Hessian symmetries of multifield cosmological models

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Outline

Definition

A scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is an oriented Riemannian n-manifold (the *scalar manifold*)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function defined on $\mathcal{M}$ (the scalar potential).

Assumptions

1. $\mathcal{M}$ is connected.
2. $(\mathcal{M}, \mathcal{G})$ is complete.
3. $V > 0$ on $\mathcal{M}$.

Each scalar triple defines a cosmological model with n real scalar fields:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|} \left[\frac{R(g)}{2} - \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) - V(\varphi)\right] \quad (1)$$

for a simply-connected and spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \;\;,\;\; \vec{x} = (x^1, x^2, x^3) \;\;,\;\; a(t) > 0 \;\forall t) \quad (2)$$

in which φ depends only on the cosmological time t:

$$\varphi = \varphi(t) \;\;. \quad (3)$$

Substituting (2) and (3) in (1) and ignoring the integration over $\vec{x}$ gives the minisuperspace action:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[a,\varphi] = \int_{-\infty}^{\infty} \mathrm{d}t\, L_{\mathcal{M},\mathcal{G},V}(a(t),\varphi(t),\dot{\varphi}(t)) \quad ,$$

where the minisuperspace Lagrangian is:

$$L_{\mathcal{M},\mathcal{G},V}(a,\varphi,\dot{\varphi}) \stackrel{\mathrm{def.}}{=} -3a\dot{a}^2 + a^3\left[\frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi)\right] = a^3\left[-3H^2 + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi)\right] \quad .$$

Here $\dot{\varphi} \stackrel{\mathrm{def.}}{=} \frac{\mathrm{d}\varphi}{\mathrm{d}t}$ and $H \stackrel{\mathrm{def.}}{=} \frac{\dot{a}}{a}$ is the Hubble parameter. This Lagrangian describes a classical system with $n+1$ degrees of freedom and configuration space $\mathcal{N} \stackrel{\mathrm{def.}}{=} \mathbb{R}_{>0} \times \mathcal{M}$. The Euler-Lagrange equations are equivalent with:

$$3H^2 + 2\dot{H} + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi) = 0$$
$$(\nabla_t + 3H)\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}} V)(\varphi) = 0 \quad .$$

We must also impose the Friedmann constraint:

$$\frac{1}{2}||\dot{\varphi}||^2 + V \circ \varphi = 3H^2 \quad ,$$

which amounts to the zero energy condition.

Proposition

When supplemented with the Friedmann constraint, the Euler-Lagrange equations of $L_{\mathcal{M},\mathcal{G},V}$ are equivalent with the cosmological equations:

$$\begin{aligned} \nabla_t \dot{\varphi} + 3H\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}} V)\circ\varphi &= 0 \\ \dot{H} + 3H^2 - V\circ\varphi &= 0 \\ \dot{H} + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} &= 0 \ . \end{aligned}$$

Remark

One can eliminate H algebraically from the cosmological equations as:

$$H(t) = \frac{1}{\sqrt{6}}\epsilon(t)\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2V(\varphi(t))\right]^{1/2} \ , \text{ where } \epsilon(t) \stackrel{\text{def.}}{=} \mathrm{sign} H(t) \in \{-1,0,1\} \ ,$$

thereby reducing the latter to the n-component second order autonomous nonlinear ODE:

$$\nabla_t \dot{\varphi}(t) + \sqrt{\frac{3}{2}}\epsilon(t)\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2V(\varphi(t))\right]^{1/2}\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0$$

which defines a (dissipative) geometric dynamical system on $T\mathcal{M}$.

Definition

A strict variational symmetry $\mathbf{X} \in \mathcal{X}(\mathbb{R} \times \mathcal{N})$ of the minisuperspace system is called strong if it is the lift of a vector field $X \in \mathcal{X}(\mathcal{N})$ defined on $\mathcal{N}$.

We have a natural decomposition:

$$T\mathcal{N} = T_\perp \mathcal{N} \oplus T_\parallel \mathcal{N} \quad ,$$

where:

$$T_\perp \mathcal{N} \stackrel{\text{def.}}{=} p_1^*(T\mathbb{R}) \quad , \quad T_\parallel \mathcal{N} \stackrel{\text{def.}}{=} p_2^*(T\mathcal{M}) \quad (p_1 : T\mathcal{N} \to \mathbb{R},\ p_2 : T\mathcal{N} \to \mathcal{M}) \quad .$$

Accordingly, $X \in \mathcal{X}(\mathcal{N})$ decomposes as:

$$X = X_\perp + X_\parallel \quad .$$

In local coordinates (U, a, φ^i) on $\mathcal{N}$, we have:

$$X_\perp(a,\varphi) = X^a(a,\varphi)\frac{\partial}{\partial a} \quad , \quad X_\parallel = X^i(a,\varphi)\frac{\partial}{\partial \varphi^i} \quad .$$

The Noether symmetry condition is:

$$\mathcal{L}_{\mathbf{X}^1}(L) = 0 \quad , \quad \text{where} \quad \mathbf{X}^1 = j^1(\mathbf{X}) \quad .$$

Theorem

For the minisuperspace Lagrangian, the Noether symmetry condition amounts to the requirement that $X_\perp$ and $X_\parallel$ have the following forms:

$$X_\perp(a,\varphi) = \frac{\Lambda(\varphi)}{\sqrt{a}}\partial_a \quad , \quad X_\parallel(a,\varphi) = Y(\varphi) - \frac{4}{a^{3/2}}(\mathrm{grad}_\mathcal{G}\Lambda)(\varphi) \quad ,$$

where $\Lambda \in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ and $Y \in \mathcal{X}(\mathcal{M})$ satisfy the characteristic system of the scalar triple $(\mathcal{M},\mathcal{G},V)$:

$$\mathrm{Hess}_\mathcal{G}(\Lambda) = \frac{3}{8}\mathcal{G}\Lambda \quad , \quad \mathcal{K}_\mathcal{G}(Y) = 0$$
$$\langle \mathrm{d}V, \mathrm{d}\Lambda\rangle_\mathcal{G} = \frac{3}{4}V\Lambda \quad , \quad Y(V) = 0 \quad .$$

which can also be written in the index-full form:

$$\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)\Lambda = \frac{3}{8}\mathcal{G}_{ij}\Lambda \quad , \quad \nabla_i Y_j + \nabla_j Y_i = 0$$
$$\mathcal{G}^{ij}\partial_i V\partial_j\Lambda = \frac{3}{4}V\Lambda \qquad\qquad , \quad Y^i\partial_i V = 0 \quad .$$

The characteristic system naturally separates into the Λ-system:

$$\mathrm{Hess}_{\mathcal{G}}(\Lambda) = \frac{3}{8}\mathcal{G}\Lambda$$
$$\langle \mathrm{d}V, \mathrm{d}\Lambda\rangle_{\mathcal{G}} = \frac{3}{4}V\Lambda$$

and the Y-system:

$$\mathcal{K}_{\mathcal{G}}(Y) = 0$$
$$Y(V) = 0 \quad .$$

Definition

A non-trivial solution $\Lambda \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ of the Λ-system is called a Hessian symmetry of $(\mathcal{M}, \mathcal{G}, V)$. A scalar triple which admits Hessian symmetries is called a Hessian triple.

Definition

A non-trivial solution $Y \in \mathcal{X}(\mathcal{M})$ of the Y-system is is called a visible symmetry of $(\mathcal{M}, \mathcal{G}, V)$. A triple which admits visible symmetries is called a visibly symmetric triple.

Let $N(\mathcal{M},\mathcal{G},V)$ denote the vector space of solutions of the characteristic system and $N_H(\mathcal{M},\mathcal{G},V)$, $N_V(\mathcal{M},\mathcal{G},V)$ denote the vector spaces of solutions of the Λ- and Y-systems.

Proposition

There exists a linear isomorphism:

$$N(\mathcal{M},\mathcal{G},V) \simeq_{\mathbb{R}} N_H(\mathcal{M},\mathcal{G},V) \oplus N_V(\mathcal{M},\mathcal{G},V) \quad .$$

In particular, a scalar triple $(\mathcal{M},\mathcal{G},V)$ admits strong Noether symmetries iff it is Hessian or visibly symmetric (or both).

A generic scalar triple $(\mathcal{M},\mathcal{G},V)$ does not admit any strong Noether symmetries. Since existence of visible symmetries is a classical problem, we focus on characterizing Hessian symmetries and Hessian scalar triples.

Remark

Existence of a Hessian Noether symmetry simplifies various cosmological problems, for example allows one to give a useful exact formula for the total number of e-folds during inflation, which is valid without 'slow roll' or 'slow turn' assumptions.

One can solve second equation in the Λ-system through the method of characteristics. Let γ be a gradient flow curve of Λ with gradient flow parameter q:

$$\frac{\mathrm{d}\gamma(q)}{\mathrm{d}q} = -(\mathrm{grad}_{\mathcal{G}}\Lambda)(\gamma(q)) \quad .$$

The second equation of the Λ-system (4) implies:

$$\frac{\mathrm{d}}{\mathrm{d}q}V(\gamma(q)) = -\langle \mathrm{d}V, \mathrm{d}\Lambda\rangle\Big|_{\gamma(q)} = -\frac{3}{4}V(\gamma(q)) \quad ,$$

which gives:

$$V(\gamma(q)) = V(\gamma(q_0))e^{-\frac{3}{4}\int_{q_0 \, \gamma}^{q} \Lambda \mathrm{d}q} \quad .$$

This determines V along the stable and unstable manifolds (with respect to Λ) of any critical point c of Λ. If p_+ lies in the stable manifold $\mathcal{M}_+^\Lambda(c)$ of c, then there exists a gradient flow line $\gamma : [0, +\infty) \to \mathcal{M}$ of Λ such that $\gamma(0) = p_+$ and $\gamma(+\infty) = c$. If p_- lies in the unstable manifold $\mathcal{M}_-^\Lambda(c)$ of c, then there exists a gradient flow line $\gamma : (-\infty, 0] \to \mathcal{M}$ of Λ such that $\gamma(0) = p_-$ and $\gamma(-\infty) = c$. In these cases, we have:

$$V(p_\pm) = V(c)e^{\pm\frac{3}{4}\int_\gamma \Lambda \mathrm{d}q} \quad .$$

Definition

The Hessian endomorphism $\widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda) \in \mathrm{End}^s_{\mathcal{G}}(T\mathcal{M})$ of $(\mathcal{M}, \mathcal{G})$ is defined by:

$$\mathrm{Hess}_{\mathcal{G}}(\Lambda)(X, Y) = \mathcal{G}(X, \widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda)(Y)) \ , \ \ \forall X, Y \in \mathcal{X}(T\mathcal{M}) \ .$$

The ($\mathbb{R}$-linear) Hesse operator $\widehat{\mathrm{Hess}}_{\mathcal{G}} : \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \to \mathrm{End}^s_{\mathcal{G}}(T\mathcal{M})$ of $(\mathcal{M}, \mathcal{G})$ associates to $\Lambda \in \mathcal{C}^\infty(T\mathcal{M}, \mathbb{R})$ its Hessian endomorphism $\widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda)$.

In local coordinates $(U, x^1, \ldots, x^n)$ on $\mathcal{M}$, we have:

$$\widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda)|_U = \widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda)_i{}^j \mathrm{d}x^i \otimes \partial_j \ , \ \text{where} \ \ \widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda)_i{}^j = \mathcal{G}^{jk}\left[\partial_i \partial_k - \Gamma^l_{ik}\partial_l\right]\Lambda \ .$$

Definition

A Hesse function of $(\mathcal{M}, \mathcal{G})$ with parameter $\beta > 0$ is a smooth function $\Lambda \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ which satisfies the Hesse equation with parameter β:

$$\mathrm{Hess}_{\mathcal{G}}(\Lambda) = \beta^2 \Lambda \mathcal{G} \Longleftrightarrow \widehat{\mathrm{Hess}}_{\mathcal{G}}(\Lambda) = \beta^2 \Lambda \, \mathrm{id}_{T\mathcal{M}} \ .$$

A unit Hesse function of $(\mathcal{M}, \mathcal{G})$ is a Hesse function Λ of $(\mathcal{M}, \mathcal{G})$ for $\beta = 1$.

Remark. In our application to cosmological models we have $\beta = \sqrt{3/8}$.

Definition

Let $\mathcal{S}_\beta(\mathcal{M},\mathcal{G})$ be the vector space of solutions of the Hesse equation with parameter β. The Hesse index of $(\mathcal{M},\mathcal{G})$ is $\mathrm{ind}(\mathcal{M},\mathcal{G}) \stackrel{\mathrm{def.}}{=} \dim \mathcal{S}_1(\mathcal{M},\mathcal{G})$. A Hesse manifold is a Riemannian manifold $(\mathcal{M},\mathcal{G})$ for which $\mathrm{ind}(\mathcal{M},\mathcal{G}) > 0$.

Proposition

We have $\mathcal{S}_\beta(\mathcal{M},\mathcal{G}) = \mathcal{S}_1(\mathcal{M},\beta^2\mathcal{G})$.

Proposition

The following statements hold for any $\Lambda \in \mathcal{S}_\beta(\mathcal{M},\mathcal{G}) \setminus \{0\}$:

1. *The set $Z(\Lambda) \stackrel{\mathrm{def.}}{=} \{p \in \mathcal{M} | \Lambda(p) = (\mathrm{d}\Lambda)(p) = 0\}$ is empty.*
2. *The $\mathbb{R}$-linear map $e_p : \mathcal{S}_\beta(\mathcal{M},\mathcal{G}) \to \mathbb{R} \oplus T_p\mathcal{M}$ defined through:*

$$e_p(\Lambda) \stackrel{\mathrm{def.}}{=} \beta\Lambda(p) + (\mathrm{grad}_\mathcal{G}\Lambda)(p) \in \mathbb{R} \oplus T_p\mathcal{M} \quad , \quad \forall \Lambda \in \mathcal{S}_\beta(\mathcal{M},\mathcal{G})$$

is injective for every $p \in \mathcal{M}$. Thus $\mathrm{ind}(\mathcal{M},\mathcal{G}) \leq 1 + \dim \mathcal{M}$.

Definition

We say that $(\mathcal{M},\mathcal{G})$ is maximally Hesse if $\mathrm{ind}(\mathcal{M},\mathcal{G}) = 1 + \dim \mathcal{M}$.

Let $r_{\mathcal{G}}(p) = \mathrm{injrad}_p(\mathcal{M}, \mathcal{G})$, $(r, \theta) \in (0, r_{\mathcal{G}}(p)) \times [0, \pi]^{n-2} \times (\mathbb{R}/2\pi)$ be normal spherical coordinates at p, $\nu(\theta) \in \mathrm{S}^{n-1}$, $u(r, \theta) = r\nu(\theta) \in \mathrm{B}^n(r_{\mathcal{G}}(p))$ and $q(r, \theta) \stackrel{\mathrm{def.}}{=} \exp_p(u(r, \theta)) \in U_p \stackrel{\mathrm{def.}}{=} \exp_p(u(B^n(r_{\mathcal{G}}(p)))$.

Proposition (Local radial expansion of Hesse functions)

Let $\Lambda \in \mathcal{C}^\infty(U_p, \mathbb{R})$ be a Hesse function of $(U_p, \mathcal{G}|_{U_p})$. Then:

$$\Lambda(q(r, \theta)) = \Lambda(p) \cosh(\beta r) + \frac{1}{\beta}(\partial_r \Lambda)(p) \sinh(\beta r) \quad .$$

Moreover, Λ is identically zero or is a Morse function.

Proposition (Sectional curvature condition on Hesse manifolds)

Let $\Lambda \in \mathcal{S}_\beta(\mathcal{M}, \mathcal{G}) \setminus \{0\}$. Then the following open set is non-empty:

$$U(\Lambda) \stackrel{\mathrm{def.}}{=} \{p \in \mathcal{M} | (\mathrm{d}\Lambda)(p) \neq 0\} = \{p \in \mathcal{M} | (\mathrm{grad}_{\mathcal{G}} \Lambda)(p) \neq 0\}$$

For any $p \in U(\Lambda)$ and any 2-plane $\pi \subset T_p\mathcal{M}$ containing $(\mathrm{grad}_{\mathcal{G}}\Lambda)(p)$, we have $K_p(\pi) = -\beta^2$, where $K_p(\pi)$ denotes the sectional curvature.

Definition

The β-Hesse pairing of two smooth functions $f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ is the smooth function $\mathcal{B}_\beta(f_1,f_2) \in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ defined through:

$$\mathcal{B}_\beta(f_1,f_2) \stackrel{\mathrm{def.}}{=} \beta^2 f_1 f_2 - \langle \mathrm{grad}_\mathcal{G} f_1, \mathrm{grad}_\mathcal{G} f_2\rangle_\mathcal{G} \quad .$$

Proposition

Let $\Lambda_1, \Lambda_2 \in \mathcal{S}_\beta(\mathcal{M},\mathcal{G})$. Then $\mathcal{B}_\beta(\Lambda_1,\Lambda_2)$ is constant on $\mathcal{M}$.

Hence restriction to $\mathcal{S}_\beta(\mathcal{M},\mathcal{G})$ gives a real-valued bilinear pairing:

$$(\ ,\ )_\beta : \mathcal{S}_\beta(\mathcal{M},\mathcal{G}) \times \mathcal{S}_\beta(\mathcal{M},\mathcal{G}) \to \mathbb{R} \quad , \quad (\Lambda_1,\Lambda_2)_\beta \stackrel{\mathrm{def.}}{=} \mathcal{B}_\beta(\Lambda_1,\Lambda_2) \quad .$$

For $p \in \mathcal{M}$, let $(\ ,\ )_p : (\mathbb{R} \oplus T_p\mathcal{M}) \times (\mathbb{R} \oplus T_p\mathcal{M}) \to \mathbb{R}$ be the Minkowski pairing:

$$(\tau_1 + v_1, \tau_2 + v_2)_p \stackrel{\mathrm{def.}}{=} \tau_1\tau_2 - \mathcal{G}_p(v_1,v_2) \quad , \quad \forall \tau_1,\tau_2 \in \mathbb{R} \quad , \quad \forall v_1, v_2 \in T_p\mathcal{M} \quad .$$

Proposition

For any $\Lambda \in \mathcal{S}_1(\mathcal{M},\mathcal{G})$ and $p \in \mathcal{M}$, the map $e_p : (\mathcal{S}_1(\mathcal{M},\mathcal{G}), (\ ,\ )_1) \to (\mathbb{R} \oplus T_p\mathcal{M}, (\ ,\ )_p)$ is an injective isometry.

The isometry group $\mathrm{Iso}(\mathcal{M},\mathcal{G})$ acts naturally on $\mathcal{C}^\infty(\mathcal{M},\mathbb{R})$:

$$\varphi^*(f) \stackrel{\text{def.}}{=} f\circ\varphi^{-1}~,~~\forall \varphi\in \mathrm{Iso}(\mathcal{M},\mathcal{G})~~\forall f\in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})~~.$$

This action preserves the subspace $\mathcal{S}_\beta(\mathcal{M},\mathcal{G})\subset \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ and restricts to a linear representation $\mathcal{R}_\mathcal{G}:\mathrm{Iso}(\mathcal{M},\mathcal{G})\to \mathrm{Aut}_\mathbb{R}(\mathcal{S}_\beta(\mathcal{M},\mathcal{G}))$ of the isometry group, called the Hesse representation:

$$\mathcal{R}_\mathcal{G}(\varphi)(f) \stackrel{\text{def.}}{=} \varphi^*(\Lambda)=\Lambda\circ\varphi^{-1}~~,~~\forall \Lambda\in \mathcal{S}_\beta(\mathcal{M},\mathcal{G})~~.$$

Proposition

The Hesse representation is $(~,~)_\beta$-orthogonal, i.e. any representation operator $R_\mathcal{G}(\varphi)$ $(\varphi\in \mathrm{Iso}(\mathcal{M},\mathcal{G}))$ preserves the Hesse pairing.

Definition

A Hesse function $\Lambda\in\mathcal{S}_\beta(\mathcal{M},\mathcal{G})$ is called:

- time-like, if $(\Lambda,\Lambda)_\mathcal{S}>0$
- space-like, if $(\Lambda,\Lambda)_\mathcal{S}<0$
- light-like, if $(\Lambda,\Lambda)_\mathcal{S}=0$.

Let $\mathrm{D}^n \stackrel{\text{def.}}{=} \{u \in \mathbb{R}^n \mid 0 \le ||u||_E < 1\}$ ($||u||_E \stackrel{\text{def.}}{=} \sqrt{u_1^2 + \ldots + u_n^2}$).

Definition

The hyperbolic ball is the complete Riemannian manifold:

$$\mathbb{D}^n \stackrel{\text{def.}}{=} (\mathrm{D}^n, G_n) \quad ,$$

where G_n be the Poincaré ball metric, i.e. the unique complete metric on D^n of constant sectional curvature $K_{G_n} = -1$.

We have:

$$\mathrm{d}s^2_{G_n} = \frac{4}{(1 - ||u||_E^2)^2} \sum_{i=1}^{n} \mathrm{d}u_i^2 = \frac{4}{(1-\rho^2)^2}(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2) \quad ,$$

where (ρ, θ) are spherical coordinates in $\mathbb{R}^n$, with $\theta = (\theta_1, \ldots, \theta_{n-1})$ and:

$$\mathrm{d}\theta^2 = \sum_{i=1}^{n-1} h_i^2(\theta) \mathrm{d}\theta_i^2$$

is the squared line element on S^{n-1}, with Lamé coefficients $h_1(\theta) = 1$ and:

$$h_i(\theta) = \prod_{j \le i-1} \sin(\theta_j) \quad , \quad \forall i = 2, \ldots, n \quad .$$

Consider the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{1,n} \stackrel{\text{def.}}{=} (\mathbb{R}^{n+1}, (\ ,\ ))$, where:

$$(x,y) \stackrel{\text{def.}}{=} x_0 y_0 - \sum_{i=1}^{n} x_i y_i = \eta^{\mu\nu} x_\mu x_\nu \quad .$$

Define $\vec{x} \stackrel{\text{def.}}{=} (x^1, \ldots, x^n)$, so that $x = (x^0, \vec{x})$ and $(x,y) = x^0 y^0 - \vec{x} \cdot \vec{y}$, where $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^n$. Let S_n^+ denote the future sheet of the hyperboloid $(x,x) = 1$:

$$S_n^+ \stackrel{\text{def.}}{=} \{x \in \mathbb{R}^{n+1} | (x,x) = 1 \ \&\ x_0 > 0\} = \left\{x \in \mathbb{R}^{n+1} | x_0 = \sqrt{1 + x_1^2 + \ldots + x_n^2}\right\} \ .$$

For any $x \in S_n^+$, define $u_i \stackrel{\text{def.}}{=} \frac{x_i}{x_0+1}$. Then S_n^+ is diffeomorphic with D^n through the Weierstrass map $\Xi : \mathrm{D}^n \to S_n^+$, which is given by:

$$\Xi(u) \stackrel{\text{def.}}{=} \left(\frac{1 + ||u||_E^2}{1 - ||u||_E^2}, \frac{2u_1}{1 - ||u||_E^2}, \ldots, \frac{2u_n}{1 - ||u||_E^2} \right) \quad .$$

The components $\Xi^\nu(u)$satisfy the relation $\eta_{\mu\nu} \Xi^\mu(u) \Xi^\nu(u) = -1$ and are Weierstrass coordinates of $u \in \mathrm{D}^n$. Moreover, Ξ is an isometry from $\mathbb{D}^n$ to S_n^+, when S_n^+ is endowed with the Riemannian metric induced by the opposite of the Minkowski metric $(\ ,\ )$.

Theorem

The hyperbolic ball $\mathbb{D}^n$ is a maximally Hesse manifold. Moreover, the family $\mathcal{E} \stackrel{\mathrm{def.}}{=} (\Xi^\mu)_{\mu=0,\ldots,n}$ is an orthonormal basis of $(\mathcal{S}_1(\mathbb{D}^n), (\ ,\ )_1)$:

$$(\Xi^\mu, \Xi^\nu)_1 = \eta^{\mu\nu} \quad , \quad \forall \mu, \nu = 0, \ldots, n \quad .$$

In particular, we have $(\mathcal{S}_1(\mathbb{D}^n), (\ ,\ )_1) \simeq \mathbb{R}^{1,n}$.

Hence a Hesse function on $\mathbb{D}^n$ has the general form:

$$\Lambda_B = (B, \Xi) = B_\mu \Xi^\mu = \eta_{\mu\nu} B^\mu \Xi^\nu \quad (B \in \mathbb{R}^{n+1}) \quad .$$

The map $\mathbf{\Lambda} : \mathbb{R}^{n+1} \to \mathcal{S}_1(\mathbb{D}^n)$ given by $\mathbf{\Lambda}(B) \stackrel{\mathrm{def.}}{=} \Lambda_B$ is a linear isomorphism.

Proposition

The Hesse pairing of $\mathbb{D}^n$ is given by:

$$(\Lambda_{B_1}, \Lambda_{B_2})_{\mathcal{S}} = (B_1, B_2) \quad , \quad \forall B_1, B_2 \in \mathbb{R}^{n+1} \quad .$$

Thus $\mathbf{\Lambda} : \mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, (\ ,\ )) \to (\mathcal{S}_1(\mathbb{D}^n), (\ ,\ )_1)$ is an isometry.

$\mathrm{Iso}^+(\mathbb{D}^n) = \mathrm{Iso}_o(\mathbb{D}^n) \simeq \mathrm{SO}^+(1,n) = \mathrm{O}_o(1,n)$ acts on $\mathbb{R}^{n+1}$ (and hence on the hyperboloid model S_n^+) through the fundamental representation:

$$\mathbb{R}^{n+1} \ni x \to Ux \in \mathbb{R}^{n+1} \quad , \quad \forall U \in \mathrm{SO}^+(1,n) \quad .$$

The natural action $u \to \varphi_U(u)$ of $\mathrm{Iso}^+(\mathbb{D}^n)$ on $\mathbb{D}^n$ is uniquely determined by:

$$\Xi(\varphi_U(u)) = U\Xi(u) \ , \quad \forall U \in \mathrm{SO}^+(1,n) \quad .$$

Proposition

The Hesse representation of $\mathrm{Iso}^+(\mathbb{D}^n) = \mathrm{SO}^+(1,n)$ is equivalent with the fundamental representation of $\mathrm{SO}^+(1,n)$ through the map $\mathbf{\Lambda}$:

$$\mathcal{R}(U)(\Lambda_B) = \Lambda_{U(B)} \quad , \quad \forall B \in \mathbb{R}^{n+1} \quad , \quad \forall U \in \mathrm{SO}^+(1,n) \quad .$$

Proposition

1. *Two time-like Hesse functions* $\Lambda_B, \Lambda_{B'} \in \mathcal{S}_1(\mathbb{D}^n)$ *lie on the same orbit iff* $(B, B) = (B', B')(> 0)$ *and* $\mathrm{sign}(B^0) = \mathrm{sign}(B'^0)$*. The set of time-like orbits of* $\mathrm{Iso}_+(\mathbb{D})$ *on* $\mathcal{S}_1(\mathbb{D}^n)$ *is in bijection with* $\{-1, 1\} \times \mathbb{R}_{>0}$*. The time-like orbit corresponding to* $(\epsilon, K) \in \{-1, 1\} \times \mathbb{R}_{>0}$ *is diffeomorphic with the sheet* $\mathrm{sign}(B^0) = \epsilon$ *of the two-sheeted hyperboloid* $(B, B) = K$*.*
2. *Two space-like Hesse functions* $\Lambda_B, \Lambda_{B'} \in \mathcal{S}_1(\mathbb{D}^n)$ *lie on the same orbit iff* $(B, B) = (B', B')(< 0)$*. The set of space-like orbits of* $\mathrm{Iso}_+(\mathcal{M}, \mathcal{G})$ *on* $\mathcal{S}_1(\mathbb{D}^n)$ *is in bijection with* $\mathbb{R}_{<0}$*. The space-like orbit corresponding to* $K \in \mathbb{R}_{<0}$ *is diffeomorphic with the one-sheeted hyperboloid* $(B, B) = K$*.*
3. *There exist exactly two non-trivial light-like orbits of* $\mathrm{Iso}^+(\mathbb{D}^n)$ *on* $\mathcal{S}_1(\mathbb{D}^n)$*, which coincide with the connected components of the complement of the origin in the light cone.*

Up to conjugation $\sim$ *in* $\mathrm{SO}^+(1, n)$*, we have:*

$$\mathrm{Stab}_{\mathrm{Iso}^+(\mathbb{D}^n)}(\Lambda_B) \sim \begin{cases} \mathrm{SO}(n) & \text{if } (B,B) > 0 \ (\mathrm{time-like}) \\ \mathrm{SO}^+(1, n-1) & \text{if } (B,B) < 0 \ (\mathrm{space-like}) \\ \mathrm{ISO}_o(n-1) & \text{if } (B,B) = 0 \ (\mathrm{light-like}) \end{cases},$$

where $\mathrm{ISO}_o(n-1)$ *is the special Euclidean group in* $n-1$ *dimensions.*

Any hyperbolic n-manifold is a Riemannian quotient $(\mathcal{M}, G) = \mathbb{D}^n/\Gamma$ of the Poincaré ball $\mathbb{D}^n$ by a discrete subgroup $\Gamma \subset \mathrm{Iso}^+(\mathbb{D}^n) \simeq \mathrm{SO}^+(1,n)$. Let $\pi : \mathrm{D}^n \to \mathcal{M}$ be the canonical projection. When $n \geq 3$, Mostow's rigidity theorem implies that $(\mathcal{M}, G)$ is determined by $\pi_1(\mathcal{M}) \simeq \Gamma$ up to isometry; when $n = 2$, the hyperbolic metric G can have moduli. Let $\mathcal{S}_1(\mathbb{D}^n)^\Gamma$ be the space of Γ-invariant Hesse functions on $\mathbb{D}^n$.

Theorem

The map $\pi^* : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathrm{D}^n)^\Gamma$ defined through:

$$\pi^*(\Lambda) \stackrel{\mathrm{def.}}{=} \Lambda \circ \pi$$

induces a linear isomorphism from $\mathcal{S}_1(\mathcal{M}, G)$ to $\mathcal{S}_1(\mathbb{D}^n)^\Gamma$. In particular, $(\mathcal{M}, G)$ is a Hesse manifold iff Γ conjugates in $\mathrm{SO}^+(1,n)$ to a discrete subgroup of $\mathrm{SO}^+(1,n)$, $\mathrm{ISO}_o(n-1)$ or $\mathrm{SO}(n)$.

Consider the open disk of unit Euclidean radius in the complex u-plane:

$$\mathrm{D}^2 \stackrel{\text{def.}}{=} \{u \in \mathbb{C} \mid 0 \leq |u| < 1\} \quad .$$

The *Poincaré disk metric* is:

$$\mathrm{d}s^2_{G_2} = \frac{4}{(1-|u|^2)^2}|\mathrm{d}u|^2 = \frac{4}{(1-\rho^2)^2}(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2) \quad ,$$

where $\rho \stackrel{\text{def.}}{=} |u|$ and $\theta \stackrel{\text{def.}}{=} \arg(u) \in [0, 2\pi)$ are polar coordinates on the u-plane. We have:

$$\mathrm{Iso}^+(\mathbb{D}^2) \simeq \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{PSU}(1,1) \stackrel{\text{def.}}{=} \mathrm{SU}(1,1)/\{-I_2, I_2\} \quad ,$$

where $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$ is the closed subgroup of $\mathrm{SL}(2,\mathbb{C})$ defined through:

$$\mathrm{SU}(1,1) \stackrel{\text{def.}}{=} \{U \in \mathrm{Mat}(2,\mathbb{C}) | U^\dagger = JU^{-1}J \;\;\&\;\; \det U = +1\} \quad .$$

The matrix $J \stackrel{\text{def.}}{=} \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]$ satisfies $J^\dagger = J = J^{-1}$. We have:

$$\mathrm{SU}(1,1) = \{U(\eta,\sigma) \Big| \eta, \sigma \in \mathbb{C} : |\eta|^2 - |\sigma|^2 = 1\} \quad ,$$

where $U(\eta,\sigma) \stackrel{\text{def.}}{=} \left[\begin{array}{cc} \eta & \sigma \\ \bar{\sigma} & \bar{\eta} \end{array}\right]$ and $\mathrm{SU}(1,1)$ acts on D^2 through:

$$\varphi_U(u) = \frac{\eta u + \sigma}{\bar{\sigma} u + \bar{\eta}} \quad (u \in \mathrm{D}) \quad .$$

Consider the future sheet of the hyperboloid $(x,x)=1$ in $\mathbb{R}^{1,2}=(\mathbb{R}^3,(\ ,\ ))$:

$$S_2^+ \stackrel{\text{def.}}{=} \{x\in\mathbb{R}^3|(x,x)=1 \ \ \& \ \ x_0>0\}=\{x\in\mathbb{R}^3|x_0=\sqrt{1+x_1^2+x_2^2}\} \ .$$

The Weierstrass map $\Xi:\mathrm{D}^2\to S_2^+$ is given by:

$$\Xi(u)\stackrel{\text{def.}}{=}\left(\frac{1+|u|^2}{1-|u|^2},\frac{2\mathrm{Re}u}{1-|u|^2},\frac{2\mathrm{Im}u}{1-|u|^2}\right) \ .$$

The components $\Xi^\mu(u)$ are the Weierstrass coordinates of $u\in\mathrm{D}^2$.

Proposition

$\mathcal{E}\stackrel{\text{def.}}{=}(\Xi^0,\Xi^1,\Xi^2)$ is an orthonormal basis of the 3-dimensional Minkowski space $(\mathcal{S}_1(\mathbb{D}^2),(\ ,\ )_1)$:

$$(\Xi^\mu,\Xi^\nu)_1=\eta^{\mu\nu} \ \ , \ \ \forall\mu,\nu=0,1,2 \ .$$

Hence the general Hesse function on D^2 has the form:

$$\Lambda_B(u)=(B,\Xi(u))=B_\mu\Xi^\mu(u)=B^0\frac{1+|u|^2}{1-|u|^2}-2B^1\frac{\mathrm{Re}u}{1-|u|^2}-2B^2\frac{\mathrm{Im}u}{1-|u|^2}$$

with arbitrary $B=(B^0,B^1,B^2)\in\mathbb{R}^3$. The unit Hesse pairing is given by:

$$(\Lambda_B,\Lambda_{B'})_1=(B,B')=\eta_{\mu\nu}B^\mu B'^\nu \ .$$

Proposition

The Hesse representation of $\mathrm{Iso}^+(\mathrm{D}^2) = \mathrm{PSU}(1,1)$ *is induced by the adjoint representation of* $\mathrm{SU}(1,1)$.

We have:

$$\Xi(\varphi_U(u)) = \mathrm{Ad}_0(U)(\Xi(u)) \ , \quad \forall u \in \mathrm{D} \ , \quad \forall U \in \mathrm{SU}(1,1) \quad ,$$

where **the Hesse representation** $\mathrm{Ad}_0 : \mathrm{SU}(1,1) \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3)$ **is given by**:

$$\mathrm{Ad}_0(U) = \begin{bmatrix} |\eta|^2 + |\sigma|^2 & -2\mathrm{Re}(\eta\bar{\sigma}) & 2\mathrm{Im}(\eta\bar{\sigma}) \\ -2\mathrm{Re}(\eta\sigma) & \mathrm{Re}(\eta^2 + \sigma^2) & -\mathrm{Im}(\eta^2 - \sigma^2) \\ -2\mathrm{Im}(\eta\sigma) & \mathrm{Im}(\eta^2 + \sigma^2) & \mathrm{Re}(\eta^2 - \sigma^2) \end{bmatrix} \quad .$$

The Killing pairing on the Lie algebra $\mathrm{su}(1,1) \simeq \mathbb{R}^3$ identifies with the Minkowski pairing $(\ , \ )$ through an isomorphism which identifies Ad_0 with the adjoint representation of $\mathrm{SU}(1,1)$. Ad_0 preserves this pairing and descends to an isomorphism of groups $\overline{\mathrm{Ad}_0} : \mathrm{PSU}(1,1) \xrightarrow{\sim} \mathrm{SO}_o(1,2)$.

Proposition

1. *Two time-like Hesse functions* $\Lambda_B, \Lambda_{B'}$ *belong to the same (time-like) orbit of* $\mathrm{PSU}(1,1)$ *iff* $(B,B) = (B',B')(>0)$ *and* $\mathrm{sign}(B^0) = \mathrm{sign}(B'^0)$*. The set of time-like orbits is in bijection with* $\{-1,1\} \times \mathbb{R}_{>0}$*. The time-like orbit corresponding to* $(\epsilon, K) \in \{-1,1\} \times \mathbb{R}_{>0}$ *is diffeomorphic with the sheet* $\mathrm{sign}(B^0) = \epsilon$ *of the two-sheeted hyperboloid* $(B,B) = K$*.*
2. *Two space-like Hesse functions* $\Lambda_B, \Lambda_{B'}$ *belong to the same orbit of* $\mathrm{PSU}(1,1)$ *iff* $(B,B) = (B',B')(<0)$*. The set of space-like orbits is in bijection with* $\mathbb{R}_{<0}$*. The space-like orbit corresponding to* $K \in \mathbb{R}_{<0}$ *is diffeomorphic with the one-sheeted hyperboloid* $(B,B) = K$*.*
3. *The non-trivial light-like orbits of* $\mathrm{PSU}(1,1)$ *on* $\mathcal{S}_1(\mathbb{D}^2)$ *coincide with the connected components of the complement of the origin in the light cone.*

The stabilizer subgroup of a nontrivial solution Λ_B *in* $\mathrm{PSU}(1,1) \simeq \mathrm{SO}_o(1,2)$ *is:*

$$\mathrm{Stab}_{\mathrm{PSU}(1,1)}(\Lambda_B) \simeq \begin{cases} \mathrm{SO}(2) & \text{if } (B,B) > 0 \ (\mathrm{time-like}) \\ \mathrm{SO}_o(1,1) \simeq (\mathbb{R},+) & \text{if } (B,B) < 0 \ (\mathrm{space-like}) \\ \mathrm{ISO}_o(1) \simeq (\mathbb{R},+) & \text{if } (B,B) = 0 \ (\mathrm{light-like}) \end{cases} .$$

Proposition

Let Γ be a non-trivial discrete subgroup of $\mathrm{PSU}(1,1)$. Suppose that there exists a non-trivial unit Hesse function $\Lambda \in \mathcal{S}_1(\mathbb{D}^2)$ such that $\Gamma \subset \mathrm{Stab}_{\mathrm{PSU}(1,1)}(\Lambda)$. Then Γ is a cyclic group. Moreover:

1. *If Λ is time-like then Γ is a finite cyclic group and hence is generated by an elliptic element of $\mathrm{PSU}(1,1)$.*
2. *If Λ is light-like then Γ is an infinite cyclic group generated by a parabolic element of $\mathrm{PSU}(1,1)$*
3. *If Λ is space-like then Γ is an infinite cyclic group generated by a hyperbolic element of $\mathrm{PSU}(1,1)$.*

Corollary

Let Γ be a non-trivial surface group (i.e. a non-trivial Fuchsian group without elliptic elements) which stabilizes a non-trivial unit Hesse function $\Lambda \in \mathcal{S}_1(\mathbb{D}^2)$. Then Λ is light-like or space-like and Γ is an infinite cyclic group. Moreover:

1. *If Λ is light-like then Γ is a parabolic cyclic group and $\mathbb{D}^2/\Gamma$ is a hyperbolic cusp.*
2. *If Λ is space-like then Γ is a hyperbolic cyclic group and $\mathbb{D}^2/\Gamma$ is a hyperbolic annulus.*

Proposition

Any complete Hesse surface (Σ, G) is hyperbolic.

Theorem

A hyperbolic surface $(\Sigma, G) \simeq \mathbb{D}^2/\Gamma$ with $\Gamma \neq 1$ is Hesse iff Γ is a parabolic or a hyperbolic cyclic group, in which case we have $\mathrm{ind}(\Sigma, G) = 1$. Moreover:

1. *If Γ is a parabolic cyclic group (i.e. if (Σ, G) is a hyperbolic cusp), then $\mathcal{S}_1(\mathbb{D}^2)^\Gamma \simeq \mathcal{S}_1(\Sigma, G)$ is a light ray of $(\mathcal{S}_1(\mathbb{D}^2), (\ ,\ )_\mathcal{S}) \simeq \mathbb{R}^{1,2}$.*
2. *If Γ is a hyperbolic cyclic group (i.e. if (Σ, G) is a hyperbolic annulus), then $\mathcal{S}_1(\mathbb{D}^2)^\Gamma \simeq \mathcal{S}_1(\Sigma, G)$ is a space-like ray of $(\mathcal{S}_1(\mathbb{D}^2), (\ ,\ )_\mathcal{S}) \simeq \mathbb{R}^{1,2}$.*

Corollary

The two-field cosmological model defined by a complete surface $(\Sigma, \mathcal{G})$ uniformized by $\Gamma \subset \mathrm{PSU}(1,1)$ **admits Hessian Noether symmetries iff** $(\Sigma, \mathcal{G})$ **is an elementary surface of Gaussian curvature** $K_\mathcal{G} = -3/8$. *In this case, the space of Hesse functions is three-dimensional when $\Gamma \simeq 1$ (hyperbolic disk) and one-dimensional when $\Gamma \simeq \mathbb{Z}$ (hyperbolic cusp or annulus).*