Matrix factorizations over Bezout and elementary divisor domains

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Outline

Motivation

We want to understand matrix factorizations over non-Noetherian rings, such as the ring of holomorphic functions defined on a complex manifold. This has a few possible applications:

- The homotopy category of matrix factorizations provides a "derived" version of the arithmetic of such rings
- Applications to *holomorphic* (as aopposed to algebraic) Landau-Ginzburg models.

Matrix factorizations over an integral domain

Let R be an integral domain and $W \in R^\times$ be a non-zero element of R. Consider the following R-linear and $\mathbb{Z}_2$-graded categories of matrix factorizations of W over R:

1. $\mathrm{MF}(R, W)$=dg category of R-valued matrix factorizations of W of finite rank. Objects: $a = (M, D)$, where M is a free $\mathbb{Z}_2$-graded R-module of finite rank and D is an odd endomorphism of M such that $D^2 = W\mathrm{id}_M$. Morphisms:

$$\mathrm{Hom}_{\mathrm{MF}(R,W)}(a_1, a_2) = \underline{\mathrm{Hom}}_R(M_1, M_2) = \mathrm{Hom}_R^{\hat{0}}(M_1, M_2) \oplus \mathrm{Hom}_R^{\hat{1}}(M_1, M_2) \quad ,$$

   with differential $\mathfrak{d}_{a_1,a_2}$ determined by:

$$\mathfrak{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1 \quad , \quad \forall f \in \mathrm{Hom}_R^\kappa(M_1, M_2) \ \ \forall \kappa \in \mathbb{Z}_2 \quad .$$

2. $\mathrm{ZMF}(R, W)$=cocycle category of $\mathrm{MF}(R, W)$. Same objects as $\mathrm{MF}(R, W)$ but:

$$\mathrm{Hom}_{\mathrm{ZMF}(R,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \{f \in \mathrm{Hom}_{\mathrm{MF}(R,W)}(a_1, a_2) | \mathfrak{d}_{a_1,a_2}(f) = 0\} \quad .$$

3. $\mathrm{BMF}(R, W)$=coboundary category of $\mathrm{MF}(R, W)$. Same objs. as $\mathrm{MF}(R, W)$ but:

$$\mathrm{Hom}_{\mathrm{BMF}(R,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \{\mathfrak{d}_{a_1,a_2}(f) | f \in \mathrm{Hom}_{\mathrm{MF}(R,W)}(a_1, a_2)\} \quad .$$

4. $\mathrm{HMF}(R, W)$=total cohomology category of $\mathrm{MF}(R, W)$. Same objs. but:

$$\mathrm{Hom}_{\mathrm{HMF}(R,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathrm{ZMF}(R,W)}(a_1, a_2) / \mathrm{Hom}_{\mathrm{BMF}(R,W)}(a_1, a_2) \quad .$$

Also consider the subcategories obtained by restricting to morphisms of even degree:

1. $\mathrm{mf}(R, W) \stackrel{\text{def.}}{=} \mathrm{MF}^{\hat{0}}(R, W)$
2. $\mathrm{zmf}(R, W) \stackrel{\text{def.}}{=} \mathrm{ZMF}^{\hat{0}}(R, W)$
3. $\mathrm{bmf}(R, W) \stackrel{\text{def.}}{=} \mathrm{BMF}^{\hat{0}}(R, W)$
4. $\mathrm{hmf}(R, W) \stackrel{\text{def.}}{=} \mathrm{HMF}^{\hat{0}}(R, W)$.

Some facts:

- $\mathrm{MF}(R, W)$, $\mathrm{BMF}(R, W)$ and $\mathrm{ZMF}(R, W)$ admit double direct sums but do not have zero objects.
- $\mathrm{HMF}(R, W)$ is additive, the matrix factorization $\begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix}$ being a zero object.
- $\mathrm{hmf}(R, W)$ is triangulated with involutive shift functor and $\mathrm{HMF}(R, W)$ is the *graded completion* of $\mathrm{HMF}(R, W)$.

Definition

Let $a = (M, D)$ be a finite rank matrix factorization of $W \in R^\times$. Then $\mathrm{rk} M^{\hat{0}} = \mathrm{rk} M^{\hat{1}}$ is called the *reduced rank* of a and is denoted by $\rho(a)$.

Elementary matrix factorizations

Definition

A matrix factorization $a = (M, D)$ of W over R is called *elementary* if it has unit reduced rank, i.e. if $\rho(a) = 1$.

Any elementary matrix factorization is isomorphic in $\mathrm{zmf}(R, W)$ to one of the form $e_v \stackrel{\mathrm{def.}}{=} (R^{1|1}, D_v)$, where v is a divisor of W and $D_v \stackrel{\mathrm{def.}}{=} \left[\begin{array}{cc} 0 & v \\ u & 0 \end{array}\right]$, with $u \stackrel{\mathrm{def.}}{=} W/v \in R$.

Let $\mathrm{EF}(R, W)$ be the full subcategory of $\mathrm{MF}(R, W)$ whose objects are the elementary factorizations. Let $\mathrm{ZEF}(R, W)$ and $\mathrm{HEF}(R, W)$ denote the cocycle and total cohomology categories of $\mathrm{EF}(R, W)$. Let $\mathrm{zef}(R, W) \stackrel{\mathrm{def.}}{=} \mathrm{ZEF}^{\hat{0}}(R, W)$ and $\mathrm{hef}(R, W) \stackrel{\mathrm{def.}}{=} \mathrm{HEF}^{\hat{0}}(R, W)$.

Remark

An elementary factorization is indecomposable in $\mathrm{zmf}(R, W)$, but it need not be indecomposable in the triangulated category $\mathrm{hmf}(R, W)$.

Bezout domains

Definition

An integral domain R is called a *GCD domain* if any two elements f, g admit a greatest common divisor (gcd).

We say that the *Bézout identity* holds for two elements f and g of a GCD domain R if for one/any gcd d of f and g, there exist $a, b \in R$ such that $d = af + bg$.

Definition

An integral domain R is called a *Bézout domain* if any (and hence all) of the following equivalent conditions hold:

- R is a GCD domain and the Bézout identity holds for any two non-zero elements $f, g \in R$.
- The ideal generated by any two elements of R is principal.
- Any finitely-generated ideal of R is principal.

Proposition

Every finitely-generated projective module over a Bézout domain is free.

Bézout domains

Proposition

Let R be a Bézout domain. Then the following statements are equivalent:

- *R is Noetherian*
- *R is a principal ideal domain (PID)*
- *R is a unique factorization domain (UFD)*
- *R satisfies the ascending chain condition on principal ideals (ACCP)*

Some examples of non-Noetherian Bézout domains:

- Any generalized valuation domain is a Bézout domain.
- The ring $\mathrm{O}(\Sigma)$ of holomorphic complex-valued functions defined on any[1] smooth connected non-compact Riemann surface Σ is a non-Noetherian Bézout domain. In particular, the ring $\mathrm{O}(\mathbb{C})$ of entire functions is a non-Noetherian Bézout domain.
- The ring $\mathbb{A}$ of all algebraic integers (the integral closure of $\mathbb{Z}$ inside $\mathbb{C}$) is a non-Noetherian Bézout domain which has no prime elements.

[1] Notice that Σ need not be algebraic. In particular, Σ can have infinite genus and an infinite number of ends.

Elementary matrix factorizations over Bezout domains

Let R be a Bézout domain and $W \in R^\times$. Let v_1, v_2 be divisors of W and $e_1 := e_{v_1}$, $e_2 := e_{v_2}$ be the corresponding elementary matrix factorizations of W. Let $u_1 \stackrel{\text{def.}}{=} W/v_1$, $u_2 = W/v_2$ and a be a gcd of v_1 and v_2. Define:

$$b \stackrel{\text{def.}}{=} v_1/a \ , \ c \stackrel{\text{def.}}{=} v_2/a \ , \ d \stackrel{\text{def.}}{=} \frac{W}{abc} \ , \ a' \stackrel{\text{def.}}{=} a/s \ , \ d' \stackrel{\text{def.}}{=} d/s \ , \tag{1}$$

where s is a gcd of a and d. In this notation, we have:

$$\begin{aligned} D_{v_1} &= \begin{bmatrix} 0 & v_1 \\ u_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ cd & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'b \\ cd' & 0 \end{bmatrix} \\ D_{v_2} &= \begin{bmatrix} 0 & v_2 \\ u_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix} = s \begin{bmatrix} 0 & a'c \\ bd' & 0 \end{bmatrix} \ . \end{aligned}$$

Define:

$$\epsilon_{\hat{0}}(v_1, v_2) \stackrel{\text{def.}}{=} \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} \frac{(v_2)}{(v_1, v_2)} & 0 \\ 0 & \frac{(v_1)}{(v_1, v_2)} \end{bmatrix} \stackrel{\text{def.}}{=} \boldsymbol{\epsilon}_{\hat{0}}(v_1, v_2)$$

$$\epsilon_{\hat{1}}(v_1, v_2; W) \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & a' \\ -d' & 0 \end{bmatrix} \in \begin{bmatrix} 0 & \frac{(v_2)}{(u_1, v_2)} \\ -\frac{(u_1)}{(u_1, v_2)} & 0 \end{bmatrix} \stackrel{\text{def.}}{=} \boldsymbol{\epsilon}_{\hat{1}}(v_1, v_2; W) \ ,$$

where the matrices in the right hand side are equivalence classes under the relation:

$$A \sim_n B \ \text{iff} \ \forall i, j \in \{1, \ldots, n\} : \exists q_{ij} \in U(R) \ \text{such that} \ B_{ij} = q_{ij} A_{ij} \ \ (A, B \in \mathrm{Mat}(n, R)) \ .$$

Morphisms in $\mathrm{HEF}(R, W)$

Proposition

Let v_i be as above. Then $\mathrm{Hom}^{\hat{0}}_{\mathrm{HMF}(R,W)}(e_1, e_2)$ and $\mathrm{Hom}^{\hat{1}}_{\mathrm{HMF}(R,W)}(e_1, e_2)$ are cyclically presented cyclic R-modules generated respectively by the matrices $\epsilon_{\hat{0}}(v_1, v_2)$ and $\epsilon_{\hat{1}}(v_1, v_2; W)$, whose annihilators are equal to each other and given by:

$$\alpha_W(v_1, v_2) \stackrel{\mathrm{def.}}{=} \langle v_1, u_1, v_2, u_2 \rangle = \langle s \rangle \quad .$$

Proposition

Let v be a divisor of W and $u = W/v$. Then:

1. *The R-algebra $\mathrm{End}_{\mathrm{zmf}(R,W)}(e_v)$ is isomorphic with R.*
2. *We have an isomorphism of $\mathbb{Z}_2$-graded R-algebras:*

$$\mathrm{End}_{\mathrm{ZMF}(R,W)}(e_v) \simeq \frac{R[\omega]}{\langle u^2 + t \rangle} \quad ,$$

where ω is an odd generator and $t \in \frac{[u,v]}{(u,v)}$. In particular, $\mathrm{End}_{\mathrm{ZMF}(R,W)}(e_v)$ is a commutative $\mathbb{Z}_2$-graded ring.

Let $\mathfrak{Div}(W) \stackrel{\text{def.}}{=} \{d \in R \,\big|\, d|W\}$ and $\alpha_W : \mathfrak{Div}(W) \times \mathfrak{Div}(W) \to G_+(R)$ be the function defined in the previous proposition. This function is symmetric since $\alpha_W(v_1, v_2) = \alpha_W(v_2, v_1)$. Let $1_{G(R)} = \langle 1 \rangle = R$ denote the neutral element of the group of divisibility $G(R)$, whose group operation we write multiplicatively.

Proposition

The symmetric function $\alpha_W(v_1, v_2)$ is multiplicative with respect to each of its arguments in the following sense:

- *For any two relatively prime elements v_2 and $\widetilde{v}_2$ of R such that $v_2\widetilde{v}_2$ is a divisor of W, we have:*
$$\alpha_W(v_1, v_2\widetilde{v}_2) = \alpha_W(v_1, v_2)\alpha_W(v_1, \widetilde{v}_2) \tag{2}$$
and $\alpha_W(v_1, v_2) + \alpha_W(v_1, \widetilde{v}_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of R.
- *For any two relatively prime elements v_1 and $\widetilde{v}_1$ of R such that $v_1\widetilde{v}_1$ is a divisor of W, we have:*
$$\alpha_W(v_1\widetilde{v}_1, v_2) = \alpha_W(v_1, v_2)\alpha_W(\widetilde{v}_1, v_2) \tag{3}$$
and $\alpha_W(v_1, v_2) + \alpha_W(\widetilde{v}_1, v_2) = 1_{G(R)}$, where $+$ denotes the sum of ideals of R.

Isomorphisms in $\mathrm{HEF}(R, W)$

Proposition

With the notations above, we have:

1. *e_1 and e_2 are isomorphic in $\mathrm{hef}(R, W)$ iff a', b, c, d' are pairwise coprime and $(bc, s) = (1)$.*
2. *An odd isomorphism between e_1 and e_2 in $\mathrm{HEF}(R, W)$ exists iff a', b, c, d' are pairwise coprime and $(a'd', s) = (1)$.*

Corollary

An elementary matrix factorization e_v is a zero object of $\mathrm{hmf}(R, W)$ iff $(u, v) = (1)$, where $u = W/v$.

Primary matrix factorizations

Recall that an element of R is called primary if it is a power of a prime element.

Definition

An elementary factorization e_v of W is called *primary* if v is a primary divisor of W.

Let e_v be a primary matrix factorization of W. Then $v = p^i$ for some prime divisor p of W and some integer $i \in \{0, \ldots, n\}$, where n is the order of p as a divisor of W. We have $W = p^n W_1$ for some element $W_1 \in R$ such that p does not divide W_1 and $u = p^{n-i} W_1$. Thus $(u, v) = (p^{\min(i, n-i)})$.

Definition

The prime divisor p of W is called the *prime locus* of e_v. The order n of p is called the *order* of e_v while the integer $i \in \{0, \ldots, n\}$ is called the *size* of e_v.

Let R be a Bézout domain and $p \in R$ be a prime element. Fix an integer $n \geq 2$ and consider the quotient ring:

$$A_n(p) \stackrel{\mathrm{def.}}{=} R/\langle p^n \rangle \quad .$$

Let $\mathbf{m}_n(p) = pA_n(p) = \langle p \rangle / \langle p^n \rangle$ and $\mathbf{k}_p = R/\langle p \rangle$.

Lemma

The following statements hold:

1. *The principal ideal $\langle p \rangle$ generated by p is maximal.*
2. *The primary ideal $\langle p^n \rangle$ is contained in a unique maximal ideal of R.*
3. *The quotient $A_n(p)$ is a quasi-local ring with maximal ideal $\mathbf{m}_n(p)$ and residue field $\mathbf{k}_p$.*
4. *$A_n(p)$ is a generalized valuation ring.*
5. *$A_n(p)$ is an Artinian local principal ideal ring, whose ideals are $\langle p^i \rangle / \langle p^n \rangle$ for $i = 0, \ldots, n$.*

Proposition

Let e_v be a primary factorization of W with prime locus p, order n and size i. Then e_v is an indecomposable object of $\mathrm{hmf}(R, W)$ whose endomorphism ring $\mathrm{End}_{\mathrm{hmf}(R,W)}(e_v)$ is a quasi-local ring isomorphic with $A_{\min(i,n-i)}(p)$.

Critically-finite superpotentials

Definition

A non-zero non-unit W of R is called:

- *non-critical*, if W has no critical divisors;
- *critically-finite* if it has a factorization of the form:

$$W = W_0 W_c \quad \text{with} \quad W_c = p_1^{n_1} \ldots p_N^{n_N} \quad , \tag{4}$$

where $n_j \geq 2$, $p_1, \ldots, p_N$ are critical prime divisors of W (with $p_i \not\sim p_j$ for $i \neq j$) and W_0 is non-critical and coprime with W_c.

Notice that the elements W_0, W_c and p_i in the factorization (4) are determined by W up to association, while the integers n_i are uniquely determined by W. The factors W_0 and W_c are called respectively the *non-critical* and *critical* parts of W. The integers $n_i \geq 2$ are called the *orders* of the critical prime divisors p_i.

Proposition

Let e_v be an elementary factorization of W over R such that $v = \prod_{i=1}^{n} v_i$, where $v_i \in R$ are mutually coprime divisors of W. Then there exists a natural isomorphism in $\mathrm{hmf}(R, W)$:

$$e_v \simeq_{\mathrm{hmf}(R,W)} \bigoplus_{i=1}^{n} e_{v_i}$$

In particular, an elementary factorization e_v for which v is finitely-factorizable divisor of W is isomorphic in $\mathrm{hmf}(R, W)$ with a direct sum of primary factorizations.

Let $\mathbf{hef}(R, W)$ denote the full subcategory of $\mathrm{hef}(R, W)$ which is additively generated by the elementary matrix factorizations of W. Recall that a *Krull-Schmidt category* is an additive category for which every object decomposes into a finite direct sum of objects having quasi-local endomorphism rings.

Theorem

Suppose that W is critically-finite. Then the additive category $\mathbf{hef}(R, W)$ is Krull-Schmidt and its non-zero indecomposable objects are the non-trivial primary matrix factorizations of W.

Theorem

Suppose that W is critically-finite with decomposition (4). Then there exists an equivalence of categories:

$$\mathbf{hef}(R, W) \simeq \vee_{i=1}^{N} \mathbf{hef}(R, p_i^{n_i}) \quad ,$$

where $\vee$ denotes the coproduct of additive categories.

Consider the inclusion functor:

$$\iota : \mathbf{hef}(R, W) \to \mathrm{hmf}(R, W)$$

Conjecture

The inclusion functor ι is an equivalence of R-linear categories.

This conjecture and the previous theorem imply:

Conjecture

Let R be a Bézout domain and W be a critically-finite element of R. Then $\mathrm{hmf}(R, W)$ is a Krull-Schmidt category.

Counting elementary factorizations over Bézout domains

Let R be a Bezout domain and W be a critically-finite element with the decomposition:

$$W = W_0 W_c \quad \text{with} \quad W_c = p_1^{n_1} \ldots p_N^{n_N} \quad ,$$

where $n_j \geq 2$, $p_1, \ldots, p_N$ are critical prime divisors of W (with $p_i \not\sim p_j$ for $i \neq j$) and W_0 is non-critical and coprime with W_c.

Theorem

The number of isomorphism classes of objects in the category $\mathrm{hef}(R, W)$ *is given by:*

$$\check{N}(R, W) = \sum_{k=0}^{r^{\hat{1}}} \sum_{\substack{K \subsetneq I, \\ |K^{\hat{1}}| = k}} 2^{r^{\hat{0}}+k} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \quad . \tag{5}$$

while the number if isomorphism classes of objects in the category $\mathrm{HEF}(R, W)$ *is given by:*

$$N(R, W) = 2^{r^{\hat{0}}} + \sum_{k=0}^{r^{\hat{1}}} 2^{r^{\hat{0}}+k-1} \sum_{\substack{K \subsetneq I \\ |K^{\hat{1}}| = k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \quad . \tag{6}$$

Elementary divisor domains

Definition

An integral domain R is called an *elementary divisor domain* (EDD) if for any three elements $a, b, c \in R$, there exist $p, q, x, y \in R$ such that $(a, b, c) = pxa + pyb + qyc$ is a GCD of a, b and c.

Definition

Let R be a commutative ring. We say that R satisfies *Kaplansky's condition* if for any three elements a, b, c in R such that $(a, b, c) \sim 1$, there exist elements $p, q \in R$ such that $(pa, pb + qc) \sim 1$.

Proposition

An integral domain R is an EDD iff it satisfies the following two conditions:

- *R is a Bézout domain*
- *R satisfies Kaplansky's condition.*

Smith normal form theorem over an EDD

Definition

Let $A \in \mathrm{Mat}(m, n, R)$ be an m by n matrix with coefficients from a GCD domain R. For any $k \in \{1, \ldots, r\}$, the *k-th determinantal invariant* $\boldsymbol{\delta}_k(A) \in R/U(R)$ of A is defined to be the gcd class of all $k \times k$ minors of A. We also define $\boldsymbol{\delta}_0(A) = (1)$.

Proposition

Let R be a GCD domain. For any $A \in \mathrm{Mat}(m, n, R)$, we have:

$$\boldsymbol{\delta}_{k-1}(A) | \boldsymbol{\delta}_k(A) \quad \forall k \in \{1, \ldots, \mathrm{rk} A\}$$

Defining the invariant factors $d_k(A) \in R/U(R)$ *by:*

$$\mathbf{d}_k(A) \stackrel{\mathrm{def.}}{=} \begin{cases} \frac{\boldsymbol{\delta}_k(A)}{\boldsymbol{\delta}_{k-1}(A)} & \text{if } \boldsymbol{\delta}_{k-1}(A) \neq 0 \\ (1) & \text{if } \boldsymbol{\delta}_{k-1}(A) = 0 \end{cases} \quad \forall k \in \{1, \ldots, \mathrm{rk} A\} \quad ,$$

we have:

$$\mathbf{d}_{k-1}(A) | \mathbf{d}_k(A) \quad \forall k \in \{2, \ldots, \mathrm{rk} A\} \quad .$$

Theorem

Let R be an EDD. For any matrix $A \in \mathrm{Mat}(m,n,R)$, there exist matrices $U \in \mathrm{GL}(m,R)$ and $V \in \mathrm{GL}(n,R)$ such that:

$$UAV^{-1} = D \ ,$$

where $D_{ij} = 0$ for all $i \neq j$ and the diagonal entries $d_i \stackrel{\mathrm{def.}}{=} D_{ii}$ (with $i \in \{1, \ldots, r\}$, where $r \stackrel{\mathrm{def.}}{=} \mathrm{rk} A \leq \min(m,n)$) are non-zero elements which satisfy the condition:

$$d_1 | d_2 | \ldots | d_k \quad .$$

In this case, the matrix D is called the Smith normal form *of A. Moreover, the association classes of d_k coincide with the invariant factors of A:*

$$(d_k) = \mathbf{d}_k(A) \quad , \quad \forall k \in \{1, \ldots, r\} \quad .$$

Proposition

Let R be an EDD and $A, B \in \mathrm{Mat}(m,n,R)$. Then A and B are equivalent iff they have the same rank and their invariant factors coincide.

Some examples of EDDs

- Any Bezout domain which is an F-domain (i.e. for which any non-zero element is contained in at most a finite number of maximal ideals) is an EDD. In particular, any PID is an EDD.
- The ring $\mathrm{O}(\Sigma)$ of holomorphic functions on any connected and non-compact borderless Riemann surface is an EDD.
- The ring $\mathbb{A}$ of all algebraic integers is an EDD which has no prime elements.
- If R is an EDD with quotient field K and J is any integral domain such that $R \subset J \subset K$, then J is an EDD.
- Any generalized valuation domains is an EDD. If $V_1, \ldots, V_n$ are generalized valuation domains with the same quotient field K, then $R \stackrel{\mathrm{def.}}{=} \cap_{i=1}^n V_i$ is an EDD.
- Let B be an EDD with quotient field K and let m be the maximal ideal of the power series ring $K[[x]]$ in one variable. Then $R := B + m$ is an EDD.
- Let B be an EDD with quotient field K and X be an indeterminate. Then $R := B + XK[X]$ is an EDD.
- Let K be an algebraically closed field of characteristic different from two and let x_1 be an indeterminate over K. Let x_2 be a square root of x_1, x_3 be a square root of x_2 and so on. Then the ring $R := \cup_{n=1}^\infty K[x_n, 1/x_n]$ is an EDD.

Matrix factorizations over elementary divisor domains

Let R be an EDD and $W \in R^\times$ be a non-zero element.

Proposition

Let $a = (R^{\rho|\rho}, D)$ and $a' = (R^{\rho'|\rho'}, D')$ be two matrix factorizations of W over R, where $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$ and $D' = \begin{bmatrix} 0 & v' \\ u' & 0 \end{bmatrix}$. Let $\mathbf{d}_1(v), \ldots, \mathbf{d}_\rho(v)$ and $\mathbf{d}_1(v'), \ldots, \mathbf{d}_{\rho'}(v')$ be respectively the invariant factors of the matrices $v \in \mathrm{Mat}(\rho, \rho, R)$ and $v' \in \mathrm{Mat}(\rho', \rho', R)$. Then the following statements are equivalent:

(a) *a and a' are strongly isomorphic, i.e. isomorphic in the category $\mathrm{zmf}(R, W)$.*
(b) *We have $\rho = \rho'$ and the invariant factors of v and v' are equal:*

$$\mathbf{d}_i(v) = \mathbf{d}_i(v') \quad \forall i \in \{1, \ldots, \rho\} \quad .$$

Proposition

There exists an autoequivalence Ψ of $\mathrm{hmf}(R, W)$ such that:

1. *Ψ is isomorphic with the identity functor $\mathrm{id}_{\mathrm{zmf}(R,W)}$*
2. *For any matrix factorization $a = (R^{\rho|\rho}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, we have:*

Theorem

Let W be a critically-finite element of R. Then $\mathrm{hmf}(R, W)$ is a Krull-Schmidt category whose non-zero indecomposables are the nontrivial primary matrix factorizations of W.

Theorem

Let W be a critically-finite element of R with factorization (4). *Then there exist equivalences of triangulated categories:*

$$\mathrm{hmf}(R, W) \simeq \vee_{i=1}^{N} \underline{\mathrm{mod}}_{A_{n_i}(p_i)} \simeq \vee_{i=1}^{N} \mathrm{D_{sing}}(A_{n_i}(p_i)) \quad ,$$

where $\vee$ denotes orthogonal sum and, for any prime element $p \in R$:

- $\underline{\mathrm{mod}}_{A_n(p)}$ *denotes the stable category of the category of finitely-generated modules over the commutative ring $A_n(p) \stackrel{\mathrm{def.}}{=} R/(p^n)$ (which is a Frobenius ring)*
- $\mathrm{D_{sing}}(A_n(p)) \simeq \underline{\mathrm{mod}}_{A_n(p)}$ *is the category of singularities of this ring.*

The category mod_Λ

For simplicity, we denote $A_n(p) \stackrel{\mathrm{def.}}{=} R/\langle p^n \rangle$ by Λ, the residue field $\mathbf{k}_n(p)$ by $\mathbf{k}$ and the maximal ideal $\mathbf{m}_n(p)$ by $\mathbf{m}$. Let mod_Λ be the category of finitely-generated modules over Λ. Since Λ is Artinian, the following statements are equivalent for a Λ-module M by the Akizuki-Hopkins-Lewitzki theorem:

- M is Noetherian
- M is Artinian
- M is finitely-generated
- M has finite composition length.

Let $\Lambda_i = \langle p^{n-i} \rangle / \langle p^n \rangle = p^{n-i}\Lambda$ with $i \in \{0, \ldots, n\}$ be the ideals of Λ, thus $\Lambda_0 = 0$, $\Lambda_{n-1} = \mathbf{m}$ and $\Lambda_n = \Lambda$. Let $V_i \stackrel{\mathrm{def.}}{=} \Lambda/\Lambda_{n-i} \simeq_R R/\langle p^i \rangle$ (with $i = 0, \ldots, n$) be the cyclically-presented cyclic Λ-modules with annihilators $\mathrm{Ann}(V_i) = \Lambda_{n-i}$.

Recall that a commutative ring R is called an *FGC ring* if every finitely-generated R-module is isomorphic with a finite direct sum of cyclic modules.

Proposition

Λ is an FGC ring whose indecomposable non-zero finitely-generated Λ-modules are the cyclic modules $V_1, \ldots, V_n$. Moreover, the decomposition of a finitely-generated Λ-module into non-zero cyclic modules is unique up to permutation and isomorphism of factors, hence mod_Λ is a Krull-Schmidt category.

Proposition

The only non-zero indecomposable Λ-module which is projective is $V_n \simeq \Lambda_n = \Lambda$.

Notice that Λ is a uniserial ring and that the indecomposable cyclic modules $V_i \simeq_R \Lambda_i$ are uniserial modules of length i. The unique composition series of Λ_i is given by:

$$0 = \Lambda_0 \subset \ldots \subset \Lambda_i \quad .$$

In particular, the only simple Λ-module is $\Lambda_1 \simeq_R V_1 \simeq_R \mathbf{k}$. We have:

$$V_{i+1}/V_i \simeq \Lambda_{i+1}/\Lambda_i \simeq \mathbf{k}$$

and the only composition factor of $\Lambda_i \simeq_R V_i$ is $\mathbf{k}$, with multiplicity i.

Proposition

The ring Λ is a commutative Frobenius ring. In particular, Λ is self-injective and hence it is a Gorenstein ring of dimension zero. Thus:

$$\mathrm{Ext}^i_R(\mathbf{k}, \Lambda) \simeq_R \begin{cases} \mathbf{k} & \textit{if } i = 0 \\ 0 & \textit{if } i \neq 0 \end{cases} \quad .$$

Let $\underline{\mathrm{mod}}_\Lambda$ denote the projectively-stable category of finitely-generated Λ-modules. Since any projective Λ-module is free, this category has the same objects as mod_Λ and morphisms given by:

$$\underline{\mathrm{Hom}}_\Lambda(M,N) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_\Lambda(M,N)/\mathcal{P}_\Lambda(M,N) \quad \forall M,N \in \mathrm{Ob}(\mathrm{mod}_\Lambda) \quad ,$$

where $\mathcal{P}_\Lambda(M,N) \subset \mathrm{Hom}_\Lambda(M,N)$ is the submodule consisting of those morphisms from M to N which factor through a free module of finite rank. Since mod_Λ is a Frobenius category, the stable category $\underline{\mathrm{mod}}_\Lambda$ has a natural triangulated structure. Let:

$$\delta_n(i) \stackrel{\mathrm{def.}}{=} \min(i, n-i) \in \{1, \ldots, n-1\} \ , \ \ \mu_n(i,j) \stackrel{\mathrm{def.}}{=} \min[\delta_n(i), \delta_n(j)] = \begin{cases} i & \text{if } i+j \leq n \ \& \ i \leq j \\ j & \text{if } i+j \leq n \ \& \ i > j \\ n-i & \text{if } i+j > n \ \& \ i > j \\ n-j & \text{if } i+j > n \ \& \ i \leq j \end{cases} \quad . \tag{7}$$

Notice the relations $\delta_n(i) = \delta_n(n-i)$ and $\delta_n(n) = 0$ as well as:

$$\mu_n(i,j) = \mu_n(j,i) = \mu_n(n-i,j) = \mu_n(i,n-j) \quad , \quad \mu_n(i,n) = 0 \quad . \tag{8}$$

Proposition

For any $1 \leq i,j \leq n-1$, we have:

$$\underline{\mathrm{Hom}}_\Lambda(V_i, V_j) \simeq_\Lambda V_{\mu_n(i,j)} \quad .$$

Proposition

The triangulated category $\underline{\mathrm{mod}}_\Lambda$ *is Krull-Schmidt and has indecomposable objects* $V_1, \ldots, V_{n-1}$. *Moreover, it has Auslander-Reiten triangles given by:*

$$V_i \xrightarrow{g_i} V_{i-1} \oplus V_{i+1} \xrightarrow{f_i} V_i \to \Omega(V_i) \ \text{ for all } \ i \in \{1, \ldots, n-1\} \quad . \tag{9}$$

and its AR quiver is:

$$\begin{tikzcd} V_1 \arrow[r, shift left] & V_2 \arrow[l, shift left] \arrow[r, shift left] & V_3 \arrow[l, shift left] \arrow[r, shift left] & V_4 \arrow[l, shift left] \end{tikzcd}$$

Figure: Auslander-Reiten quiver for $\underline{\mathrm{mod}}_\Lambda$ when $n = 5$. The translation fixes all vertices.

Proposition

The category $\underline{\mathrm{mod}}_\Lambda$ *is Calabi-Yau with idempotent translation functor given on objects by* $\Omega(V_i) = V_{n-i}$.

Proposition

The smallest full subcategory of $\underline{\mathrm{mod}}_\Lambda$ *which contains the object* $V_1 = \mathbf{k}_p$ *and is closed under isomorphisms, direct sums, direct summands and extensions coincides with* $\underline{\mathrm{mod}}_\Lambda$. *Hence:*

$$\langle V_1 \rangle = \underline{\mathrm{mod}}_\Lambda \quad .$$

Examples: Non-compact Riemann surfaces

Let Σ be any connected, smooth and borderless non-compact Riemann surface. The non-Noetherian ring $R = \mathrm{O}(\Sigma)$ of holomorphic complex-valued functions defined on X is an elementary divisor domain whose prime elements are those holomorphic functions which have a single simple zero and no other zeros. A critically-finite superpotential is a holomorphic function $W \in \mathrm{O}(\Sigma)$ of the form $W = W_0 W_c$, where $W_0 \in \mathrm{O}(\Sigma)$ has only simple zeros while $W_c \in \mathrm{O}(\Sigma)$ has a finite number of zeros, each of multiplicity at least two.

Remark

1. Σ need not be algebraic (e.g.: the unit disk) It is (affine) algebraic iff it can be written as a compact Riemann surface minus a finite number of points.
2. W_0 can have a (countable) infinity of simple zeros.
3. Σ may have infinite genus as well as an infinite number of Freudenthal ends.

In this situation, our results allow one to:

- Present the Krull-Schmidt category $\mathrm{hmf}(R, W)$ in terms of its indecomposable objects, which are the primary matrix factorizations
- Count the number of isomorphism classes of elementary matrix factorizations
- Completely describe the triangulated structure of $\mathrm{hmf}(R, W)$ using the Auslader-Reiten theory.

This has applications to the theory of Landau-Ginzburg models defined on non-compact Riemann surfaces (notice that any such surface is a Stein manifold).

Other examples

The class of non-Noetherian Bezout domains is very large. Indeed, the Jaffards-Ohm theorem shows that there exists a Bézout domain whose group of divisibiity is isomorphic with any given lattice ordered group. Our methods can be applied to Bezout and elementary divisor domains obtained by this and other constructions:

- Bézout and elementary divisor domains obtained through the Jaffards-Ohm construction
- Bézout and elementary divisor domains obtained from spectral posets
- Robba rings.

We discuss a few such applications in our preprints.