Hesse manifolds and Hesse functions

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Definition

An n-dimensional scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected Riemannian n-manifold (called scalar manifold)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

$(\mathcal{M}, \mathcal{G})$ is oriented and complete.

Each n-dimensional scalar triple defines a cosmological model with n real scalar fields:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g, \varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|} \left[\frac{R(g)}{2} - \frac{1}{2} \mathrm{Tr}_g \varphi^*(\mathcal{G}) - V(\varphi) \right] \quad . \tag{1}$$

Take g to describe a simply-connected and spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \quad , \quad \vec{x} = (x^1, x^2, x^3) \quad , \quad a(t) > 0 \ \forall t) \tag{2}$$

and φ to depends only on the cosmological time t:

$$\varphi = \varphi(t) \quad . \tag{3}$$

Substituting (2) and (3) in (1) and ignoring the integration over $\vec{x}$ gives the minisuperspace action:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[a,\varphi] = \int_{-\infty}^{\infty} \mathrm{d}t \, L_{\mathcal{M},\mathcal{G},V}(a(t),\varphi(t),\dot{\varphi}(t)) \quad ,$$

where the minisuperspace Lagrangian is:

$$L_{\mathcal{M},\mathcal{G},V}(a,\varphi,\dot{\varphi}) \stackrel{\mathrm{def.}}{=} -3a\dot{a}^2 + a^3\left[\frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi)\right] = a^3\left[-3H^2 + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi)\right] \quad .$$

Here $\dot{} \stackrel{\mathrm{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}t}$ and $H \stackrel{\mathrm{def.}}{=} \frac{\dot{a}}{a}$ is the Hubble parameter. This Lagrangian describes a classical system with $n+1$ degrees of freedom and configuration space $\mathcal{N} \stackrel{\mathrm{def.}}{=} \mathbb{R}_{>0} \times \mathcal{M}$. The E-L equations are equivalent with:

$$\begin{aligned} &3H^2 + 2\dot{H} + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi) = 0 \\ &(\nabla_t + 3H)\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}} V)(\varphi) = 0 \quad . \end{aligned}$$

We must also impose the Friedmann constraint:

$$\frac{1}{2}||\dot{\varphi}||^2 + V \circ \varphi = 3H^2 \quad ,$$

which amounts to the zero energy condition.

Proposition

When supplemented with the Friedmann constraint, the Euler-Lagrange equations of $L_{\mathcal{M},\mathcal{G},V}$ are equivalent with the cosmological equations:

$$\begin{aligned}\nabla_t\dot{\varphi} + 3H\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}} V)\circ\varphi &= 0\\ \dot{H} + 3H^2 - V\circ\varphi &= 0\\ \dot{H} + \frac{1}{2}||\dot{\varphi}||_{\mathcal{G}}^2 &= 0\ .\end{aligned}$$

Remark

One can eliminate H algebraically from the cosmological equations as:

$$H(t) = \frac{1}{\sqrt{6}}\epsilon(t)\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2V(\varphi(t))\right]^{1/2}\ ,\ \text{where}\ \ \epsilon(t) \stackrel{\text{def.}}{=} \mathrm{sign} H(t)\ \ ,$$

thereby obtaining the reduced cosmological equation:

$$\nabla_t\dot{\varphi}(t) + \sqrt{\frac{3}{2}}\epsilon(t)\left[||\dot{\varphi}(t)||_{\mathcal{G}}^2 + 2V(\varphi(t))\right]^{1/2}\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0\ \ ,$$

which defines a (dissipative) geometric dynamical system on $T\mathcal{M}$.

The characteristic system for strong variational symmetries

We have a natural decomposition $T\mathcal{N} = T_1\mathcal{N} \oplus T_2\mathcal{N}$, where:

$$T_1\mathcal{N} \stackrel{\text{def.}}{=} p_1^*(T\mathbb{R}_{>0}) \ , \ \ T_2\mathcal{N} \stackrel{\text{def.}}{=} p_2^*(T\mathcal{M}) \ \ (p_1 : T\mathcal{N} \to \mathbb{R}_{>0}, \ p_2 : T\mathcal{N} \to \mathcal{M}) \ .$$

Theorem

A vector field $X \in \mathcal{X}(\mathcal{N})$ is a time-independent Noether symmetry iff:

$$X(a,\varphi) = X_{\Lambda,Y}(a,\varphi) = \frac{\Lambda(\varphi)}{\sqrt{a}}\partial_a \ + Y(\varphi) - \frac{4}{a^{3/2}}(\mathrm{grad}_{\mathcal{G}}\Lambda)(\varphi) \ ,$$

where $\Lambda \in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ and $Y \in \mathcal{X}(\mathcal{M})$ satisfy the Λ-system:

$$\mathrm{Hess}_{\mathcal{G}}(\Lambda) = \frac{3}{8}\mathcal{G}\Lambda \ , \ \ \langle \mathrm{d}V, \mathrm{d}\Lambda\rangle_{\mathcal{G}} = \frac{3}{4}V\Lambda$$

and Y-system:

$$\mathcal{K}_{\mathcal{G}}(Y) = 0 \ , \ \ Y(V) = 0 \ .$$

which can also be written as follows:

$$\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)\Lambda = \frac{3}{8}\mathcal{G}_{ij}\Lambda \ , \ \ \nabla_i Y_j + \nabla_j Y_i = 0$$

$$\mathcal{G}^{ij}\partial_i V\partial_j\Lambda = \frac{3}{4}V\Lambda \ , \ \ Y^i\partial_i V = 0 \ .$$

Visible and Hessian symmetries

Definition

A time-independent Noether symmetry $X = X_{\Lambda,Y}$ is called:

- visible if $\Lambda = 0$.
- Hessian if $Y = 0$.

The scalar triple $(\mathcal{M}, \mathcal{G}, V)$ and cosmological model are called visibly-symmetric or Hessian if they admit visible or Hessian symmetries, respectively.

Let $N_H(\mathcal{M}, \mathcal{G}, V)$, $N_V(\mathcal{M}, \mathcal{G}, V)$ and $N(\mathcal{M}, \mathcal{G}, V)$ be respetcvely the linear spaces of Hessian, visible and time-independent symmetries.

Proposition

We have a linear isomorphism $N(\mathcal{M}, \mathcal{G}, V) \simeq_{\mathbb{R}} N_H(\mathcal{M}, \mathcal{G}, V) \oplus N_V(\mathcal{M}, \mathcal{G}, V)$.

Remark

Existence of a Hessian symmetry simplifies various cosmological problems. For example, it gives:

$$\left[\frac{a(t)}{a(t_0)}\right]^{3/2} \Lambda(\varphi(t)) - \Lambda_0 = \left(\frac{3}{2} H_0 \Lambda_0 + (\mathrm{d}_{\varphi_0}\Lambda)(\dot{\varphi}_0)\right)(t - t_0) \quad .$$

Let $\beta = \sqrt{\mathbf{3/8}}$ and $G = \beta^2 \mathcal{G}$.

Definition

The rescaled scalar manifold is the Riemannian manifold $(\mathcal{M}, G)$.

Then Λ-system of $(\mathcal{M}, \mathcal{G}, V)$ is equivalent with:

- $\mathrm{Hess}_G(\Lambda) = \Lambda G$ (the Hesse equation of the rescaled scalar manifold)
- $\langle \mathrm{d}V, \mathrm{d}\Lambda \rangle_G = 2V\Lambda$ (the Λ-V equation of the rescaled scalar manifold)

Definition

Let $(\mathcal{M}, G)$ be a complete Riemannian manifold. A Hesse function of $(\mathcal{M}, G)$ is a smooth solution of the Hesse equation of $(\mathcal{M}, G)$:

$$\mathrm{Hess}_G(\Lambda) = \Lambda G \quad .$$

Let $\mathcal{S}(\mathcal{M}, G)$ be the linear space of Hesse functions of $(\mathcal{M}, G)$. The Hesse index of $(\mathcal{M}, G)$ is defined through:

$$\mathfrak{h}(\mathcal{M}, \mathcal{G}) \stackrel{\mathrm{def.}}{=} \dim \mathcal{S}(\mathcal{M}, G) \quad .$$

The complete Riemannian manifold $(\mathcal{M}, G)$ is called a Hesse manifold if it admits non-trivial Hesse functions, i.e. if $\mathfrak{h}(\mathcal{M}, G) > 0$.

Proposition

We have $\mathfrak{h}(\mathcal{M}, G) \leq n+1$. We say that $(\mathcal{M}, G)$ is maximally Hesse if equality is attained.

Definition

The Hesse pairing of $(\mathcal{M}, G)$ is the symmetric $\mathbb{R}$-bilinear map $(\ ,\ )_G : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ defined through:

$$(f_1, f_2)_G \stackrel{\text{def.}}{=} f_1 f_2 - \langle \mathrm{d}f_1, \mathrm{d}f_2 \rangle_G = f_1 f_2 - \langle \mathrm{grad}_G f_1, \mathrm{grad}_G f_2 \rangle_G \ , \ \forall f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}) \ .$$

Proposition

Let $\Lambda_1, \Lambda_2 \in \mathcal{S}(\mathcal{M}, G)$ be two Hesse functions on $(\mathcal{M}, G)$. Then the Hesse pairing $(\Lambda_1, \Lambda_2)_G$ is constant on $\mathcal{M}$.

Hence the Hesse pairing restricts to a symmetric $\mathbb{R}$-bilinear map:

$$(\ ,\ )_G : \mathcal{S}(\mathcal{M}, G) \times \mathcal{S}(\mathcal{M}, G) \to \mathbb{R} \ , \ (\Lambda_1, \Lambda_2)_G \stackrel{\text{def.}}{=} (\Lambda_1, \Lambda_2)_G$$

on the vector space $\mathcal{S}(\mathcal{M}, G)$.

General solution of the Λ-V equation

Theorem

For any non-trivial Hesse function $\Lambda \in \mathcal{S}(\mathcal{M}, G)$, any smooth solution of the Λ-V-equation of $(\mathcal{M}, G)$ takes the following form:

$$V = \Omega ||\mathrm{d}\Lambda||_G^2 = \Omega \left[\Lambda^2 - (\Lambda, \Lambda)_G\right] \quad , \tag{4}$$

where $\Omega \in \mathcal{C}^\infty(\mathcal{M} \setminus \mathrm{Crit}(\Lambda))$ is constant along the gradient flow of Λ:

$$\langle \mathrm{d}\Omega, \mathrm{d}\Lambda \rangle_G = 0 \quad . \tag{5}$$

Definition

The cosmological model defined by the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is called *weakly Hessian* if the rescaled scalar manifold $(\mathcal{M}, G)$ is Hesse. It is called *Hessian* if it admits non-trivial Hessian symmetries.

Proposition

The cosmological model defined by the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is Hessian iff it is weakly Hessian and the scalar potential V has the form (4)*, with Ω a solution of* (5).

Theorem

Let $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ be a Hesse function of $(\mathcal{M}, G)$ and suppose that V satisfies the Λ-V- equation with respect to Λ. Then the minisuperspace Lagrangian takes the following form in natural local coordinates (a_0, y, υ) on the configuration space $\mathcal{N}$:

$$L(y, \dot{y}, \upsilon, \dot{\upsilon}) = -3(\Lambda, \Lambda)_G \dot{\upsilon}^2 - 3a_0 \dot{a}_0^2 + a_0^3 L_\lambda(y, \dot{y}) \quad , \tag{6}$$

where the reduced Lagrangian L_λ *is given by:*

$$L_\lambda(y, \dot{y}) \stackrel{\text{def.}}{=} \frac{1}{2\beta^2} ||\dot{y}||_g^2 - V_\lambda(y) \quad . \tag{7}$$

with $V_\lambda \stackrel{\text{def.}}{=} V|_{\mathcal{M}_\Lambda(\lambda)}$ the restriction of V to the level set $\mathcal{M}_\Lambda(\lambda)$ and g the metric induced by G on this level set. We have:

$$V_\lambda(y) = \left[\lambda^2 - (\Lambda, \Lambda)_G\right] \Omega(y) \quad , \tag{8}$$

with Ω a smooth arbitrary function defined on $\mathcal{M}_\Lambda(\lambda)$.

Definition

The Hesse norm of a Hesse function $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ is the non-negative number $\kappa_\Lambda \stackrel{\text{def.}}{=} \sqrt{|(\Lambda, \Lambda)_G|}$, while its type indicator is the sign factor $\epsilon_\Lambda \stackrel{\text{def.}}{=} \mathrm{sign}(\Lambda, \Lambda)_G$. A non-trivial Hesse function Λ is called timelike, *spacelike* or lightlike when ϵ_Λ equals $+1$, -1 or 0.

Proposition

Let $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ be a non-trivial Hesse function. Then:

1. *If Λ is timelike, then $Z(\Lambda) = \emptyset$ and Λ has constant sign η_Λ on $\mathcal{M}$. Moreover, Λ has exactly one critical point, with critical value $\eta_\Lambda \kappa_\Lambda$, which is a global minimum or maximum according to whether $\eta_\Lambda = +1$ or -1.*
2. *If Λ is spacelike, then $\mathrm{Crit}(\Lambda) = \emptyset$. Moreover, the zero locus of Λ is the following non-singular hypersurface in $\mathcal{M}$:*

$$Z(\Lambda) = \{m \in \mathcal{M} \,|\, ||\mathrm{d}_m \Lambda||_G = \kappa_\Lambda\}$$

   and coincides with the locus where $\mathrm{grad}_G \Lambda$ has minimal norm.
3. *If Λ is lightlike, then $Z(\Lambda) = \mathrm{Crit}(\Lambda) = \emptyset$ and Λ has constant sign on $\mathcal{M}$, which we denote by η_Λ.*

Definition

A timelike or lightlike non-trivial Hesse function $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ is called future or past pointing when $\eta_\Lambda = +1$ or -1.

Definition

Let $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ be a non-trivial Hesse function of $\mathcal{M}$. The characteristic set of Λ is the following closed subset of $\mathcal{M}$:

$$Q_\Lambda \stackrel{\text{def.}}{=} \begin{cases} \mathrm{Crit}(\Lambda) , & \text{if } \Lambda \text{ is timelike} \\ Z(\Lambda) , & \text{if } \Lambda \text{ is spacelike} \\ \mathcal{M}_{|\Lambda|}(1) , & \text{if } \Lambda \text{ is lightlike} \end{cases} .$$

The characteristic constant of Λ is defined through:

$$C_\Lambda \stackrel{\text{def.}}{=} \begin{cases} \kappa_\Lambda , & \text{if } \epsilon = +1 \\ 0 , & \text{if } \epsilon = -1 \\ 1 , & \text{if } \epsilon = 0 \end{cases} .$$

Setting $\mathcal{U}_\Lambda \stackrel{\text{def.}}{=} \mathcal{M} \setminus \mathrm{Crit}(\Lambda)$, we have:

$$Q_\Lambda = \{m \in \mathcal{U}_\Lambda \mid |\Lambda(m)| = C_\Lambda\} .$$

Definition

The characteristic sign function $\Theta_\Lambda : \mathcal{M} \to \mathbb{R}$ of a non-trivial Hesse function $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ is defined through:

$$\Theta_\Lambda(m) \stackrel{\text{def.}}{=} \begin{cases} 1 \ , & \text{if } \epsilon_\Lambda = +1 \\ \mathrm{sign}(\Lambda(m)) \ , & \text{if } \epsilon_\Lambda = -1 \\ \mathrm{sign}(|\Lambda(m)| - 1) \ , & \text{if } \epsilon_\Lambda = 0 \end{cases} \ .$$

The Λ-distance function $d_\Lambda : \mathcal{M} \to \mathbb{R}$ is defined through:

$$d_\Lambda(m) \stackrel{\text{def.}}{=} \Theta_\Lambda(m)\mathrm{dist}_G(m, Q_\Lambda) \ .$$

Theorem

Let $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ be a non-trivial Hesse function. Then the following relation holds for all $m \in \mathcal{M}$:

$$\Lambda(m) = \begin{cases} \mathrm{sign}(\Lambda)\kappa_\Lambda \cosh d_\Lambda(m) \ , & \text{if } \epsilon_\Lambda = +1 \\ \kappa_\Lambda \sinh d_\Lambda(m) \ , & \text{if } \epsilon_\Lambda = -1 \\ \mathrm{sign}(\Lambda) e^{d_\Lambda(m)} \ , & \text{if } \epsilon_\Lambda = 0 \end{cases} \ .$$

The isometry group $\mathrm{Iso}(\mathcal{M}, G)$ acts on $\mathcal{C}^\infty(\mathcal{M})$ through:

$$\psi^*(f) \stackrel{\text{def.}}{=} f \circ \psi^{-1} \ , \ \ \forall \psi \in \mathrm{Iso}(\mathcal{M}, G) \ , \ \ \forall f \in \mathcal{C}^\infty(\mathcal{M}) \ \ .$$

This action preserves the subspace of Hesse functions and hence corestricts to the Hesse representation:

$$\mathcal{H}_G(\psi)(f) \stackrel{\text{def.}}{=} \psi^*(\Lambda) = \Lambda \circ \psi^{-1} \ \ , \ \ \forall \Lambda \in \mathcal{S}(\mathcal{M}, G) \ \ .$$

Proposition

The Hesse representation is $(\ ,\ )_G$-orthogonal, i.e. any representation operator $\mathcal{H}_G(\psi)$ preserves the Hesse pairing:

$$(\mathcal{H}_G(\psi)\Lambda_1, \mathcal{H}_G(\psi)\Lambda_2)_G = (\Lambda_1, \Lambda_2)_G \ , \ \ \forall \Lambda_1, \Lambda_2 \in \mathcal{S}(\mathcal{M}, G) \ \ .$$

Definition

A Riemannian n manifold $(\mathcal{M}, \mathcal{G})$ is called locally maximally Hesse if the germ of local Hesse functions of $\mathcal{M}$ has dimension $n+1$ at any point.

Theorem

A Remannian manifold is locally maximally Hesse iff it is hyperbolic.

Theorem

Let $(\mathcal{M}, G)$ be a complete Riemannian manifold. The following are equivalent:

- *$(\mathcal{M}, G)$ is Hesse and locally maximally Hesse.*
- *$(\mathcal{M}, G)$ is isometric with the Poincaré n-ball or with an elementary hyperbolic space form.*

Moreover, $(\mathcal{M}, G)$ is maximally Hesse iff it is isometric with the Poincaré n-ball.

Definition

An n-dimensional elementary hyperbolic space form is a complete hyperbolic n-manifold uniformized by a non-trivial torsion-free elementary discrete subgroup $\Gamma \subset \mathrm{SO}_o(1, n)$.

Any torsion-free elementary discrete subgroups of $\mathrm{SO}_o(1, n)$ is:

- hyperbolic, if it conjugates to a subgroup of the canonical squeeze group $\mathcal{T}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1,n)}(E_n) \simeq \mathrm{SO}(1, n-1)$. In this case, Γ is a hyperbolic cyclic group.
- parabolic if it conjugates to a subgroup of the canonical shear group $\mathcal{P}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1,n)}(E_0 + E_n) \simeq \mathrm{ISO}(n)$. In this case, Γ is a free Abelian group of rank at most $n-1$.

Definition

An elementary hyperbolic space form is said to be of hyperbolic or parabolic type according to the type of its uniformizing group.

Example. The two-dimensional elementary hyperbolic space forms are:

- The hyperbolic annuli $\mathbb{A}(R)$ (hyperbolic type, $\Gamma \simeq \mathbb{Z}$, $\mathfrak{h}(\mathbb{A}(R) = 1)$
- The hyperbolic punctured disk $\mathbb{D}^*$ (parabolic type, $\Gamma \simeq \mathbb{Z}$, $\mathfrak{h}(\mathbb{D}^*) = 1$.)

Hesse surfaces and Hessian 2-field models

Theorem

Any Hesse surface (Σ, G) is locally maximally Hesse and hence is isometric with one of the following:

- *The hyperbolic disk $\mathbb{D} := \mathbb{D}^2$ (Hesse index 3)*
- *The hyperbolic punctured disk $\mathbb{D}^*$ (Hesse index 1)*
- *A hyperbolic annulus $\mathbb{A}(R)$ (Hesse index 1)*

Theorem

The two-field cosmological model defined by the two-dimensional scalar triple $(\Sigma, \mathcal{G}, V)$ is weakly-Hessian iff its rescaled scalar manifold $(\mathcal{M}, G)$ is isometric with the hyperbolic disk, the hyperbolic punctured disk or a hyperbolic annulus $\mathbb{A}(R)$ of arbitrary modulus $\mu = 2\log R > 0$. In this case, the model is Hessian iff the scalar potential V has the form $V = \Omega[\Lambda^2 - (\Lambda, \Lambda)_G]$, where $\Lambda \in \mathcal{S}(\Sigma, G)$ is a non-trivial Hesse function and Ω is a smooth function which is constant along the gradient flow of Λ.

The space $\mathcal{S}(\Sigma, G)$ can be determined in each of the three cases. This leads to an explicit classification of all Hessian two-field cosmological models.

- One can descibe explicitly the space of Hesse functions on the Poincaré n-ball $\mathbb{D}^n$ of any dimension $n \geq 2$ and obtain the explicit form of all scalar potentials on $\mathbb{D}^n$ which solve the Λ-V equation for any non-trivial Hesse function Λ. This leads to an explicit classification of all Hessian n-field models on the Poincaré ball.
- Deeper analysis allow one to characterize all Hessian n-field models.

Elementary discrete subgroups of $\mathrm{SO}_o(1, n)$

The stabilizer of a nontrivial $(n+1)$-vector $X \in \mathbb{R}^{n+1} \setminus \{0\}$ in the fundamental representation of $\mathrm{SO}_o(1, n)$ conjugates to one of the canonical subgroups:

- X timelike: $\mathrm{Stab}_{\mathrm{SO}_o(1,n)}(X) \sim \mathcal{R}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1,n)}(E_0) \simeq \mathrm{SO}(n)$ (elliptic, rotation).
- X spacelike: $\mathrm{Stab}_{\mathrm{SO}_o(1,n)}(X) \sim \mathcal{T}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1,n)}(E_n) \simeq \mathrm{SO}(1, n-1)$ (hyperbolic, squeeze)
- X lightlike: $\mathrm{Stab}_{\mathrm{SO}_o(1,n)}(X) \sim \mathcal{P}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1,n)}(E_0 + E_n) \simeq \mathrm{ISO}(n)$ (parabolic, shear).

Definition

A discrete subgroup Γ of $\mathrm{SO}_0(1, n)$ is called elementary if its action on the closure of the Poincaré ball fixes at least one point in $\overline{\mathrm{D}^n}$.

An elementary discrete subgroup $\Gamma \subset \mathrm{SO}_o(1, n)$ is:

- elliptic if it conjugates to a subgroup of $\mathcal{R}_n$. In this case, Γ is finite.
- hyperbolic if it conjugates to a subgroup of $\mathcal{T}_n$. In this case, Γ contains a hyperbolic cyclic group of finite index, to which it reduces iff Γ is torsion-free.
- parabolic if it conjugates to a subgroup of $\mathcal{P}_n$. In this case, Γ is a finite extension of a free Abelian group of rank at most $n - 1$.