat{\mathrm{PV}}(X) \to \hat{\Omega}^{0,\bullet}(X, \wedge T^*X) \ , \ \Omega\wedge : \mathrm{PV}(X) \to \Omega^{d,\bullet}(X, \wedge TX) \ .$$

Proposition

For any $\mathbb{Z}$-homogeneous element $\omega \in \mathrm{PV}(X)$ and any $T \in \widehat{\mathrm{PV}}(X)$, we have:

$$\langle \omega, T \rangle = (-1)^{d \deg \omega} \mathbb{S}_{\wedge TX}(\Omega \wedge \omega, \Omega_\lrcorner T) \ .$$

Proposition

The extended off-shell bulk pairing $\langle \cdot, \cdot \rangle$ is a perfect topological pairing between the bounded tccs $(\mathrm{PV}(X), \delta_W)$ and $(\widehat{\mathrm{PV}}(X), \hat{\delta}_W)$.

Suppose that the critical set Z_W is compact.

Lemma

$\mathrm{HPV}^k(X, W)$ and $\widehat{\mathrm{HPV}}^k(X, W)$ are finite-dimensional for every $k \in \{-d, \ldots, d\}$ and the extended canonical off-shell bulk pairing $\langle \cdot, \cdot \rangle$ determined by any holomorphic volume form Ω is cohomologically perfect, hence we have Ω-dependent natural isomorphisms:

$$\widehat{\mathrm{HPV}}^k(X, W) \simeq_{\mathbb{C}} \mathrm{HPV}^{-k}(X, W)^{\vee} \ , \ \ \forall k \in \{-d, \ldots, d\} \ .$$

Let $s : \mathrm{PV}_c(X) \to \widehat{\mathrm{PV}}(X)$ be the inclusion map.

Proposition

Suppose that the critical set Z_W is compact. Then $\mathrm{HPV}_c(X, W)$ and $\widehat{\mathrm{HPV}}(X, W)$ are finite-dimensional over $\mathbb{C}$ and s_* is an isomorphism of $\mathbb{C}$-vector spaces. Moreover, for any $k \in \{-d, \ldots, d\}$, we have a natural isomorphism of vector spaces:

$$\widehat{\mathrm{HPV}}^k(X, W) \simeq_{\mathbb{C}} \mathbb{H}^k_c(\mathcal{K}_W) \ ,$$

where $\mathbb{H}_c(\mathcal{K}_W)$ denotes compactly-supported hypercohomology of $\mathcal{K}_W$.

Proof of Theorem A

Proof.

By Proposition 52, the inclusion $s : \mathrm{PV}_c(X) \hookrightarrow \widehat{\mathrm{PV}}(X)$ is a morphism of complexes which induces an isomorphism $s_* : \mathrm{HPV}_c(X, W) \xrightarrow{\sim} \widehat{\mathrm{HPV}}(X, W)$. Since $\langle \omega, \eta \rangle_B = \langle \omega, s(\eta) \rangle$ for all $\omega \in \mathrm{PV}(X)$ and all $\eta \in \mathrm{PV}_c(X)$, we have $\langle u_1, u_2 \rangle_B^H = \langle u_1, s_*(u_2) \rangle^H$ for all $u_1 \in \mathrm{HPV}(X, W)$ and all $u_2 \in \mathrm{HPV}_c(X, W)$. Since $\langle \cdot, \cdot \rangle^H$ is non-degenerate by Lemma 51 and s_* is injective, the pairing $\langle \cdot, \cdot \rangle_B^H : \mathrm{HPV}(X, W) \times \mathrm{HPV}_c(X, W) \to \mathbb{C}$ induced by $\langle \cdot, \cdot \rangle_B$ is also non-degenerate. On the other hand, we have $\langle u_1, u_2 \rangle_c = \langle i_*(u_1), u_2 \rangle_B^H$ for all $u_1, u_2 \in \mathrm{HPV}_c(X, W)$, where $i_* : \mathrm{HPV}_c(X, W) \to \mathrm{HPV}(X, W)$ is the map induced by the inclusion $i : \mathrm{PV}_c(X) \to \mathrm{PV}(X)$ and $\langle \cdot, \cdot \rangle_c$ is the pairing induced by Tr_B on cohomology. Since Z_W is compact, the map i_* is an isomorphism. This shows that the pairing $\langle \cdot, \cdot \rangle_c : \mathrm{HPV}_c(X, W) \times \mathrm{HPV}_c(X, W) \to \mathbb{C}$ is non-degenerate. By the results of our previous work, we also have $\langle u_1, u_2 \rangle_c = \langle i_*(u_1), i_*(u_2) \rangle_\Omega$ for all $u_1, u_2 \in \mathrm{HPV}_c(X, W)$, which shows that $\langle \cdot, \cdot \rangle_\Omega$ is non-degenerate since i_* is bijective. □ □

Proof.

The isomorphism $\mathrm{HPV}^k(X, W) \simeq \mathbb{H}^k(\mathcal{K}_W)$ was proved in our previous work. Since Z_W is compact, we also have an isomorphism $\mathrm{HPV}_c^k(X, W) \simeq \mathrm{HPV}^k(X, W)$ which follows from our previous work. The isomorphism $\mathrm{HPV}^k(X, W) \simeq \widehat{\mathrm{HPV}}^{-k}(X, W)$ follows from Lemma 51. □ □

The proof of theorems B and B' is similar to those of theorem A and A'.