Systematics of Constant Roll Inflation

Lilia Anguelova

INRNE, Bulgarian Academy of Sciences

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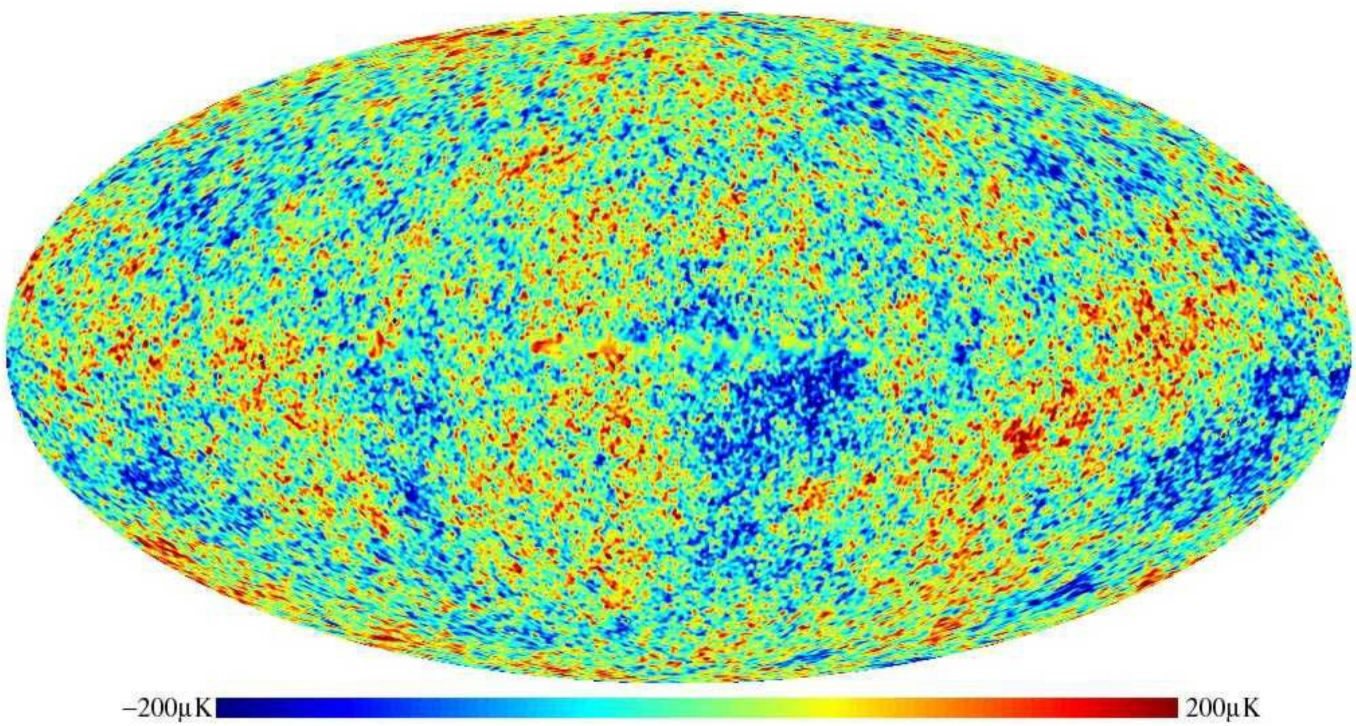
(with P. Suranyi, L.C.R. Wijewardhana)

Motivation

Cosmic Microwave Background (CMB) radiation:

WMAP (2003-2012) and Planck (2013) satellites:

Detailed map of CMB temperature fluctuations on the sky



$\overline{T} = 2.7\text{K}$

According to CMB data:

- On large scales:

  Universe is homogeneous and isotropic

- In Early Universe:

  Small perturbations that seed structure formation

  [(Clusters of) Galaxies]

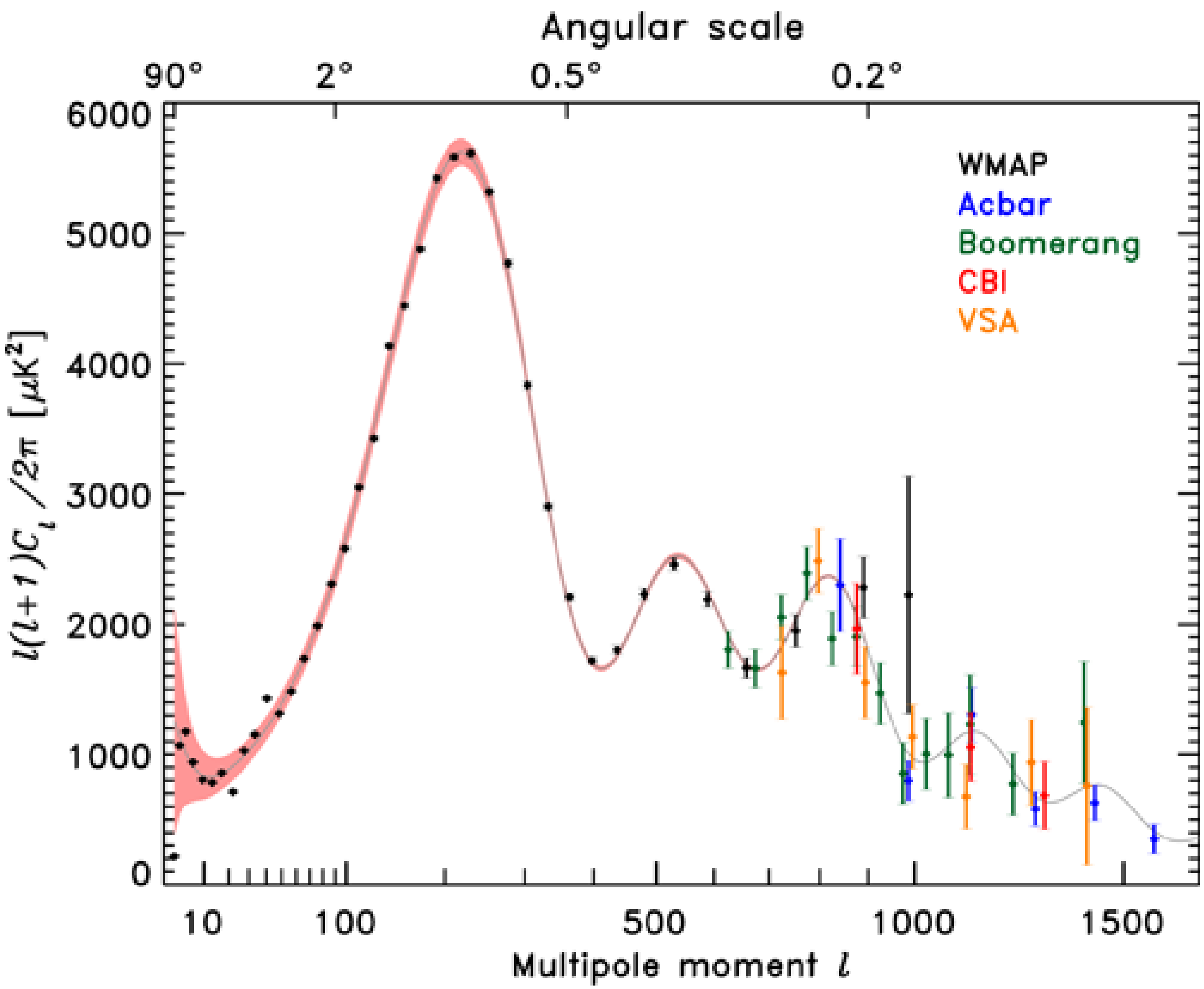
- Spectrum of temperature fluctuations:

  Expand: $$\frac{\delta T(\theta,\varphi)}{\overline{T}} = \sum_{l,m} a_{lm} Y_{lm}(\theta,\varphi) \ ,$$

  Y_{lm} - standard spherical harmonics

Rotationally invariant angular power spectrum:

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{l} |a_{lm}|^2$$



Cosmological inflation:

Period of very fast expansion of space in Early Universe

(faster than speed of light)

$\Rightarrow$ homogeneity and isotropy observed today

CMB power spectrum:

$\rightarrow$ Can extract values of cosmological observables

In particular, scalar spectral index: $n_s \approx 1$

$\Rightarrow$ Long thought that this requires

slow roll approximation

Will see that the slow roll approximation is not necessary !

Cosmological inflation:

Standard description:

– expansion driven by the potential energy of a scalar field φ, called inflaton, with action:

$$S = \int d^4x \sqrt{-\det g} \left[\frac{R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \, \partial_\nu\varphi - V(\varphi) \right]$$

– slow roll approximation:

$$\epsilon_{\rm v} = \frac{1}{2} \left[\frac{V'(\varphi)}{V(\varphi)} \right]^2 \ll 1 \quad , \quad \eta_{\rm v} = \frac{V''(\varphi)}{V(\varphi)} \ll 1$$

(almost exponential expansion)

Constant roll inflation

Constant roll regime: modification of ultra-slow roll

Ultra-slow roll inflation: (arXiv:gr-qc/0503017, W. Kinney)

$$\epsilon \ll 1 \qquad , \qquad \eta = 3$$

Gives $n_s = 1$ (i.e. scale-invariant spectrum), but does not last for more than a few e-foldings

Constant roll inflation: $\eta = const$

For some values of $\eta = const \neq 3$: stable expansion

(arXiv:1411.5021 [astro-ph.CO], H. Motohashi, A. Starobinsky, J. Yokoyama)

Goal: to study the constant roll regime systematically

General set-up:

Standard metric ansatz: $$ds_{4d}^2 = -dt^2 + a(t)^2 d\vec{x}^2$$

EoMs (Friedman equations):

$$3H^2 = \frac{\dot{\varphi}^2}{2} + V \qquad \text{and} \qquad 2\dot{H} = -\dot{\varphi}^2 \qquad ,$$

H - Hubble parameter , $$H(t) = \frac{\dot{a}(t)}{a(t)}$$

[Note: φ-EoM , $\ddot{\varphi} + 3H\dot{\varphi} + \partial_\varphi V = 0$, automatically satisfied for any solution of Friedman equations]

Inflationary slow roll parameters:

$$\epsilon = -\frac{\dot{H}}{H^2} \qquad \text{and} \qquad \eta = -\frac{\ddot{\varphi}}{H\dot{\varphi}}$$

General set-up:

Canonical definition: $[\, H = H(\varphi)\ ,\ H_\varphi \equiv dH/d\varphi \,]$

$$\epsilon = 2\left(\frac{H_\varphi}{H}\right)^2 \qquad \text{and} \qquad \eta = 2\,\frac{H_{\varphi\varphi}}{H}$$

EoM $2\dot{H} = -\dot{\varphi}^2$: $2H_\varphi = -\dot{\varphi}$

$$\Rightarrow \quad \epsilon = 2\left(\frac{H_\varphi}{H}\right)^2 = -\frac{\dot{H}}{H^2} \quad , \quad \eta = 2\,\frac{H_{\varphi\varphi}}{H} = -\,\frac{\ddot{\varphi}}{H\dot{\varphi}}$$

In slow roll approx.: (for all solutions of the EoMs)

$$\epsilon \approx \epsilon_{\rm v} \qquad , \qquad \eta \approx \eta_{\rm v} - \epsilon_{\rm v}$$

[Recall: $\epsilon_{\rm v} = \frac{1}{2}\left(\frac{V_\varphi}{V}\right)^2$ and $\eta_{\rm v} = \frac{V_{\varphi\varphi}}{V}$]

Constant roll inflation:

Rewrite the η-parameter, by using EoMs, as:

$$\eta = -\frac{\ddot{H}}{2H\dot{H}}$$

Then, constant roll condition becomes an ODE for $H(t)$:

$$-\frac{\ddot{H}}{2H\dot{H}} = c \qquad , \qquad c \equiv \mathrm{const}$$

$\rightarrow$ Can systematically find all solutions !

- recover the known ones
- find new solutions

Constant roll inflation:

Note: Finding the function $H(t)$ determines the inflationary model completely

Indeed, all other functions follow from $H(t)$:

- the scale factor $a(t)$ from solving $H = \dot{a}/a$
- the inflaton $\varphi(t)$ from $\varphi = \pm \int \sqrt{-2\dot{H}}\, dt$
- the scalar potential $V(\varphi)$ from inverting $\varphi(t)$ to obtain $t = t(\varphi)$ and substituting the result in EoM:

$$V = 3H^2 - \frac{\dot{\varphi}^2}{2}$$

General solution:

Unifying form of Hubble parameter:

$$H(t) = h\, \frac{k\, e^{hct} + e^{-hct}}{k\, e^{hct} - e^{-hct}} \quad ,$$

h, k - complex integration constants

Can show: above $H(t)$ real only for:

a) both h and k: real ; b) $h = i\mathbb{R}$ and $k = \pm e^{i\theta}$

Four real solutions, obtained for following (h, k) pairs:

$$(h \in \mathbb{R}\, ,\, k > 0)_{(1)} \quad , \quad (h \in \mathbb{R}\, ,\, k < 0)_{(2)} \quad ,$$
$$(h = i\mathbb{R}\, ,\, k = e^{i\theta})_{(3)} \quad , \quad (h = i\mathbb{R}\, ,\, k = -e^{i\theta})_{(4)}$$

Four real solutions:

$$H_{(1)}(t) \;=\; h\,\coth\left(hct + \frac{1}{2}\ln k\right) \;\text{ for }\; h \in \mathbb{R}\;,\; k > 0\;,$$

$$H_{(2)}(t) \;=\; h\,\tanh\left(hct + \frac{1}{2}\ln |k|\right) \;\text{ for }\; h \in \mathbb{R}\;,\; k < 0\;,$$

$$H_{(3)}(t) \;=\; -ih\,\cot\left(-ihct + \frac{\theta}{2}\right) \;\text{ for }\; h = i\mathbb{R}\;,\; k = e^{i\theta}\;,$$

$$H_{(4)}(t) \;=\; ih\,\tan\left(-ihct + \frac{\theta}{2}\right) \;\text{ for }\; h = i\mathbb{R}\;,\; k = -e^{i\theta}$$

$H_{(1)}$, $H_{(2)}$ - known (arXiv:1411.5021 [astro-ph.CO], H.M.,A.S.,J.Y.)

$H_{(3)}$, $H_{(4)}$ - new

Four real solutions:

Require real inflaton $\Rightarrow$ constraint on parameter space:

- sol. with $H_{(1)}$: $c > 0$
- sol. with $H_{(2)}$: $c < 0$

[$H_{(2)}$ solution: hilltop inflation]

- sol. with $H_{(3)}$: $c > 0$
- sol. with $H_{(4)}$: $c > 0$

Require $H > 0$ $\Rightarrow$ constraint on argument of H

$\rightarrow$ together with $c > 0$:

$\Rightarrow$ solutions (3) and (4) are equivalent

$\rightarrow$ consider (3) from now on

New solution:

Convenient to introduce: $N \equiv -ihc \in \mathbb{R}$

(set $\theta = 0$: no loss of generality)

Four-parameter family of solutions:

$$H_{(3)} = \frac{N}{c}\cot(Nt) \;\; , \;\; a_{(3)} = C_3^a \sin^{1/c}(Nt) \;\; ,$$
$$\varphi_{(3)} = \pm\sqrt{\frac{2}{c}}\ln\left[\cot\left(\frac{Nt}{2}\right)\right] + C_3^{\varphi} \;\; ,$$

$N\,,\; C_3^a\,,\; C_3^{\phi}$ - integration constants

Also: $t \in \left(0, \frac{\pi}{2N}\right)$, due to requirement that $H > 0$

Parameter space:

Note: t-interval can be made as large as needed by choosing suitably integration constant N

But Nt-interval: at most $\left(0, \frac{\pi}{2}\right)$, to ensure $H > 0$

Note: Nt-interval can be shortened at will, due to freedom to rescale N

Indeed: $N \to N_* \equiv \frac{2}{\pi}\theta_* N$, with some fixed $\theta_* < \frac{\pi}{2}$,

implies $Nt \in \left(0, \frac{\pi}{2}\right) \to N_* t \in (0, \theta_*)$

Parameter space:

Require $\ddot{a}(t) > 0$ (condition for inflation) $\Rightarrow \quad c < 1$

So, inflationary parameter space: $\quad 0 < c < 1$

In standard dS inflation: $\ddot{a}(t)$ - increasing

In present class of models:

- for $\frac{1}{2} < c < 1$, $\ddot{a}$: always decreasing
- for $0 < c < \frac{1}{2}$, $\ddot{a}$: (depends on N and c)
  - always increasing
  - first increasing, then decreasing

Inflaton potential:

Any more restrictions on parameter space from $V(\varphi)$?

From EoM $V = 3H^2 - \frac{1}{2}\dot{\varphi}^2$ upon using $t = t(\varphi)$:

$$V(\varphi) = \frac{N^2}{2c^2}\left[(3-c)\cosh\left(\sqrt{2c}\,(\varphi + \varphi_0)\right) - (3+c)\right] \;,$$

$$\varphi_0 \equiv C_3^{\varphi}$$

In principle, can choose φ_0 such that $V(\varphi)$ - positive

But, even for $\varphi_0 = 0$, can show that V - positive-definite in entire inflationary parameter space

$\rightarrow$ no new constraints from V

Scalar perturbations:

Perturbed inflaton and metric:

$$\begin{aligned} \varphi(t,\vec{x}) &= \bar{\varphi}(t) + \delta\varphi(t,\vec{x}) \ , \\ g_{\mu\nu}(t,\vec{x}) &= \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x}) \ , \end{aligned}$$

$\bar{\varphi}$, $\bar{g}_{\mu\nu}$ - classical background

Gauge transformations:

- time reparametrizations $[\,t \to t + \alpha\,]$
- spatial reparametrizations $[\,x^i \to x^i + \delta^{ij}\partial_j\beta \ , \ i = 1,2,3\,]$

$\to$ $\delta\varphi$ mixes with scalar degrees of freedom (d.o.f.) in $\delta g_{\mu\nu}$

Scalar perturbations:

One independent scalar d.o.f.: curvature perturbation ζ

In comoving gauge: $\delta\varphi = 0$, $\delta g_{ij} = a^2[(1-2\zeta)\delta_{ij} + h_{ij}]$

Fourier transform: $\zeta(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \zeta_k(t)\, e^{i\vec{k}.\vec{x}}$

Introduce $v_k \equiv \sqrt{2} z \zeta_k$ with $z^2 \equiv -a^2 \frac{\dot{H}}{H^2}$

$\rightarrow$ Mukhanov-Sasaki equation: [linearized EoMs]

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \ ,$$

$k \equiv |\vec{k}|$ and $' \equiv \partial_\tau$ with τ - conformal time [τ: $dt^2 = a^2 d\tau^2$]

Super-Hubble scales:

Inflationary model - stable, if there are no growing modes on super-Hubble scales

To verify that for our new class of models:

$\rightarrow$ Need to study Mukhanov-Sasaki equation in regime with $k^2 \ll z''/z$:

$$v_k'' - \frac{z''}{z} v_k = 0$$

General solution for ζ_k, implied by general solution for v_k:

$$\zeta_k = A_k + B_k \int \frac{H^2}{a^3 \dot{H}} \, dt \quad , \quad A_k, B_k = const$$

Super-Hubble scales:

Time-dependent part of ζ_k :

$$\int \frac{H^2}{a^3 \dot{H}} \, dt \; = \; \frac{\cos^3(Nt)}{3\,c\,N(C_3^a)^3} \;\; {}_2F_1\left(\frac{3}{2}\,,\frac{c+3}{2c}\,,\frac{5}{2}\,;\,\cos^2(Nt)\right)$$

Denote $x \equiv \cos^2(Nt)$, then functional dependence:

$$f(x) \; \equiv \; x^{\frac{3}{2}} \; {}_2F_1\left(\frac{3}{2}\,,\frac{c+3}{2c}\,,\frac{5}{2}\,;\,x\right) \; , \quad x \downarrow \; \text{as} \;\; t \uparrow$$

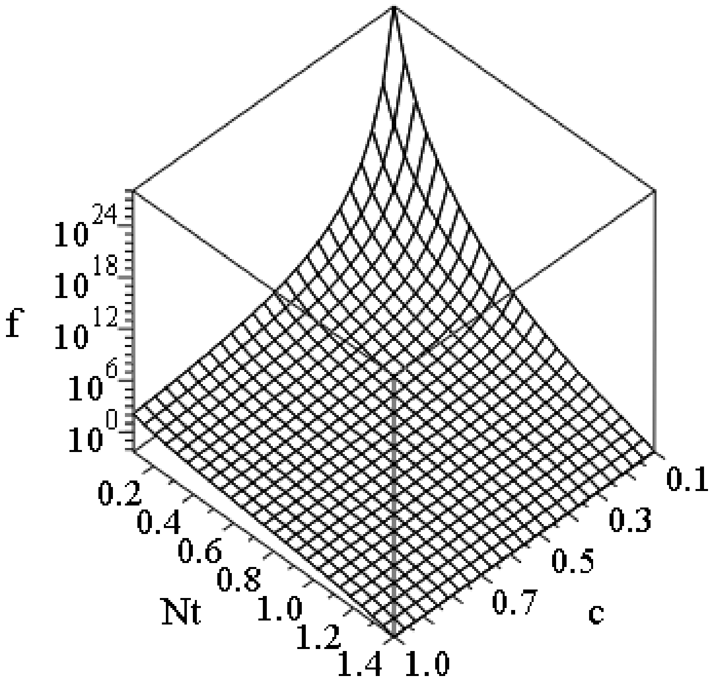
Recall: entire inflationary parameter space:

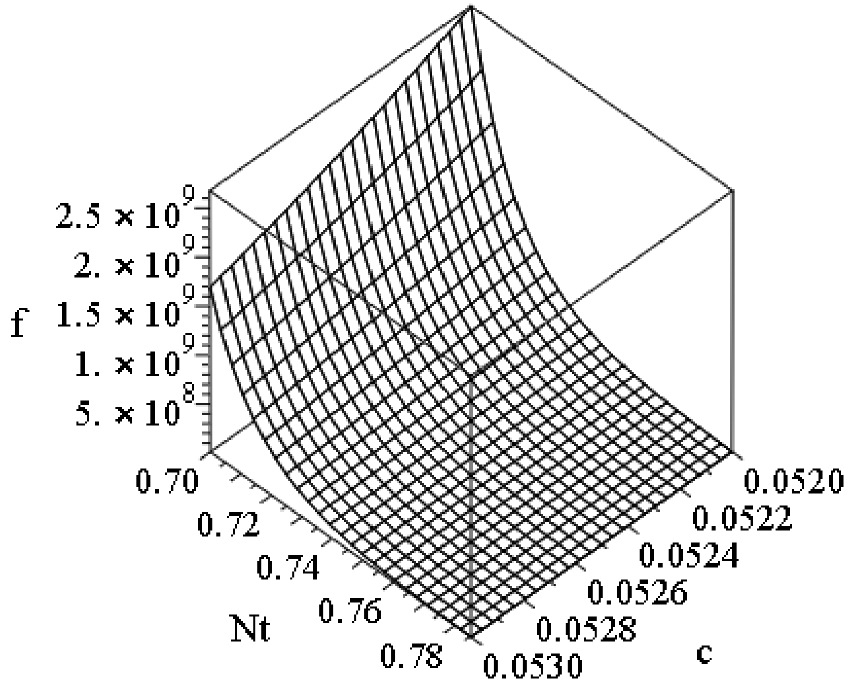
$$Nt \in \left(0, \tfrac{\pi}{2}\right) \quad \text{and} \quad c \in (0,1)$$

Showed: f - always decreasing with t $\;\Rightarrow\;$ stable expansion

Super-Hubble scales:

$f(Nt, c)$:





Inflationary parameter space:

$$Nt \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad c \in (0, 1)$$

Scalar spectral index:

From current observations: $n_s = 0.96$

Is our class of models consistent with this?

To compute n_s, need to solve Mukhanov-Sasaki equation with both k^2 and z''/z terms (around time of horizon crossing)

Conformal time $\tau = \int dt/a$:

$$\tau \,=\, -\frac{\cos(Nt)}{C_3^a N}\;{}_2F_1\!\left(\frac{1}{2}\,,\,\frac{c+1}{2c}\,,\,\frac{3}{2}\,;\,\cos^2(Nt)\right) + const$$

Can choose $const$ so that: $\tau \in (-\infty, 0]$

(just as in de Sitter case, i.e. with $H = const$)

Scalar spectral index:

Take standard initial condition:

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \qquad \text{for} \qquad \tau \to -\infty$$

Note: $$\frac{z''}{z} = a^2H^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3\right) \ ,$$

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} \quad , \quad \epsilon_{i+1} \equiv \frac{\dot{\epsilon}_i}{H\epsilon_i}$$

In general, need numerical methods to solve mode equation

But for $c \ll 1$:

- $\epsilon_1 \approx 2c \ , \ \epsilon_2 \approx 2c \ , \ \epsilon_3 \approx 4c$
- $aH \approx -1/\tau$

Scalar spectral index:

So solution for $c \ll 1$:

$$v_k(\tau) = \frac{\sqrt{\pi}}{2}\sqrt{-\tau}\; H_\nu^{(1)}(-k\tau) \quad , \quad \nu^2 \equiv \frac{9}{4} + c + 3c^2$$

Knowing $v_k \;\rightarrow\; \zeta_k \;\rightarrow\; \mathcal{P}_s(k)$ [scalar power spectrum]

Spectral index n_s: $\mathcal{P}_s(k) \sim k^{n_s - 1}$

$$\rightarrow \qquad n_s = 4 - 2\sqrt{\frac{9}{4} + c + 3c^2}$$

Impose $n_s = 0.96 \;\Rightarrow\; c = 0.052$ [2nd root < 0]

$\rightarrow$ New model - compatible with observations

Scalar spectral index:

Note:

In slow roll: all three ϵ_i - negligible in Mukhanov-Sasaki equation

In Starobinsky et al. class of solutions: $\epsilon_{1,3} \ll \epsilon_2$

(arXiv:1411.5021 [astro-ph.CO])

In our new class of solutions: all three ϵ_i - non-negligible and of the same order

$[\, \epsilon_i \sim \mathcal{O}(0.1) \,]$

$\rightarrow$ (More) Genuine deviation from slow roll !

Summary

Found so far:

- Studied systematically constant roll inflation
- Found new class of solutions
- Showed that they produce stable inflationary expansion

  [in entire parameter space]
- Showed that they give $n_s \approx 1$ [in part of parameter space]

Open issues:

- Compute (numerically) n_s for entire parameter space ...
- Other stable non-slow roll regimes ?...
- Constant roll in composite inflation models ?...

Thank you!