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Canonical construction of invariant differential operators for Lie algebras and quantum groups

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Introduction

Invariant differential equations play a very important role in the description of physical symmetries - recall, e.g., the examples of Dirac, Maxwell equations, (for more examples cf., e.g., [BR]). It is important to construct systematically such equations for the setting of quantum groups, where they are expected as (multiparameter) q-difference equations.

In the present talk we consider the construction of deformed analogs of some conformally invariant equations, in particular, the Maxwell equations, following the approach of [Da,D1]. We start with the classical situation and we first write the Maxwell equations in an indexless formulation, trading the indices for two conjugate variables $z, \bar{z}$. This formulation has two advantages. First, it is very simple, and in fact, just with the introduction of an additional parameter, we can describe a whole infinite hierarchy of equations, which we call the *Maxwell hierarchy* . Second, we can easily identify the variables $z, \bar{z}$ and the four Minkowski coordinates with the six local coordinates of a flag manifold of $SL(4)$ and $SU(2,2)$. Thus, one may look at this as a nice example of unifying internal and external degrees of freedom.

Next we need the deformed analogs of the above constructions. The specifics of the approach of [Da,DI] is that one needs also the complexification of the algebra in consideration. Thus we have used the deformations $U_q(gl(m))$ and $U_q(sl(m))$ in the case $m = 4$. Using the corresponding representations and intertwiners of deformed $U(sl(4))$ we also derive infinite hierarchies of deformed Maxwell and related equations.

Classical setting

It is well known that Maxwell equations may be written in several equivalent forms:

$$\partial^\mu F_{\mu\nu} = J_\nu \ , \quad \partial^{\mu *} F_{\mu\nu} = 0 \tag{1}$$

or,

$$\begin{aligned}
&\partial_k E_k = J_0 \ (= 4\pi\rho), \\
&\partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m = J_k \ (= -4\pi j_k), \\
&\partial_k H_k = 0 \ , \\
&\partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m = 0
\end{aligned} \tag{2}$$

where $E_k \equiv F_{k0}$, $H_k \equiv (1/2)\varepsilon_{k\ell m}F_{\ell m}$,
or

$$\partial_k F_k^\pm = J_0 \ , \quad \partial_0 F_k^\pm \pm i\varepsilon_{k\ell m}\partial_\ell F_m^\pm = J_k \tag{3}$$

where

$$F_k^\pm \equiv E_k \pm i H_k \tag{4}$$

Not so well known is the fact that the eight equations in (3) can be rewritten as two conjugate scalar equations in the following way:

$$I^{+}\ F^{+}(z)\ =\ J(z,\bar{z})\ , \tag{5a}$$

$$I^{-}\ F^{-}(\bar{z})\ =\ J(z,\bar{z}) \tag{5b}$$

where

$$\begin{aligned} I^{+} \quad &= \quad \bar{z}\partial_{+} + \partial_{v}\ - \qquad\qquad (6a)\\ &\quad - \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_{v} + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_{z}\ , \\ I^{-} \quad &= \quad z\partial_{+} + \partial_{\bar{v}}\ - \qquad\qquad (6b)\\ &\quad - \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_{v} + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_{\bar{z}} \end{aligned}$$

$$\begin{aligned} &x_{\pm} \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar{v} \equiv x_1 + ix_2, \\ &\partial_{\pm} \equiv \partial/\partial x_{\pm}, \quad \partial_{v} \equiv \partial/\partial v, \quad \partial_{\bar{v}} \equiv \partial/\partial\bar{v} \end{aligned} \tag{7}$$

$$\begin{aligned} F^{+}(z) \quad &\equiv \quad z^2(F_1^{+} + iF_2^{+}) - 2zF_3^{+}\ - \\ &\quad - \ (F_1^{+} - iF_2^{+})\ , \qquad\qquad (8)\\ F^{-}(\bar{z}) \quad &\equiv \quad \bar{z}^2(F_1^{-} - iF_2^{-}) - 2\bar{z}F_3^{-}\ - \\ &\quad - \ (F_1^{-} + iF_2^{-})\ , \\ J(z,\bar{z}) \quad &\equiv \quad \bar{z}z(J_0 + J_3) + \bar{z}(J_1 - iJ_2)\ + \\ &\quad + \ z(J_1 + iJ_2) + (J_0 - J_3) \end{aligned}$$

where we continue to suppress the x_μ, resp., $x_\pm, v, \bar{v}$, dependence in F and J. (The conjugation mentioned above is standard and in our terms it is : $I^+ \longleftrightarrow I^-$, $F^+(z) \longleftrightarrow F^-(\bar{z})$.)

It is easy to recover (3) from (5) - just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (5) one gets at once (3).

Writing the Maxwell equations in the simple form (5) has also important conceptual meaning. The point is that each of the two scalar operators I^+, I^- is indeed a single object, namely it is an intertwiner of the conformal group, while the individual components in (1) - (3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations T^χ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1+1)(n_2+1)$), and d is the conformal dimension (or energy). Then the intertwining properties of the operators in (6) are given by:

$$\begin{aligned} I^+ &: C^+ \longrightarrow C^0 \ , \\ I^+ \circ T^+ &= T^0 \circ I^+ \ , \end{aligned} \tag{9a}$$

$$\begin{aligned} I^- &: C^- \longrightarrow C^0 \ , \\ I^- \circ T^- &= T^0 \circ I^- \end{aligned} \tag{9b}$$

where $T^a = T^{\chi^a}$, $a = 0, +, -$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2,0;2] \ , \quad \chi^- = [0,2;2] \ , \quad \chi^0 = [1,1;3] \tag{10}$$

as anticipated. Indeed, $(n_1, n_2) = (1, 1)$ is the four-dimensional Lorentz representation, (carried by J_μ above), and $(n_1, n_2) = (2, 0), (0, 2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F_k^{\pm}$ above), while the conformal dimensions are the canonical dimensions of a current ($d = 3$), and of the Maxwell field ($d = 2$).

We see that the variables $z, \bar{z}$ are related to the spin properties and we shall call them 'spin variables'. More explicitly, a Lorentz spin-tensor $G(z, \bar{z})$ with signature (n_1, n_2) is a polynomial in $z, \bar{z}$ of order n_1, n_2, resp.

$$
\begin{tikzcd}
\phi \arrow[r, leftrightarrow, "/" marking] \arrow[d, "\partial_\mu"'] & \Phi \\
A_\mu \arrow[r, leftrightarrow, "/" marking] \arrow[dr, "\partial_{[\lambda,\cdot]}"] & J_\mu \arrow[u, "\partial^\mu"'] \\
 & F_{[\lambda,\mu]} \arrow[u, "\partial^\lambda"]
\end{tikzcd}
$$

Fig. 1. Simplest example of diagram with conformal invariant operators (arrows are differential operators, dashed arrows are integral operators)

$$\partial_\mu = \frac{\partial}{\partial x_\mu}\,,\ \ A_\mu \text{ electromagnetic potential,} \quad \partial_\mu \phi = A_\mu$$

$$F \text{ electromagnetic field,} \quad \partial_{[\lambda} A_{\mu]} = \partial_\lambda A_\mu - \partial_\mu A_\lambda \ = \ F_{\lambda\mu}$$

$$J_\mu \text{ electromagnetic current,} \ \ \partial^\lambda F_{\lambda\mu} = J_\mu,\ \ \partial^\mu J_\mu = \Phi$$

$$
\begin{tikzcd}
\phi \arrow[d, "\partial_\mu"'] \arrow[r, leftrightarrow] & \Phi \\
A_\mu \arrow[d, "d_{12}"'] \arrow[dr, "d_{23}"' near start] \arrow[r, leftrightarrow] & J_\mu \cong \chi^0 \arrow[u, "\partial^\mu"'] \\
F^-_{[\lambda,\mu]} \cong \chi^- \arrow[ur, "d_{23} = I^-" near end] \arrow[r, leftrightarrow] & F^+_{[\lambda,\mu]} \cong \chi^+ \arrow[u, "d_{12} = I^+"']
\end{tikzcd}
$$

Fig. 2. More precise showing of the simplest example,
$F = F^+ \oplus F^-$ shows the parity splitting of the electromagnetic field,
d_{12}, d_{23} linear invariant operators corresponding to the roots α_{12}, α_{23}

Formulae (9), (10) are part of an infinite hierarchy of couples of first order intertwiners. Explicitly, instead of (9), (10) we have [Db,DI]:

$$\begin{aligned} I^+_n \ &: \ C^+_n \longrightarrow C^0_n \ , \\ I^+_n \circ T^+_n \ &= \ T^0_n \circ I^+_n \ , \end{aligned} \tag{11a}$$

$$\begin{aligned} I^-_n \ &: \ C^-_n \longrightarrow C^0_n \ , \\ I^-_n \circ T^-_n \ &= \ T^0_n \circ I^-_n \end{aligned} \tag{11b}$$

where $T^a_n = T^{\chi^a_n}$, $C^a_n = C^{\chi^a_n}$, and the signatures are:

$$\begin{aligned} &\chi^+_n = [n+2, n; 2] \ , \qquad \chi^-_n = [n, n+2; 2] \ , \\ &\chi^0_n = [n+1, n+1; 3] \ , \qquad n \in \mathbb{Z}_+ \end{aligned} \tag{12}$$

while instead of (5) we have:

$$I^+_n \ F^+_n(z,\bar{z}) \ = \ J_n(z,\bar{z}) \ , \tag{13a}$$

$$I^-_n \ F^-_n(z,\bar{z}) \ = \ J_n(z,\bar{z}) \tag{13b}$$

where $(n \in \mathbb{Z}_+)$

$$I_n^+ = \frac{n+2}{2}\Big(\bar{z}\partial_+ + \partial_v\Big) - \tag{14a}$$
$$- \frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_z \ ,$$
$$I_n^- = \frac{n+2}{2}\Big(z\partial_+ + \partial_{\bar{v}}\Big) - \tag{14b}$$
$$- \frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_{\bar{z}}$$

while $F_n^+(z,\bar{z})$, $F_n^-(z,\bar{z})$, $J_n(z,\bar{z})$, are polynomials in $z,\bar{z}$ of degrees $(n+2,n)$, $(n,n+2)$, $(n+1,n+1)$, resp., as explained above.

If we want to use the notation with indices as in (1), then $F_n^+(z,\bar{z})$ and $F_n^-(z,\bar{z})$ correspond to $F_{\mu\nu,\alpha_1,\ldots,\alpha_n}$ which is antisymmetric in the indices μ,ν, symmetric in $\alpha_1,\ldots,\alpha_n$, and traceless in every pair of indices, while $J_n(z,\bar{z})$ corresponds to $J_{\mu,\alpha_1,\ldots,\alpha_n}$ which is symmetric and traceless in every pair of indices. Note, however, that the analogs of (1) would be much more complicated if one wants to write explicitly all components. The crucial advantage of (13) is that the operators $I_n^\pm$ are given just by a slight generalization of $I_0^\pm \equiv I^\pm$.

We call the hierarchy of equations (13) the *Maxwell hierarchy*. The Maxwell equations are the zero member of this hierarchy.

Formulae (13),(11),(12) are part of a much more general classification scheme [Dh,DI], involving also other intertwining operators, and of arbitrary order.

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\Lambda^-_{p\nu n} \arrow[r, leftrightarrow, "/" marking] \arrow[d, "d_2^\nu"'] & \Lambda^+_{p\nu n} \\
\Lambda'^-_{p\nu n} \arrow[r, leftrightarrow, "/" marking] \arrow[d, "d_{12}^n"'] \arrow[dr, "d_{23}^p"] & \Lambda'^+_{p\nu n} \arrow[u, "d_{13}^\nu"'] \\
\Lambda''^-_{p\nu n} \arrow[r, leftrightarrow, "/" marking] \arrow[ur, "d_{23}^p"'] & \Lambda''^+_{p\nu n} \arrow[u, "d_{12}^n"']
\end{tikzcd}
$$

Fig. 3. The general classification of invariant differential operators valid for $so(4,2)$, $so(5,1)$ and $so(3,3) \cong sl(4,\mathbb{R})$.

p, ν, n are three natural numbers, the shown simplest case is when $p = \nu = n = 1$,

d_2^ν, d_{13}^ν linear invariant operators of order ν corresponding to the roots α_2, α_{13}

d_{12}^n, d_{23}^p linear invariant operators of order n, p corresponding to the roots α_{12}, α_{23}

To proceed further we rewrite (14) in the following form:

$$I_n^+ = \frac{1}{2}\Big((n+2)I_1 I_2 - (n+3)I_2 I_1\Big),$$
$$I_n^- = \frac{1}{2}\Big((n+2)I_3 I_2 - (n+3)I_2 I_3\Big) \quad (15)$$

where

$$I_1 \ \equiv\ \partial_z\,,\ I_2 \ \equiv\ \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\,,\ I_3 \ \equiv\ \partial_{\bar{z}} \quad (16)$$

It is important to note that group-theoretically the operators I_a correspond to the right action of the three simple roots $\alpha_1, \alpha_2, \alpha_3$ of the root system of $sl(4)$, while the operators $I_n^{\pm}$ are obtained from the lowest possible singular vectors corresponding to the two non-simple non-highest roots $\alpha_{12} \equiv \alpha_1 + \alpha_2$, $\alpha_{23} \equiv \alpha_2 + \alpha_3$.

In particular, the operator d_2^{ν} on Figure 3 has a very simple expression since it corresponds to the simple root α_2, namely:

$$d_2^\nu = \Big(I_2\Big)^\nu$$

The form (15) is that we generalize for the deformed case. In fact, we can write at once the q-deformed form [Dh,DII]:

$$\begin{aligned}\widehat{I}_n^+ &= \frac{1}{2}\Big([n+2]_q\,\widehat{I}_1\widehat{I}_2 - [n+3]_q\,\widehat{I}_2\widehat{I}_1\Big),\\ \widehat{I}_n^- &= \frac{1}{2}\Big([n+2]_q\,\widehat{I}_3\widehat{I}_2 - [n+3]_q\,\widehat{I}_2\widehat{I}_3\Big)\end{aligned}\tag{17}$$

where $[m]_q \equiv \frac{q^m - q^{-m}}{q - q^{-1}}$ are the ubiquitous q-numbers.

Here $\widehat{I}_n^\pm$ are obtained from the lowest possible singular vectors of $U_q(sl(4))$, corresponding (as above) to the roots α_{12}, α_{23} [Dh,DII].

Quantum Minkowski space-time

The variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the flag manifold
$\mathcal{Y} = GL(4)/\tilde{B} = SL(4)/B$,
where $\tilde{B}, B$ are the Borel subgroups of $GL(4)$, $SL(4)$, respectively, consisting of all upper diagonal matrices. Under a natural conjugation (cf. also below) this is also a flag manifold of the conformal group $SU(2,2)$.

Explicitly, for this is used the triangular Gauss decomposition:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \tag{18}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ v & x_- & 1 & 0 \\ x_+ & \bar{v} & \bar{z} & 1 \end{pmatrix} \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[This decomposition is valid for a dense subset of the group (more precisely, it is not valid for submanifolds of lower dimensions). But this is not essential especially for the deformed quantum group case.]

We know from [Dh,DII] what are the properties of the non-commutative coordinates on the $SL_q(4)$ coset. Thus, we obtain for the commutation rules of the q-Minkowski space-time coordinates

$$\begin{aligned} x_\pm v \ &= \ q^{\pm 1} v x_\pm \,, \quad x_\pm \bar{v} \ = \ q^{\pm 1} \bar{v} x_\pm \,, \\ x_+ x_- - x_- x_+ \ &= \ \lambda v \bar{v} \,, \quad \bar{v} v \ = \ v \bar{v} \end{aligned} \tag{19}$$

The q-Minkowski length ℓ_q is defined as the q-determinant of $M = \begin{pmatrix} x_+ & \bar{v} \\ v & x_- \end{pmatrix}$:

$$\ell_q \ \doteq \ \mathrm{det}_q \ M \ = \ x_+ x_- - q \bar{v} v \tag{20}$$

and hence it commutes with the q-Minkowski coordinates. It has the correct classical limit $\ell_{q=1} = x_0^2 - \vec{x}^2$.

We know from [Dh,DII] that for q phase ($|q| = 1$) the commutation relations (19) are preserved by an anti-linear anti-involution ω acting as:

$$\omega(x_\pm) = x_\pm\,, \quad \omega(v) = \bar{v} \tag{21}$$

from which follows also that $\omega(\ell_q) = \ell_q$.

The commutation rules involving the spin variables $z, \bar{z}$ are:

$$\begin{aligned}
&\bar{z}z = z\bar{z}\,,\\
&x_+z = q^{-1}zx_+\,, \quad x_-z = qzx_- - \lambda v\,,\\
&vz = q^{-1}zv\,, \quad \bar{v}z = qz\bar{v} - \lambda x_+\,,\\
&\bar{z}x_+ = qx_+\bar{z}\,, \quad \bar{z}x_- = q^{-1}x_-\bar{z} + \lambda\bar{v}\,,\\
&\bar{z}v = q^{-1}v\bar{z} + \lambda x_+\,, \quad \bar{z}\bar{v} = q\bar{v}\bar{z}\,,\\
&z\ell_q = \ell_q z\,, \quad \bar{z}\ell_q = \ell_q\bar{z}
\end{aligned} \tag{22}$$

Certainly, the commutation relations (22) are also preserved (for q phase) by the conjugation ω - supplementing (21) by $\omega(z) = \bar{z}$.

With this conjugation $\mathcal{Y}_q$ becomes a coset of $SU_q(2,2)$.

Quantum Maxwell equations hierarchy

The normally ordered basis of the q - coset $\mathcal{Y}_q$ considered as an associative algebra is :

$$\begin{aligned} \widehat{\varphi}_{ijk\ell mn} \quad &= \quad z^i\ v^j\ x_-^k\ x_+^\ell\ \bar{v}^m\ \bar{z}^n\,, \qquad (23) \\ & \qquad i,j,k,\ell,m,n \in \mathbb{Z}_+ \end{aligned}$$

We introduce now the representation spaces C^χ , $\chi = [n_1, n_2; d]$. The elements of C^χ , which we shall call (abusing the notion) functions, are polynomials in $z, \bar{z}$ of degrees n_1, n_2, resp., and formal power series in the quantum Minkowski variables. Namely, these functions are given by:

$$\widehat{\varphi}_{n_1,n_2}(\bar{Y}) \quad = \quad \sum_{\substack{i,j,k,\ell,m,n\in\mathbb{Z}_+ \\ i\leq n_1,\ n\leq n_2}} \mu_{ijk\ell mn}^{n_1,n_2}\ \widehat{\varphi}_{ijk\ell mn} \tag{24}$$

where $\bar{Y}$ denotes the set of the six coordinates on $\mathcal{Y}_q$. Thus the quantum analogs of $F_n^{\pm}$, J_n, cf. (13), are :

$$\widehat{F}_n^{+} = \widehat{\varphi}_{n+2,n}(\bar{Y}) , \quad \widehat{F}_n^{-} = \widehat{\varphi}_{n,n+2}(\bar{Y}) ,$$
$$\widehat{J}_n = \widehat{\varphi}_{n+1,n+1}(\bar{Y}) \tag{25}$$

Using the above machinery we can present a deformed version of the Maxwell hierarchy of equations. For this we use that the operators $\widehat{I}_a$ are given by the right action of $U_q(sl(4))$ on $\mathcal{Y}_q$:

$$\widehat{I}_a = \pi_R(X_a^-) \tag{26}$$

Explicitly, we have:

$$\widehat{I}_1 = \widehat{\mathcal{D}}_z \; T_z \; T_v \; T_+ \; (T_- T_{\bar{v}})^{-1} \tag{27a}$$

$$\begin{aligned} \widehat{I}_2 = \Big(& q \; \widehat{M}_z \; \widehat{\mathcal{D}}_v \; T_-^2 + \widehat{\mathcal{D}}_- \; T_- + \\ & + \widehat{M}_z \; \widehat{M}_{\bar{z}} \; \widehat{\mathcal{D}}_+ \; T_- \; T_{\bar{v}} \; T_v^{-1} + \\ & + q^{-1} \; \widehat{M}_{\bar{z}} \; \widehat{\mathcal{D}}_{\bar{v}} - \\ & - \lambda \; \widehat{M}_v \; \widehat{M}_{\bar{z}} \; \widehat{\mathcal{D}}_- \; \widehat{\mathcal{D}}_+ \; T_{\bar{v}} \Big) \; T_{\bar{v}} \; T_{\bar{z}}^{-1} \end{aligned} \tag{27b}$$

$$\widehat{I}_3 = \widehat{\mathcal{D}}_{\bar{z}} \; T_{\bar{z}} \tag{27c}$$

where we us the q-shift operators T_κ:

$$
\begin{aligned}
T_z \, \widehat{\varphi}_{ijk\ell mn} &= q^i \, \widehat{\varphi}_{ijk\ell mn} \\
T_v \, \widehat{\varphi}_{ijk\ell mn} &= q^j \, \widehat{\varphi}_{ijk\ell mn} \\
T_- \, \widehat{\varphi}_{ijk\ell mn} &= q^k \, \widehat{\varphi}_{ijk\ell mn} \\
T_+ \, \widehat{\varphi}_{ijk\ell mn} &= q^\ell \, \widehat{\varphi}_{ijk\ell mn} \\
T_{\bar{v}} \, \widehat{\varphi}_{ijk\ell mn} &= q^m \, \widehat{\varphi}_{ijk\ell mn} \\
T_{\bar{z}} \, \widehat{\varphi}_{ijk\ell mn} &= q^n \, \widehat{\varphi}_{ijk\ell mn}
\end{aligned}
$$

further the q-difference operators:

$$
\widehat{\mathcal{D}}_\kappa = \frac{1}{q - q^{-1}} \, M_\kappa^{-1} \left(T_\kappa - T_\kappa^{-1} \right)
$$

where:

$$
\begin{aligned}
M_z \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{i+1,jk\ell mn} \\
M_v \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{i,j+1,k\ell mn} \\
M_- \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ij,k+1,\ell mn} \\
M_+ \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk,\ell+1,mn} \\
M_{\bar{v}} \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk\ell,m+1,n} \\
M_{\bar{z}} \, \widehat{\varphi}_{ijk\ell mn} &= \widehat{\varphi}_{ijk\ell m,n+1}
\end{aligned}
$$

Note that for $q \to 1$ we have: $T_\kappa \to \mathbf{1}$, $\widehat{\mathcal{D}}_\kappa \to \partial_\kappa$.

With this we have now the q - Maxwell hierarchy of equations - it remains just to substitute the operators of (27) in (17). In fact, we can also rewrite these in the q-analog of (13). We have :

$$
\begin{aligned}
{}_qI_n^+ &= \tfrac{1}{2}\Bigg(\Big(q\widehat{\mathcal{D}}_v + \widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_+(T_-T_v)^{-1}T_{\overline{v}}\Big)[n+2-N_z]_q \\
&\quad -q^{-n-2}\Big(\widehat{\mathcal{D}}_-T_- + q^{-1}\widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_{\overline{v}} - \qquad\qquad (28a) \\
&\quad -\lambda\widehat{M}_v\widehat{M}_{\overline{z}}\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+T_{\overline{v}}\Big)\widehat{\mathcal{D}}_z\Bigg)T_+T_-T_vT_zT_{\overline{z}}^{-1} \\
{}_qI_n^- &= \tfrac{1}{2}\Big(\widehat{\mathcal{D}}_{\overline{v}} + q\widehat{M}_z\widehat{\mathcal{D}}_+T_{\overline{v}}T_-T_v^{-1} - \qquad\qquad (28b) \\
&\quad -q\lambda\widehat{M}_v\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+T_{\overline{v}}\Big)T_{\overline{v}}[n+2-N_{\overline{z}}]_q - \\
&\quad -\tfrac{1}{2}q^{n+3}\Big(\widehat{\mathcal{D}}_- + q\widehat{M}_z\widehat{\mathcal{D}}_vT_-\Big)\widehat{\mathcal{D}}_{\overline{z}}T_-T_{\overline{v}}
\end{aligned}
$$

Clearly, for $q=1$ the operators in (27), (28) coincide with (16),(15), resp.

With this the final result for the q - Maxwell hierarchy of equations is:

$$
\begin{aligned}
{}_qI_n^+ \; {}_qF_n^+ &= {}_qJ_n \,, &\quad (29a)\\
{}_qI_n^- \; {}_qF_n^- &= {}_qJ_n &\quad (29b)
\end{aligned}
$$

Formulae (29) are part of a much more general classification scheme, cf. Figure 3, involving also other intertwining operators, and of arbitrary order. A subset of this scheme are two infinite two-parameter families of representations which are intertwined by the same operators (14). Explicitly, instead of (12) we have:

$$
\begin{aligned}
I^+_{n_1^+,n_2^+} &: C^+_{n_1^+,n_2^+} \longrightarrow C^{0+}_{n_1^+,n_2^+} \,, &\quad (30a)\\
I^+_{n_1^+,n_2^+} \circ T^+_{n_1^+,n_2^+} &= T^{0+}_{n_1^+,n_2^+} \circ I^+_{n_1^+,n_2^+} \,, \\
I^-_{n_1^-,n_2^-} &: C^-_{n_1^-,n_2^-} \longrightarrow C^{0-}_{n_1^-,n_2^-} \,, &\quad (30b)\\
I^-_{n_1^-,n_2^-} \circ T^-_{n_1^-,n_2^-} &= T^{0-}_{n_1^-,n_2^-} \circ I^-_{n_1^-,n_2^-}
\end{aligned}
$$

where $T^a_{n_1^\pm,n_2^\pm} = T^{\chi^a_{n_1^\pm,n_2^\pm}}$, $C^a_{n_1^\pm,n_2^\pm} = C^{\chi^a_{n_1^\pm,n_2^\pm}}$, $a = \pm$, or $a = 0\pm$, and

$$\begin{aligned}
\chi^+_{n_1^+,n_2^+} &= [n_1^+, n_2^+; \frac{n_1^+ - n_2^+}{2} + 1] \qquad \text{(31a)}\\
\chi^{0+}_{n_1^+,n_2^+} &= [n_1^+ - 1, n_2^+ + 1; \frac{n_1^+ - n_2^+}{2} + 2],\\
& \qquad n_1^+ \in I\!N,\ n_2^+ \in Z\!\!\!Z_+ \ ,\\
\chi^-_{n_1^-,n_2^-} &= [n_1^-, n_2^-; \frac{n_2^- - n_1^-}{2} + 1] \qquad \text{(31b)}\\
\chi^{0-}_{n_1^-,n_2^-} &= [n_1^- + 1, n_2^- - 1; \frac{n_2^- - n_1^-}{2} + 2],\\
& \qquad n_1^- \in Z\!\!\!Z_+,\ n_2^- \in I\!N
\end{aligned}$$

while instead of (13) in the $q = 1$ case and (29) in the q-deformed case, we have:

$$\begin{aligned}
{}_qI^+_{n_1^+}\ F^+_{n_1^+,n_2^+}(z,\bar z) &= J^+_{n_1^+,n_2^+}(z,\bar z), \qquad \text{(32a)}\\
{}_qI^-_{n_2^-}\ F^-_{n_1^-,n_2^-}(z,\bar z) &= J^-_{n_1^-,n_2^-}(z,\bar z) \qquad \text{(32b)}
\end{aligned}$$

where ${}_qI^+_{n_1^+}$, ${}_qI^-_{n_2^-}$, are given by (17), while $F^\pm_{n_1^\pm,n_2^\pm}(z,\bar z)$,

$J^{\pm}_{n_1^{\pm},n_2^{\pm}}(z,\bar{z})$, are polynomials in $z,\bar{z}$ of degrees $(n_1^{\pm},n_2^{\pm})$, $(n_1^{\pm}\mp 1, n_2^{\pm}\pm 1)$, resp.

The crucial feature which unifies these representations is the form of the operators ${}_qI_n^{\pm}$, which is not generalized anymore in equations (32).

We call the hierarchy of equations (32) the *generalized q - Maxwell hierarchy*. The q - Maxwell hierarchy is obtained in the partial case when $\chi^{0+}_{n_1^+,n_2^+} = \chi^{0-}_{n_1^-,n_2^-} = \chi^0_n$ which fixes three of the four parameters: $n_1^+ - 2 = n_2^+ = n_1^- = n_2^- - 2 = n$.

Another one parameter subhierarchy of the generalized q-Maxwell hierarchy involves the two signatures of $\chi_n^+ = [n+2, n; 2]$, $\chi_n^- = [n, n+2; 2]$, and in addition

$$\chi_n^{00} \ = \ [n+1, n+1; 1] \ , \qquad n \in \mathbb{Z}_+ \tag{33}$$

The intertwining relations are:

$$\begin{aligned} I^+_{n-1} \ &: \ C_n^{00} \longrightarrow C_n^- \ , & (34) \\ I^+_{n-1} \circ T_n^{00} \ &= \ T_n^- \circ I^+_{n-1} \, , & \\ I^-_{n-1} \ &: \ C_n^{00} \longrightarrow C_n^+ \, , & (35) \\ I^-_{n-1} \circ T_n^{00} \ &= \ T_n^+ \circ I^-_{n-1} & (36) \end{aligned}$$

where $T_n^{00} = T^{\chi_n^{00}}$, $C_n^{00} = C^{\chi_n^{00}}$. Here the equations are:

$$\begin{aligned} {}_qI^+_{n-1} \ {}_qA_n \quad &= \quad {}_qF_n^- \ , & (37a) \\ {}_qI^-_{n-1} \ {}_qA_n \quad &= \quad {}_qF_n^+ & (37b) \end{aligned}$$

where ${}_qI^{\pm}_n$ are given by (17), ${}_qA_n$ has the signature χ_n^{00}.

This hierarchy will be called the **potential q-Maxwell hierarchy** . The reason is that the lowest member obtained for $n = 0$ and $q = 1$ is just:

$$\partial_{[\mu} A_{\nu]} \; = \; F_{\mu\nu} \tag{38}$$

q - d'Alembert equations hierarchy

Next we consider another one parameter sub-hierarchy of the generalized q-Maxwell hierarchy which is obtained from (31) for $n_1^+ = n_2^- = r \in I\!N$, $n_1^- = n_2^+ = 0$, i.e.

$$\chi_r^{d+} = [r, 0; \frac{r}{2} + 1], \tag{39a}$$

$$\chi_r^{d0+} = [r-1, 1; \frac{r}{2} + 2], \quad r \in I\!N,$$

$$\chi_r^{d-} = [0, r; \frac{r}{2} + 1], \tag{39b}$$

$$\chi_r^{d0-} = [1, r-1; \frac{r}{2} + 2], \quad r \in I\!N,$$

where the two conjugated equations follow from (32):

$$_qI_r^+ \ F_r^{d+} = J_r^{d+} \ , \tag{40a}$$

$$_qI_r^- \ F_r^{d-} = J_r^{d-} \tag{40b}$$

where $_qI_r^\pm$ are given by (17).

For the minimal possible value of the parameter $r = 1$ we obtain the two conjugate *q - Weyl equations*.

The case $r = 2$ gives the q-Maxwell equations (note that $J_2^{d+} = J_2^{d-}$). This is the only intersection of the present hierarchy with the q-Maxwell hierarchy.

We call this hierarchy *q - d'Alembert hierarchy* following the classical case, (cf. [De,DI,DII]), due to the following. We consider the representations $\chi_a^{d\pm}$ for the excluded above value $r = 0$, when they coincide. Thus, we set: $\chi^d \equiv \chi_0^{d\pm} = [0,0;1]$, $F^d \equiv F_0^{d\pm}$. Furthermore, the relevant equation is the q-d'Alembert equation [De,DII]:

$$\Box_q \; F^d \; = \; J^d \tag{41}$$

where $\chi^J \; = \; [0,0;3]$,

$$\Box_q \; = \; \Big(\widehat{\mathcal{D}}_{\bar{v}}\widehat{\mathcal{D}}_v - q\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+ T_v T_{\bar{v}}\Big) T_v T_{\bar{v}} T_+ T_- \tag{42}$$

Weyl gravity equations hierarchy

Next we study another hierarchy which is given as follows:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & C_m^+ \arrow[dr] & \\
C_m^h \arrow[ur] \arrow[dr] & & C_m^T \\
 & C_m^- \arrow[ur] &
\end{tikzcd} \qquad (43)$$

where $m \in I\!N$, and the corresponding signatures are:

$$\begin{aligned}
\chi_m^+ &= [2m, 0; 2], \quad \chi_m^- = [0, 2m; 2], \qquad (44)\\
\chi_m^h &= [m, m; 2-m], \quad \chi_m^T = [m, m; 2+m]
\end{aligned}$$

The arrows on (43) represent invariant differential operators of order m. It is a partial case of the general conformal scheme parametrized by three natural numbers p, ν, n, (cf. Fig. 3), setting there: $\nu = 1$, $p = n = m$. This hierarchy intersects with the Maxwell hierarchy for the lowest value $m = 1$. Below we consider the linear conformal gravity which is obtained for $m = 2$.

Linear conformal gravity

Linear conformal gravity is governed by the *Weyl tensor* $C_{\mu\nu\sigma\tau}$ which is given in terms of the Riemann curvature tensor $R_{\mu\nu\sigma\tau}$, Ricci curvature tensor $R_{\mu\nu}$, scalar curvature R :

$$\begin{aligned} C_{\mu\nu\sigma\tau} &= R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} \; - \\ &- \; g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) \; + \\ &+ \; \tfrac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R \end{aligned} \tag{45}$$

where $g_{\mu\nu}$ is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, $h_{\mu\nu}$ are small so that all quadratic and higher order terms are neglected. In particular:

$$R_{\mu\nu\sigma\tau} = \tfrac{1}{2}(\partial_\mu\partial_\tau h_{\nu\sigma} + \partial_\nu\partial_\sigma h_{\mu\tau} - \partial_\mu\partial_\sigma h_{\nu\tau} - \partial_\nu\partial_\tau h_{\mu\sigma})$$

The equations of linear conformal gravity are:

$$\partial^\nu \partial^\tau C_{\mu\nu\sigma\tau} = T_{\mu\sigma} \qquad (46)$$

where $T_{\mu\nu}$ is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

$$\begin{aligned} &C_0 = C_{0123} \ , \quad C_1 = C_{2121} \ , \quad C_2 = C_{0202} \ , \\ &C_3 = C_{3012} \ , \quad C_4 = C_{2021} \ , \quad C_5 = C_{1012} \ , \\ &C_6 = C_{2023} \ , \quad C_7 = C_{3132} \ , \quad C_8 = C_{2123} \ , \\ &C_9 = C_{1213} \end{aligned} \qquad (47)$$

Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as $C^\pm$ (cf. (44) for $m = 2$). The tensors $T_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric and traceless with nine independent components.

Further, we shall use again the fact that a Lorentz irrep (spin-tensor) with signature (n_1, n_2)

may be represented by a polynomial $G(z,\bar{z})$ in $z,\bar{z}$ of order n_1, n_2, resp. More explicitly, for the Weyl gravity representations mentioned above we use:

$$
\begin{aligned}
C^{+}(z) &= z^4 C_4^{+} + z^3 C_3^{+} + z^2 C_2^{+} + z C_1^{+} + C_0^{+} \ , \\
C^{-}(\bar{z}) &= \bar{z}^4 C_4^{-} + \bar{z}^3 C_3^{-} + \bar{z}^2 C_2^{-} + \bar{z} C_1^{-} + C_0^{-} \ , \\
T(z,\bar{z}) &= z^2\bar{z}^2 T'_{22} + z^2\bar{z} T'_{21} + z^2 T'_{20} + \\
&\quad + z\bar{z}^2 T'_{12} + z\bar{z} T'_{11} + z T'_{10} + \\
&\quad + \bar{z}^2 T'_{02} + \bar{z} T'_{01} + T'_{00} \ , \\
h(z,\bar{z}) &= z^2\bar{z}^2 h'_{22} + z^2\bar{z} h'_{21} + z^2 h'_{20} + \\
&\quad + z\bar{z}^2 h'_{12} + z\bar{z} h'_{11} + z h'_{10} + \\
&\quad + \bar{z}^2 h'_{02} + \bar{z} h'_{01} + h'_{00}
\end{aligned}
\tag{48}
$$

The components $C_k^{\pm}$ are given in terms of the

Weyl tensor components as follows:

$$
\begin{aligned}
C_0^+ &= C_2 - \frac{1}{2}C_1 - C_6 + i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^+ &= 2(C_4 - C_8 + i(C_9 - C_5)) \\
C_2^+ &= 3(C_1 - iC_3) \\
C_3^+ &= 8(C_4 + C_8 + i(C_9 + C_5)) \\
C_4^+ &= C_2 - \frac{1}{2}C_1 + C_6 + i(C_0 + \frac{1}{2}C_3 - C_7) \\
C_0^- &= C_2 - \frac{1}{2}C_1 - C_6 - i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^- &= 2(C_4 - C_8 - i(C_9 - C_5)) \\
C_2^- &= 3(C_1 + iC_3) \\
C_3^- &= 2(C_4 + C_8 - i(C_9 + C_5)) \\
C_4^- &= C_2 - \frac{1}{2}C_1 + C_6 - i(C_0 + \frac{1}{2}C_3 - C_7)
\end{aligned}
\tag{49}
$$

while the components T'_{ij} are given in terms of

$T_{\mu\nu}$ as follows:

$$
\begin{aligned}
T'_{22} &= T_{00} + 2T_{03} + T_{33} \\
T'_{11} &= T_{00} - T_{33} \\
T'_{00} &= T_{00} - 2T_{03} + T_{33} \\
T'_{21} &= T_{01} + iT_{02} + T_{13} + iT_{23} \\
T'_{12} &= T_{01} - iT_{02} + T_{13} - iT_{23} \\
T'_{10} &= T_{01} + iT_{02} - T_{13} - iT_{23} \\
T'_{01} &= T_{01} - iT_{02} - T_{13} + iT_{23} \\
T'_{20} &= T_{11} + 2iT_{12} - T_{22} \\
T'_{02} &= T_{11} - 2iT_{12} - T_{22}
\end{aligned}
\tag{50}
$$

and similarly for h'_{ij} in terms of $h_{\mu\nu}$.

In these terms all linear conformal Weyl gravity equations (46) (cf. also (43)) may be written in compact form as the following pair of equations:

$$
I^+ \, C^+(z) \, = \, T(z,\bar{z}) \, , \qquad I^- \, C^-(\bar{z}) \, = \, T(z,\bar{z}) \tag{51}
$$

where the operators $I^{\pm}$ are given as follows:

$$
\begin{aligned}
I^{+} \;=\; & \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
& +2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
& +2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_- \Big)\partial_z^2 - \\
& -6\Big(z\bar{z}^2\partial_+^2 + z\partial_v^2 + 2z\bar{z}\partial_v\partial_+ + \bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_v\partial_- \Big)\partial_z + \\
& +12\Big(\bar{z}^2\partial_+^2 + \partial_v^2 + 2\bar{z}\partial_v\partial_+ \Big) \; , \qquad (52) \\
I^{-} \;=\; & \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
& +2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
& +2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
& +2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_- \Big)\partial_{\bar{z}}^2 - \\
& -6\Big(z^2\bar{z}\partial_+^2 + \bar{z}\partial_{\bar{v}}^2 + 2z\bar{z}\partial_+\partial_{\bar{v}} + z^2\partial_v\partial_+ + \\
& +z(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_-\partial_{\bar{v}} \Big)\partial_{\bar{z}} + \\
& +12\Big(z^2\partial_+^2 + \partial_{\bar{v}}^2 + 2z\partial_+\partial_{\bar{v}} \Big)
\end{aligned}
$$

To make more transparent the origin of (51) and in the same time to derive the quantum group deformation of (51), (52) we first introduce the following parameter-dependent operators:

$$\begin{aligned}
I_n^+ &= \frac{1}{2}\Big(n(n-1)I_1^2 I_2^2 - 2(n^2-1)I_1 I_2^2 I_1 \ + \\
&\quad + n(n+1)I_2^2 I_1^2\Big), \qquad (53) \\
I_n^- &= \frac{1}{2}\Big(n(n-1)I_3^2 I_2^2 - 2(n^2-1)I_3 I_2^2 I_3 \ + \\
&\quad + n(n+1)I_2^2 I_3^2\Big)
\end{aligned}$$

where $I_1 = \partial_z$, $I_2 = \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-$, $I_3 = \partial_{\bar{z}}$, are from (16). We recall that group-theoretically the operators I_a correspond to the three simple roots of the root system of $sl(4)$, while the operators $I_n^\pm$ correspond to the singular vectors for the two non-simple non-highest roots. More precisely, the operator I_n^+ is obtained from the $sl(4)$ formula for the singular vector of weight $m_{12}\alpha_{12} = 2\alpha_{12}$, while the operator I_n^- corresponds to weight $m_{23}\alpha_{23} = 2\alpha_{23}$. The parameter $n = \max(2j_1, 2j_2)$.

It is easy to check that we have the following relation:

$$I^{\pm} = I_4^{\pm} \tag{54}$$

i.e., (51) are written as:

$$I_4^+ \, C^+(z) \; = \; T(z,\bar{z}) \; , \qquad I_4^- \, C^-(\bar{z}) \; = \; T(z,\bar{z}) \tag{55}$$

Using the same operators we can write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

$$I_2^+ \, h(z,\bar{z}) \; = \; C^-(\bar{z}) \; , \qquad I_2^- \, h(z,\bar{z}) \; = \; C^+(z) \tag{56}$$

We stress again the advantage of the indexless formalism due to which two different pairs of equations, (55), (56), may be written using the same parameter-dependent operator expressions by just specializing the values of a parameter.

The above equations are is immediately generalizable to the deformed case.

Using the $U_q(sl(4))$ formula for the singular vector given in [Dh,DII] we obtain for the q-analogue of (53):

$$\begin{aligned} {}_qI_n^+ &= \frac{1}{2}\Big([n]_q\,[n-1]_q\ {}_qI_1^2\,{}_qI_2^2\ - \\ &\quad -\ [2]_q\,[n-1]_q\,[n+1]_q\,{}_qI_1\,{}_qI_2^2\,{}_qI_1 + \\ &\quad +[n]_q\,[n+1]_q\,{}_qI_2^2\,{}_qI_1^2 \Big)\ , \qquad (57) \\ {}_qI_n^- &= \frac{1}{2}\Big([n]_q\,[n-1]_q\,{}_qI_3^2\,{}_qI_2^2\ - \\ &\quad -\ [2]_q\,[n-1]_q\,[n+1]_q\,{}_qI_3\,{}_qI_2^2\,{}_qI_3 + \\ &\quad +[n]_q\,[n+1]_q\,{}_qI_2^2\,{}_qI_3^2 \Big) \end{aligned}$$

where the q-deformed ${}_qI_a$ were given above.

Then the q-Weyl gravity equations are (cf. (55)):

$$ {}_qI_4^+ \, C^+(z) \ = \ T(z,\bar{z}) \,, \qquad {}_qI_4^- \, C^-(\bar{z}) \ = \ T(z,\bar{z}) \tag{58} $$

while q-analogues of (56) are:

$$ {}_qI_2^+ \, h(z,\bar{z}) \ = \ C^-(\bar{z}) \,, \qquad {}_qI_2^- \, h(z,\bar{z}) \ = \ C^+(z) \tag{59} $$

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