Section Sigma Models and Einstein-Section-Maxwell theories

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Outline

Kaluza-Klein spaces

Let (M, g) be a Lorentzian 4-manifold. A *Lorentzian submersion* over (M, g) is a surjective pseudo-Riemannian submersion with target (M, g) in the sense of O'Neill.

Definition

A *Kaluza-Klein space* over (M, g) is a Lorentzian submersion $\pi : (E, h) \to (M, g)$ such that (E, h) is connected and geodesically complete and such that the fibers of π are totally geodesic connected submanifolds of (E, h). The Kaluza-Klein space is called *integrable* if the horizontal distribution $H(h) \subset TE$ defined by h (=the h-orthogonal complement of the vertical distribution V of π) is Frobenius integrable.

Proposition

Let g be a fixed Lorentzian metric on a four-manifold M. Then isomorphism classes of Kaluza-Klein spaces over (M, g) with typical fiber $(\mathcal{M}, \mathcal{G})$ and horizontal distribution H having Ehresmann holonomy $G \subseteq \mathrm{Iso}(\mathcal{M}, \mathcal{G})$ are in bijection with isomorphism classes of principal G-bundles Π defined over M and endowed with a principal connection θ together with an embedding $G \hookrightarrow \mathrm{Iso}(\mathcal{M}, \mathcal{G})$, considered up to conjugation. Moreover, H is integrable if and only if θ is flat.

Vertical scalar potentials and scalar bundles

Let $\pi : (E, h) \to (M, g)$ be a Kaluza-Klein space and T be its Ehresmann transport.

Definition

A *vertical scalar potential* for π is a smooth T-invariant real-valued function $\mathbf{\Phi} \in \mathcal{C}^\infty(E, \mathbb{R})$ defined on the total space E of π.

Definition

A *scalar bundle* over (M, g) is a pair $(\pi : (E, h) \to (M, g), \mathbf{\Phi})$, where π is a Kaluza-Klein space and $\mathbf{\Phi}$ is a vertical scalar potential for π. The scalar bundle is called *integrable* if the Kaluza-Klein space π is integrable.

Proposition

Integrable scalar bundles defined over (M, g) and having type $(\mathcal{M}, \mathcal{G}, \Phi)$ are classified up to isomorphism by the points of the character variety:

$$\mathcal{M}_{\mathrm{Iso}(\mathcal{M},\mathcal{G},\Phi)}(M) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(\pi_1(M), \mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi))/\mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi) \quad .$$

Section sigma models

Given a Kaluza-Klein space $\pi : (E, h) \to (M, g)$, let $P_V : TE \to V$ and $P_H : TE \to H$ be the h-orthogonal projectors and T the Ehresmann transport of $H \stackrel{\mathrm{def.}}{=} H(h)$. Let $\nabla^v \stackrel{\mathrm{def.}}{=} P_V \circ \nabla$ be the connection induced on V by the Levi-Civita connection ∇ of (E, h). Let $\mathbf{\Phi} \in \mathcal{C}^\infty(E, \mathbb{R})$ be a vertical scalar potential for π, so $(\pi, \mathbf{\Phi})$ is a bundle of scalar structures.

Definition

The *vertical Lagrange density* of $(\pi, \mathbf{\Phi})$ is the map $e^v_{\mathbf{\Phi}} : \Gamma(\pi) \to \mathcal{C}^\infty(M, \mathbb{R})$ defined, for every $s \in \Gamma(\pi)$, as follows:

$$e^v_{\mathbf{\Phi}}(g, h, s) \stackrel{\mathrm{def.}}{=} \frac{1}{2}\mathrm{Tr}_g s^*(h_V) + \mathbf{\Phi}^s \quad .$$

Here $\mathbf{\Phi}^s = \mathbf{\Phi} \circ s \in \mathcal{C}^\infty(M, \mathbb{R})$. The *section sigma model* defined by $(\pi, \mathbf{\Phi})$ has action functional $S_{\mathrm{sc}} : \mathrm{Met}_{3+n,1}(E) \times \Gamma(\pi) \to \mathbb{R}$ given by:

$$S_{\mathrm{sc}}[s] = -\int_U \nu_M(g) e^v_{\mathbf{\Phi}}(g, h, s) \qquad (1)$$

for any relatively-compact open subset $U \subset M$.

Section sigma models

Let $s \in \Gamma(\pi)$ be a smooth section. The differential $\mathrm{d}s \colon TM \to TE$ of s is an unbased morphism of vector bundles equivalent to a section $\mathrm{d}s \in \Omega^1(M, TE^s)$ which for simplicity we denote by the same symbol. We define $d^v s \stackrel{\mathrm{def.}}{=} P^s_V \circ \mathrm{d}s \in \Omega^1(M, V^s)$, where P^s_V denotes the vertical projection of the pulled-back bundle V^s. The Levi-Civita connection on (M, g) together with the s-pull-back of the connection ∇^v on V induce a connection on $T^*M \otimes V^s$, which for simplicity we denote again by ∇^v.

Definition

The *vertical tension field* of $s \in \Gamma(\pi)$ is defined through:

$$\tau^v(g, h, s) \stackrel{\mathrm{def.}}{=} \mathrm{Tr}_g \nabla^v \mathrm{d}^v s \in \Gamma(M, V^s) \quad .$$

For simplicity we will sometimes drop the explicit dependence on g and h in $e_{\boldsymbol{\Phi}}(g, h, s)$, $\tau(g, h, s)$.

Proposition

The critical points of (1) *with respect to* $s \in \Gamma(\pi)$ *are solutions of the* deformed pseudo-harmonic section equation*:*

$$\tau^v(s) = -(\mathrm{grad}_h \boldsymbol{\Phi})^s \, . \tag{2}$$

The sheaves of configurations and solutions

The local character of the model allows us to define two sheaves of sets on M, namely:

- The sheaf of configurations Conf_π is defined as the sheaf of local smooth sections of π.
- The sheaf of solutions $\mathrm{Sol}_{\pi,\mathbf{\Phi}}$ is the sub-sheaf of Conf_π whose set of sections $\mathrm{Sol}_{\pi,\Phi}(U)$ for an open subset $U \subset M$ consists of local solutions (defined on U) of the deformed harmonic section equation.

When the vertical potential $\mathbf{\Phi}$ vanishes identically, the deformed harmonic section equation reduces to the *pseudo-harmonic section equation*

$$\tau^v(s) = 0 \quad ,$$

which was introduced and studied by C. M. Wood. This generalizes the classical pseudo-harmonic map equation $\tau(\varphi) = 0$ (which is the equation of motion of the classical non-linear sigma model without a scalar potential).

Scalar-electromagnetic bundles

Let $\pi : (E, h) \to (M, g)$ be a KK space with Ehresmann transport T and horizontal distribution $H \subset TE$. Let $\boldsymbol{\Delta} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ be a flat symplectic vector bundle defined over E. Given $m \in M$, let $(\mathcal{S}_m, D_m, \omega_m)$ be the restriction of $(\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ to the fiber E_m. This is a flat symplectic vector bundle defined on the Riemannian manifold (E_m, h_m), also known as a *duality structure*. For any path $\Gamma \in \mathcal{P}(E)$ in the total space of E, let $U_\Gamma : \boldsymbol{\mathcal{S}}_{\Gamma(0)} \to \boldsymbol{\mathcal{S}}_{\Gamma(1)}$ be the parallel transport defined by $\mathbf{D}$ along Γ. Since $\mathbf{D}$ is a symplectic connection, U_Γ is a symplectomorphism between the symplectic vector spaces $(\boldsymbol{\mathcal{S}}_{\Gamma(0)}, \boldsymbol{\omega}_{\Gamma(0)})$ and $(\boldsymbol{\mathcal{S}}_{\Gamma(1)}, \boldsymbol{\omega}_{\Gamma(1)})$. For any path $\gamma \in \mathcal{P}(M)$, let $\boldsymbol{\gamma}_e \in \mathcal{P}(E)$ be its horizontal lift starting at the point $e \in E_{\gamma(0)}$. We have $T_\gamma(e) = \boldsymbol{\gamma}_e(1)$.

Definition

The *extended horizontal transport* along a path $\gamma \in \mathcal{P}(M)$ is the unbased isomorphism of vector bundles $\mathbf{T}_\gamma : \mathcal{S}_{\gamma(0)} \to \mathcal{S}_{\gamma(1)}$ defined through:

$$\mathbf{T}_\gamma(e) \stackrel{\text{def.}}{=} U_{\boldsymbol{\gamma}_e} : \boldsymbol{\mathcal{S}}_e \to \boldsymbol{\mathcal{S}}_{T_\gamma(e)}\,, \quad \forall\, e \in E_{\gamma(0)}\,,$$

which lifts the Ehresmann transport $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ along γ.

Clearly $\mathbf{T}_\gamma$ is an isomorphism of flat symplectic vector bundles:

$$\mathbf{T}_\gamma : (\mathcal{S}_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}) \xrightarrow{\sim} (\mathcal{S}_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)})\,,$$

which lifts the isometry $T_\gamma : (E_{\gamma(0)}, h_{\gamma(0)}) \to (E_{\gamma(1)}, h_{\gamma(1)})$.

Scalar-electromagnetic bundles

Definition

Let $\pi\colon (E,h)\to (M,g)$ be a Kaluza-Klein space. A *duality bundle* $\boldsymbol{\Delta}$ for π is a flat symplectic vector bundle $\boldsymbol{\Delta} = (\boldsymbol{\mathcal{S}},\boldsymbol{\omega},\mathbf{D})$ defined over the total space E of π. Let $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ be duality bundles. A *morphism of duality bundles* from $\boldsymbol{\Delta}_1$ to $\boldsymbol{\Delta}_2$ is a morphism of the underlying flat symplectic vector bundles.

Let $\boldsymbol{\Delta}$ be a duality bundle for a Kaluza-Klein space $\pi : (E,h)\to (M,g)$. A *taming* $\mathbf{J}$ of $(\mathbf{S},\boldsymbol{\omega})$ is an automorphism of the symplectic vector bundle $(\boldsymbol{\mathcal{S}},\boldsymbol{\omega})$ satisfying:

$$\mathbf{J}^2 = -\mathrm{Id}_{\boldsymbol{\mathcal{S}}}\,, \qquad \boldsymbol{\omega}(\mathbf{J}e,e) > 0\,, \qquad \forall e\in\Gamma(E,\boldsymbol{\mathcal{S}})\,.$$

Tamings always exist. Given a taming $\mathbf{J}$ of $\boldsymbol{\Delta}$ and a point $m\in M$, we denote by $J_m \stackrel{\mathrm{def.}}{=} \mathbf{J}|_{E_m}$ the taming of the symplectic vector bundle $(\mathcal{S}_m,\omega_m)$ given by the restriction of $\mathbf{J}$ to the fiber E_m of π.

Definition

A taming $\mathbf{J}$ of $\boldsymbol{\Delta} = (\boldsymbol{\mathcal{S}},\boldsymbol{\omega},\mathbf{D})$ is called *vertical* if it is $\mathbf{T}$-invariant, which means that it satisfies:

$$\mathbf{T}_\gamma\circ J_{\gamma(0)} = J_{\gamma(1)}\circ\mathbf{T}_\gamma\,, \qquad \forall\gamma\in\mathcal{P}(M)\,.$$

Scalar-electromagnetic bundles

Definition

Let $\pi\colon (E,h) \to (M,g)$ be a Kaluza-Klein space. An *electromagnetic bundle* $\mathbf{\Xi}$ for π is a duality bundle $\mathbf{\Delta}$ for π which is equipped with a vertical taming $\mathbf{J}$. We write $\mathbf{\Xi} \stackrel{\mathrm{def.}}{=} (\mathbf{\Delta}, \mathbf{J}) = (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D}, \mathbf{J})$. Let $\mathbf{\Xi}_1$ and $\mathbf{\Xi}_2$ be two electromagnetic bundles. A morphism of electromagnetic bundles $f\colon \mathbf{\Xi}_1 \to \mathbf{\Xi}_2$ from $\mathbf{\Xi}_1$ to $\mathbf{\Xi}_2$ is a morphism of the underlying duality structures that satisfies $\mathbf{J}_2 \circ f = f \circ \mathbf{J}_1$.

Definition

A *scalar-electromagnetic bundle* defined over (M,g) is a triple $\boldsymbol{\mathcal{D}} = (\pi, \mathbf{\Phi}, \mathbf{\Xi})$ consisting of a Kaluza-Klein space $\pi : (E,h) \to (M,g)$, a vertical potential $\mathbf{\Phi}$ for π and an electromagnetic bundle $\mathbf{\Xi}$ for π. The scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$ is called *integrable* if $\pi : (E,h) \to (M,g)$ is an integrable Kaluza-Klein space.

The fundamental bundle form of an electromagnetic bundle

Let $\boldsymbol{\mathcal{D}} = (\pi, \boldsymbol{\Phi}, \boldsymbol{\Xi})$ be a scalar-electromagnetic bundle with $\boldsymbol{\Xi} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D}, \mathbf{J})$ and $\pi\colon (E, h) \to (M, g)$. Let:

$$\mathbf{D}^{\mathrm{ad}} \colon \Omega^0(E, End(\boldsymbol{\mathcal{S}})) \to \Omega^1(E, End(\boldsymbol{\mathcal{S}})) \,,$$

be the connection induced by $\mathbf{D}$ on the endomorphism bundle $End(\boldsymbol{\mathcal{S}})$ of $\boldsymbol{\mathcal{S}}$.

Definition

The *fundamental bundle form* $\boldsymbol{\Theta}$ associated to $\boldsymbol{\mathcal{D}}$ is the $End(\boldsymbol{\mathcal{S}})$-valued one-form defined on E as follows:

$$\boldsymbol{\Theta} \stackrel{\mathrm{def.}}{=} \mathbf{D}^{\mathrm{ad}} \mathbf{J} \in \Omega^1(E, End(\boldsymbol{\mathcal{S}})) \,.$$

The fact $\mathbf{J}$ is vertical together with the fact that the decomposition $TE = H \oplus V$ is h-orthogonal, implies that we have:

$$\boldsymbol{\Theta} \in \Omega^0(E, V^* \otimes End(\boldsymbol{\mathcal{S}})) \,.$$

Definition

The *fundamental bundle field* $\boldsymbol{\Psi}$ associated to $\boldsymbol{\mathcal{D}}$ is the $End(\boldsymbol{\mathcal{S}})$-valued vector field defined on E as follows:

$$\boldsymbol{\Psi} \stackrel{\mathrm{def.}}{=} (\sharp_h \otimes \mathrm{Id}_{End(\boldsymbol{\mathcal{S}})}) \circ D^{\mathrm{ad}} \mathbf{J} \in \Omega^0(E, V \otimes End(\boldsymbol{\mathcal{S}})) \,.$$

Generalized Einstein-Section-Maxwell theories

Definition

Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be a scalar-electromagnetic bundle with Kaluza-Klein space $\pi \colon (E, h) \to (M, g)$ and let $s \in \Gamma(\pi)$ be a smooth global section of π. An *electromagnetic field strength* is a two-form $\mathcal{V} \in \Omega^2(M, \mathcal{S}^s)$ having the following properties:

1. $\mathcal{V}$ is positively-polarized with respect to $\mathbf{J}^s$, i.e. the following relation is satisfied:

$$*_g \mathcal{V} = -\mathbf{J}^s \mathcal{V} \, .$$

2. $\mathcal{V}$ satisfies the *electromagnetic equation* with respect to s:

$$\mathrm{d}_{\mathbf{D}^s} \mathcal{V} = 0 \quad . \tag{3}$$

We denote by $\Omega^{2+,s}_{g,\mathcal{S},\mathbf{J}}$ the sheaf of $\mathcal{S}^s$-valued two-forms which are positively-polarized with respect to $\mathbf{J}^s$. Thus $\Omega^{2+,s}_{g,\mathcal{S},\mathbf{J}}(M)$ denotes the space of $\mathcal{S}^s$-valued two-forms on M which are positively-polarized with respect to $\mathbf{J}^s$.

Definition

Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be a scalar-electromagnetic bundle with Kaluza-Klein space $\pi\colon (E, h) \to (M, g)$. The *configuration sheaf* of a GESM-theory associated to $\mathcal{D}$ is the sheaf of sets defined through:

$$\mathbf{Conf}_{\mathcal{D}}(U) \stackrel{\mathrm{def.}}{=} \left\{(g, s, \mathcal{V})\,, \mid g \in \mathrm{Met}_{3,1}(U)\,, \, s \in \Gamma(\pi|_U)\,, \, \mathcal{V} \in \Omega^{2+,s}_{g,\mathcal{S},\mathrm{J}}(U)\right\}\,,$$

for every open set $U \subset M$.

The *exterior pairing* $(\ ,\ )_g$ is the pseudo-Euclidean metric induced by g on the exterior bundle $\wedge_M \stackrel{\mathrm{def.}}{=} \wedge T^*M$.

Definition

The *twisted exterior pairing* $(\ ,\ ) := (\ ,\ )_{g,\mathbf{Q}^s}$ is the unique pseudo-Euclidean scalar product on the twisted exterior bundle $\wedge_M(\boldsymbol{\mathcal{S}}^s) \stackrel{\mathrm{def.}}{=} \wedge_M \otimes \boldsymbol{\mathcal{S}}^s$ which satisfies:

$$(\rho_1 \otimes \xi_1, \rho_2 \otimes \xi_2)_{g,\mathbf{Q}^s} = (\rho_1, \rho_2)_g \mathbf{Q}^s(\xi_1, \xi_2)\,,$$

for any $\rho_1, \rho_2 \in \Omega(M)$ and any $\xi_1, \xi_2 \in \Gamma(M, \boldsymbol{\mathcal{S}}^s)$. Here $\mathbf{Q}^s(\xi_1, \xi_2) = \boldsymbol{\omega}^s(\mathbf{J}^s\xi_1, \xi_2)$ and the superscript denotes pull-back by s.

For any vector bundle W, we trivially extend the twisted exterior pairing to a W-valued pairing (which for simplicity we denote by the same symbol) between the bundles $W \otimes (\wedge_M(\boldsymbol{\mathcal{S}}^s))$ and $\wedge_M(\boldsymbol{\mathcal{S}}^s)$. Thus:

$$(e \otimes \eta_1, \eta_2)_{g,\mathbf{Q}^s} \stackrel{\mathrm{def.}}{=} e \otimes (\eta_1, \eta_2)_{g,\mathbf{Q}^s}\,, \quad \forall\, e \in \Gamma(M, W)\,, \quad \forall\, \eta_1, \eta_2 \in \wedge_M(\boldsymbol{\mathcal{S}}^s)\,.$$

The *inner g-contraction of two-tensors* is the bundle morphism $\oslash_g : (\otimes^2 T^*M)^{\otimes 2} \to \otimes^2 T^*M$ uniquely determined by the condition:

$$(\alpha_1 \otimes \alpha_2) \oslash_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_3)_g \alpha_1 \otimes \alpha_4\,, \quad \forall\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega^1(M)\,.$$

We define the *inner g-contraction of two-forms* to be the restriction of $\oslash_g$ to a map $\wedge^2 T^*M \otimes \wedge^2 T^*M \to \otimes^2 T^*M$.

Definition

The *twisted inner contraction* of $\boldsymbol{\mathcal{S}}^s$-valued two-forms is the unique morphism of vector bundles $\oslash$: $\wedge^2_M(\boldsymbol{\mathcal{S}}^s) \times_M \wedge^2_M(\boldsymbol{\mathcal{S}}^s) \rightarrow \otimes^2(T^\ast M)$ which satisfies:

$$(\rho_1 \otimes \xi_1) \oslash (\rho_2 \otimes \xi_2) = \mathbf{Q}^s(\xi_1, \xi_2)\rho_1 \oslash_g \rho_2 ,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_2, \xi_2 \in \Gamma(M, \boldsymbol{\mathcal{S}}^s)$.

Definition

Let $\boldsymbol{\mathcal{D}} = (\pi, \boldsymbol{\Phi}, \boldsymbol{\Xi})$ be a scalar-electromagnetic bundle with $\pi\colon (E, h) \to (M, g)$. The GESM theory associated to $\boldsymbol{\mathcal{D}}$ is defined by the following set of globally well-defined partial differential equations with unknowns $(g, s, \boldsymbol{\mathcal{V}}) \in \mathrm{Conf}_{\boldsymbol{\mathcal{D}}}(M)$:

- The Einstein equations[a]:

$$\mathcal{E}_{Eins}(g, s, \boldsymbol{\mathcal{V}}) \stackrel{\text{def.}}{=} \mathrm{G}(g) - \kappa\, \mathrm{T}(g, s, \boldsymbol{\mathcal{V}}) = 0\,, \qquad (4)$$

  where $\mathrm{T}(g, s, \boldsymbol{\mathcal{V}}) \in \Gamma(M, S^2 T^* M)$ is the energy-momentum tensor of the theory, which is given by

$$\mathrm{T}(g, s, \boldsymbol{\mathcal{V}}) = g\, e_0^v(g, h, s) - h^s + \frac{g}{2} \boldsymbol{\Phi}^s + 2\, \boldsymbol{\mathcal{V}} \oslash \boldsymbol{\mathcal{V}}\,.$$

- The scalar equations:

$$\mathcal{E}_{sc}(g, s, \boldsymbol{\mathcal{V}}) \stackrel{\text{def.}}{=} \tau^v(g, h, s) + (\mathrm{grad}_h \boldsymbol{\Phi})^s - \frac{1}{2}(*\boldsymbol{\mathcal{V}}, \boldsymbol{\Psi}^s \boldsymbol{\mathcal{V}}) = 0\,. \qquad (5)$$

- The electromagnetic equations:

$$\mathcal{E}_{em}(g, s, \boldsymbol{\mathcal{V}}) \stackrel{\text{def.}}{=} \mathrm{d}_{\mathbf{D}^s} \boldsymbol{\mathcal{V}} = 0\,. \qquad (6)$$

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[a] Here $\mathrm{G}(g)$ is the Einstein tensor of g while $\kappa \stackrel{\text{def.}}{=} \frac{8\pi G_N}{c^4}$ with G_N the Newton constant.

Definition

The *solution sheaf* $\mathrm{Sol}_{\mathcal{D}}$ of a GESM-theory associated to a scalar-electromagnetic bundle $\mathcal{D} = (\pi, \Phi, \Xi)$ is the sheaf of sets given by:

$$\mathbf{Sol}_{\mathcal{D}}(U) \stackrel{\text{def.}}{=} \{(g, s, \mathcal{V}) \in \mathrm{Conf}_{\mathcal{D}}(U) \mid \mathcal{E}_E(g, s, \mathcal{V}) = 0, \mathcal{E}_S(g, s, \mathcal{V}) = 0, \mathcal{E}_K(g, s, \mathcal{V}) = 0\}.$$

for every open set $U \subset M$.

This gives an extremely general global formulation of Einstein Scalar Maxwell theories which agrees with the local formulation usually found in the literature. The global formulation depends on the choice of the scalar-electromagnetic bundle. When M is not simply-connected, there generally exists a continuous infinity of such choices, so the local formulation of ESM theories found in the physics literature has continuous infinity of physically-inequivalent globalizations on space-times which are not simply-connected.

- This is not merely an "index-free" formulation but a *global* formulation.
- It is well-known that a physical theory coupled to gravity must be defined *globally* in the sense that one must specify from the outset the *global* character of all fields and operators appearing in the equations of motion. One cannot escape this problem since there is no general "continuation" principle for solutions of non-linear hyperbolic PDEs. See the work of Y. Choquet-Bruhat (in particular, her book on GR) and of the Princeton group of Sergiu Klainerman.

Theorem

(non-technical version) *GESM theories are* locally *equivalent with the usual coordinate formulation of Einstein-Scalar-Maxwell theory (as found in the physics literature) and hence provide admissible global formulations of the latter. Moreover, global solutions of such theories can be interpreted as a supergravity version of certain "bosonic U-folds", thus providing a rigorous description of the latter within global differential geometry. Notice that such U-folds are constructed directly in a classical field theory (as opposed to within string theory).*

There exists a mathematically rigorous formulation of this result, which is proved using "special trivializing atlases" for scalar-electromagnetic bundles.

Relation to the Scherk-Schwarz construction

One can show that the Scherk-Schwarz construction **without gauging** can be recovered from the integrable case our construction when $M = \mathbb{R}^3 \times \mathrm{S}^1$. In this sense, one can think of the integrable case of our models as providing an *intrinsic formulation* of the "ultimate generalization" of the Scherk-Schwarz construction. In this interpretation, locally-geometric classical U-folds are simply global solutions of "Scherk-Schwarz supergravities" in four dimensions.