

Mathematical and physical aspects of field theories with defects

(Ingo Runkel, Univ. of Hamburg)

Outline

- defects and other observables
 - ↪ Kramers-Wannier duality
- properties and classification in 2d
 - ↪ reflection/transmission coeff.
- topological defects
 - ↪ dualities, orbifolds

Observables in QFT

- most common : localised at points $\langle \phi_1(p_1) \dots \phi_n(p_n) \rangle$

functional of some
fundamental field Ψ
e.g. $\partial/\partial x_i \Psi^\alpha$

space-time
point

- Gauge theories : localised on lines $\langle W_{\gamma_1}^{R_1} \dots W_{\gamma_n}^{R_n} \rangle$

loop in
space-time

repⁿ of
gauge group

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loop in
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repⁿ of
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- Both functionals on "fundamental field"
Other poss. : prescribed singularities at point, line, ...

↪ Intrinsic formulation of observables ?

Obs. localised on higher dim. submanifolds ?

Kramers - Wannier duality

Partition function

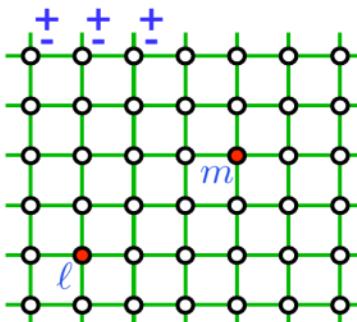
$$Z(\beta) = \sum_{\sigma} \exp\left(\beta \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j\right)$$

Correlation function

$$\langle \epsilon_e \epsilon_m \rangle_\beta = \frac{1}{Z(\beta)} \sum_{\sigma} \epsilon_e \epsilon_m \exp\left(\beta \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j\right)$$

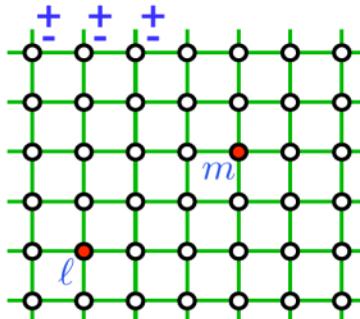
Critical point $\beta_c = \frac{1}{2} \ln(1 + \sqrt{2})$

Continuum limit is conformal field theory with $c = \frac{1}{2}$

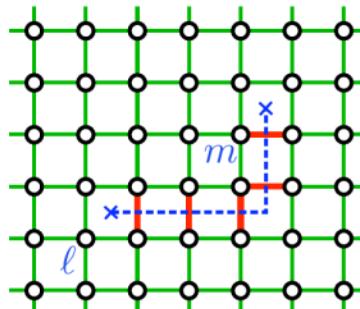


... Kramers-Wannier duality

$$\begin{aligned} & \langle \tilde{\epsilon}_e \tilde{\epsilon}_m \rangle_{\beta} \\ &= \frac{1}{Z(\beta)} \sum_{\sigma} \tilde{\epsilon}_e \tilde{\epsilon}_m \exp\left(\beta \sum_{\langle ij \rangle} \epsilon_i \epsilon_j\right) \end{aligned}$$



$$\begin{aligned} & \langle \mu_e \mu_m \rangle_{\tilde{\beta}} \\ &= \frac{1}{Z(\tilde{\beta})} \sum_{\sigma} 1 \cdot \exp\left(\tilde{\beta} \sum_{\langle ij \rangle} J_{ij} \epsilon_i \epsilon_j\right) \end{aligned}$$



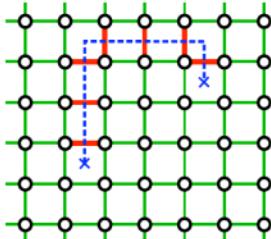
$\tilde{\beta} = \frac{1}{2} \ln(\tanh \beta)$ dual inverse temp.

J_{ij} coupling between sites i, j : $\bullet = +1$ $\circ = -1$

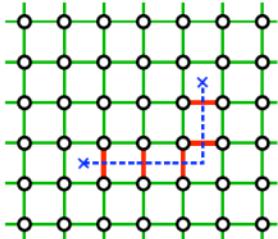
... Kramers-Wannier duality

Lessons

- Dual observable formulated in terms of boundary conditions on the lattice spin
- Dual obs. localized on a line
- Line can be deformed w/o changing expect. value.



=



Higher dim. observables / defects

k-dim. defect in n-dim. QFT :

k-dim. submanifold on which fundamental field can have discontinuities / singularities

k-dim. observable in n-dim. QFT :

k-dim. submf. on which a functional of the fund. field is localised

} same

Can try to

→ find examples

→ classify defects

use
sym.

Supersym.

conformal

diffeom.
...

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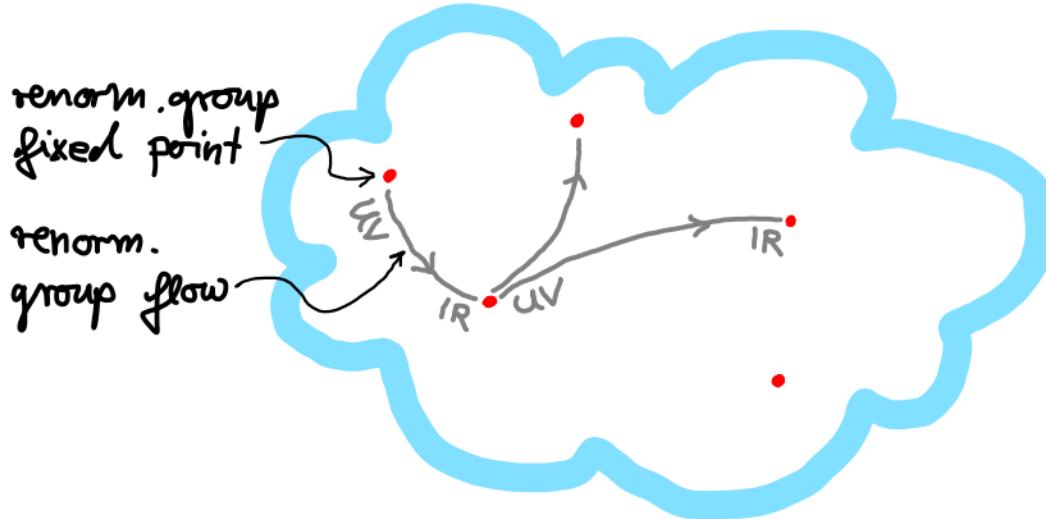
conformal (2d)

diffeom.

...

Universality

"space of n-dim. QFTs"

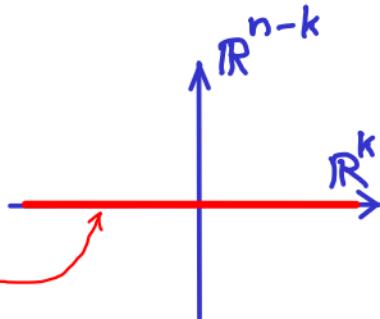


Fixed point QFTs : short / long range behaviour of QFTs,
↪ scale covariant, often **conformal**

... universality

Let C be a n -dim. CFT,
consider k -dim. defects in C .

(here : $n=2$, $k=1$)

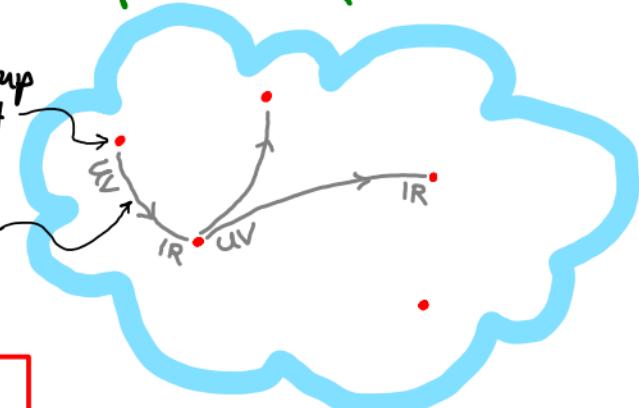


Same picture, but now : "space of defect cond. for
 k -dim defects in a fixed CFT C "

Fixed points :

scale inv. defect cond.
(often conf- inv.)

renorm. group
fixed point
renorm.
group flow



↪ Universality classes of
 k -dim. defects / obs. of C

Reflection and transmission

From here on : 1-dim. conf inv. defects in 2-dim. CFT

stress tensor T^{ab} , $a, b \in \{1, 2\}$

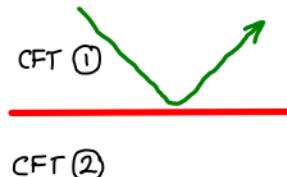
$$T^{11} + T^{22} = 0 \quad (\text{traceless})$$

$$T := \pi(T^{11} - iT^{12}) \quad \bar{T} := \pi(T^{11} + iT^{12})$$

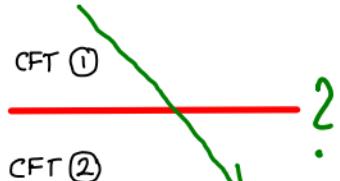
$$\frac{\partial}{\partial z} T = 0 = \frac{\partial}{\partial \bar{z}} T \quad (\text{conserved})$$

... reflection and transmission

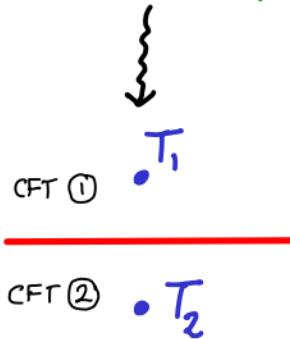
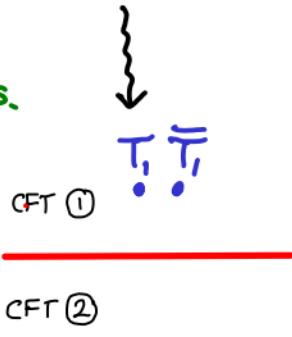
How to distinguish



vs.

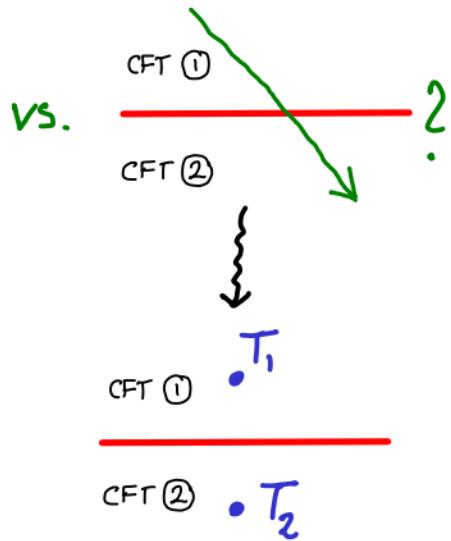
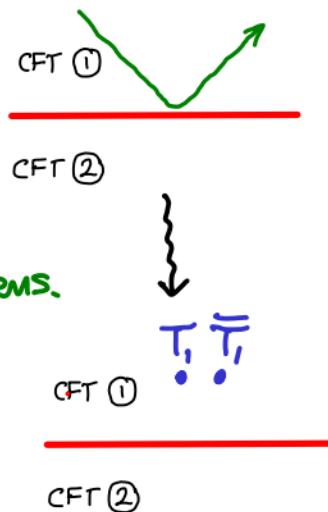


Idea: corr. of stress tens.



... reflection and transmission

How to distinguish



Idea: corr. of stress tens.

Want a number.

$$R = \frac{\langle T_1 \bar{T}_1 + T_2 \bar{T}_2 \rangle}{\langle (T_1 + \bar{T}_2)(\bar{T}_1 + T_2) \rangle}$$

reflection coeff.

$$R + T = 1$$

$$T = \frac{\langle T_1 T_2 + \bar{T}_1 \bar{T}_2 \rangle}{\langle (T_1 + \bar{T}_2)(\bar{T}_1 + T_2) \rangle}$$

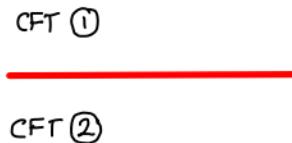
transmission coeff.

... reflection and transmission

Extremal cases

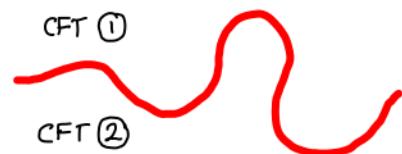
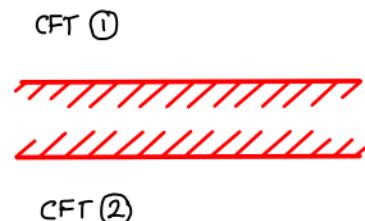
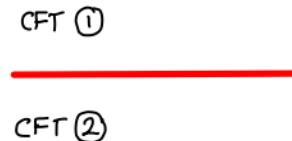
$$R=1, T=0:$$

e.g. conformal boundaries



$$R=0, T=1:$$

e.g. topological defects



... reflection and transmission

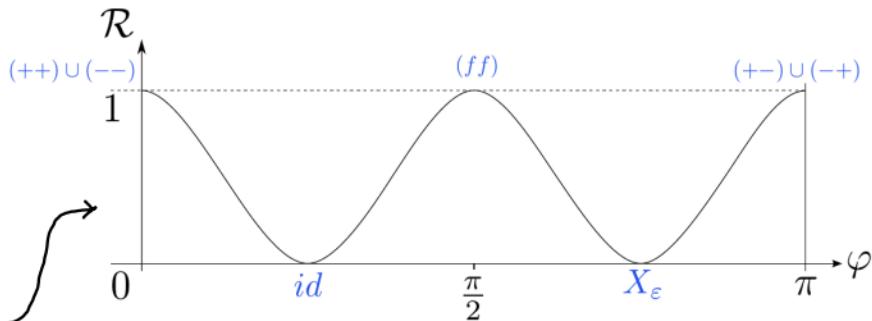
E.g. critical 2d Ising model

- class. of conf. def. via brd. cond. of $(\text{Ising}) \times (\text{Ising})$, a free boson orb.

Get

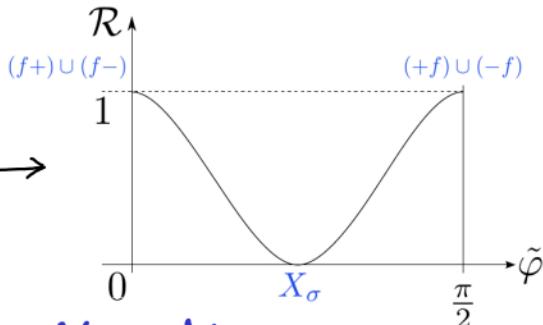
Dirichlet line

Neumann line



- 3 conf. brd. cond.:
 $+/-$: spin up/down
 f : free

- 3 topological defects: id , X_ϵ , X_σ

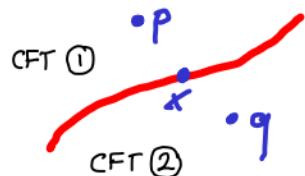


Topological defects

... are defects that are transparent to the stress tensor:

$$\lim_{p \rightarrow x} T_1(p) = \lim_{q \rightarrow x} T_2(q)$$

ditto for \bar{T}_1, \bar{T}_2



Topological defects can be deformed

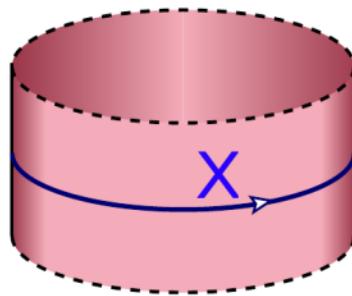
$$\left\langle \begin{array}{c} \phi_1(p_1) \\ \text{CFT ①} \\ \text{---} \\ \phi_2(p_2) \\ \cdot \psi_1(q_1) \end{array} \right\rangle = \left\langle \begin{array}{c} \phi_1(p_1) \\ \text{CFT ①} \\ \text{---} \\ \phi_2(p_2) \\ \cdot \psi_1(q_1) \\ \text{---} \\ \psi_2(q_2) \end{array} \right\rangle$$

Topological defects as operators

\mathcal{H} : state space of CFT on circle

Wrap top. def. X around cylinder
to get operator

$$D_X : \mathcal{H} \longrightarrow \mathcal{H}$$



Virasoro algebra:

$$L_m, \bar{L}_m : \mathcal{H} \rightarrow \mathcal{H} \quad - \text{Fourier modes of } T, \bar{T}$$

D_X satisfies

$$[L_m, D_X] = 0 = [\bar{L}_m, D_X]$$

Constraints from modular invariance

Petkova, Zuber '00

Compute torus partition function in two ways:

$$Z_x(R,L) = \left(\text{value of CFT on } \text{ (circle with } x \text{) } \right)$$

$$\text{tr}_X \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) = \text{tr}_X \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right)$$

$$= \text{tr}_{\mathcal{H}} D_x e^{-R H(L)}$$

$$= \text{fr}_{x_t} e^{-L H_x(R)}$$

Constraints from modular invariance

Compute torus partition function in two ways:

$$Z_x(R, L) = (\text{value of CFT on } \textcircled{O}_x)$$

$$\begin{aligned} & \text{tr}_{\mathcal{H}} \left(\text{Diagram of a rectangle with width } L \text{ and height } R \right) \\ & \quad \stackrel{=}{\swarrow} \qquad \stackrel{=}{\searrow} \\ & \text{tr}_{\mathcal{H}_X} \left(\text{Diagram of a rectangle with width } R \text{ and height } L \right) \end{aligned}$$

$$= \text{tr}_{\mathcal{H}} D_X e^{-R H(L)}$$

$$= \text{tr}_{\mathcal{H}_X} e^{-L H_X(R)}$$

green — known in CFT
w/o defects

red — not known

Constraints from modular invariance

Petkova, Zuber '00

Compute torus partition function in two ways:

$$Z_X(R, L) = (\text{value of CFT on } \textcircled{O}_X)$$

$$\text{tr}_{\mathcal{H}} \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) \quad \xleftarrow{=} \quad \text{tr}_{\mathcal{H}_X} \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right)$$

$\begin{matrix} R \\ L \end{matrix}$

$$\text{tr}_{\mathcal{H}_X} \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right) \quad \xleftarrow{=} \quad \text{tr}_{\mathcal{H}} \left(\begin{array}{c|c} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \right)$$

$\begin{matrix} L \\ R \end{matrix}$

$$= \text{tr}_{\mathcal{H}} D_X e^{-R H(L)}$$

$$= \text{tr}_{\mathcal{H}_X} e^{-L H_X(R)}$$

$$= \sum_{n=0}^{\infty} \underbrace{C_n}_{\in \mathbb{C}} e^{-R \frac{1}{L} \frac{1}{2\pi} E_n}$$

$$= \sum_{n=0}^{\infty} \underbrace{N_n}_{E \in \mathbb{Z}_{>0}} e^{-L \frac{1}{R} \frac{1}{2\pi} E_n}$$

... constraints from modular invariance

E.g. critical Ising model

$$\mathcal{H} = R_1 \otimes \bar{R}_1 \oplus R_6 \otimes \bar{R}_6 \oplus R_\varepsilon \otimes \bar{R}_\varepsilon$$


irreducible repⁿ of two copies of Virasoro algebra L_n, \bar{L}_n

Since D_x is intertwiner:

$$D_x = \lambda_1 \cdot \text{id}_{R_1 \otimes \bar{R}_1} + \lambda_6 \cdot \text{id}_{R_6 \otimes \bar{R}_6} + \lambda_\varepsilon \cdot \text{id}_{R_\varepsilon \otimes \bar{R}_\varepsilon}$$

↪ fixed up to $\lambda_1, \lambda_6, \lambda_\varepsilon \in \mathbb{C}$

Modular inv: 3 fundamental sol_y X_1, X_6, X_ε

Application : Kramers-Wannier duality at criticality

Recall : top. def. transparent to stress tensor

$$\langle \text{ } \overset{\bullet}{T(p)} \text{ } \rangle = \langle \text{ } \overset{\bullet}{T(p)} \text{ } \rangle$$

Application : Kramers-Wannier duality at criticality

Recall : top. def. transparent to stress tensor

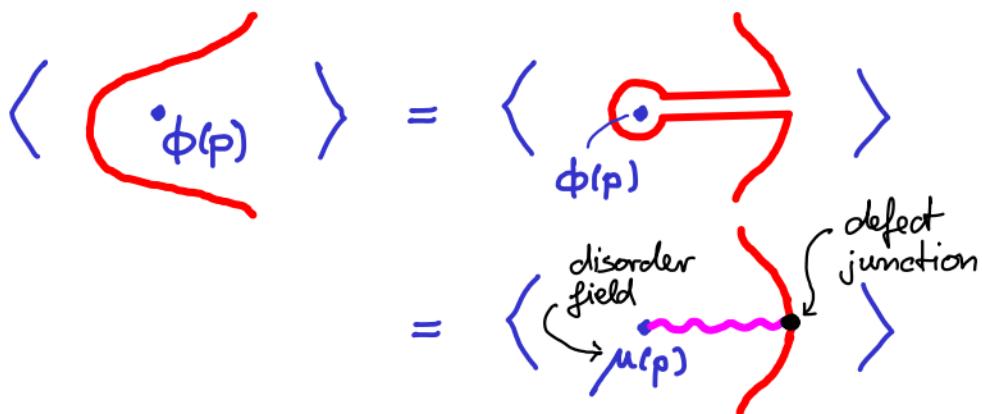
$$\langle \text{ } \overset{\circ}{T(p)} \text{ } \rangle = \langle \text{ } \overset{\circ}{T(p)} \text{ } \rangle$$

For a generic field one generates a disorder field :

$$\begin{aligned} \langle \text{ } \overset{\circ}{\phi(p)} \text{ } \rangle &= \langle \text{ } \overset{\circ}{\phi(p)} \text{ } \rangle \\ &= \langle \text{ } \overset{\circ}{\mu(p)} \text{ } \rangle \end{aligned}$$

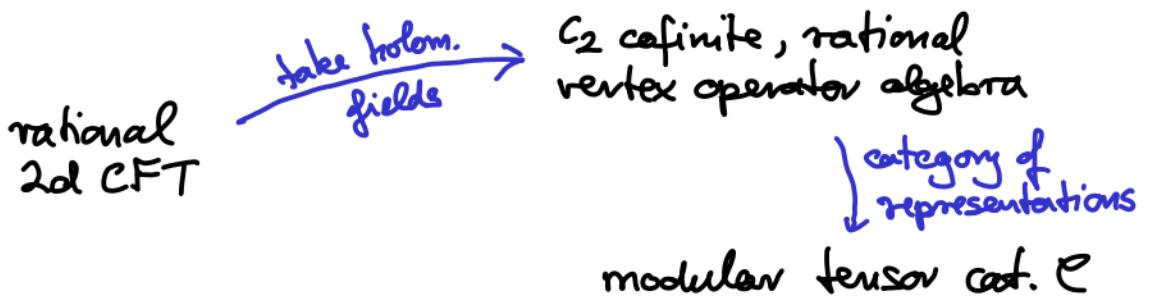
disorder field

defect junction



Aside. : 2d defect CFTs via 3d defect TFTs

Felder, Fröhlich, Fuchs, Schweigert '01
Fuchs, Schweigert, IR '02
Fröhlich, Fuchs, Schweigert, IR '06



correlators of CFT as boundary values of 3d TFT ← 3d TFT with defects evaluate TFT on bordism with 1 & 2 dim. defects

... application: Kramers-Wannier duality

In critical Ising model CFT one can show

$$\begin{array}{c} \text{Diagram: A curly line labeled } \check{\sigma}(x) \text{ with a dot labeled } \sigma \text{ at the bottom left.} \\ \text{Diagram: A curly line labeled } \mu(\check{x}) \text{ with a dot labeled } \sigma \text{ at the bottom right. A red dashed line labeled } \varepsilon \text{ connects the two dots.} \end{array} = \begin{array}{c} \text{Diagram: A curly line labeled } \check{\sigma}(x) \text{ with a dot labeled } \sigma \text{ at the bottom left.} \\ \text{Diagram: A curly line labeled } \mu(x) \text{ with a dot labeled } \sigma \text{ at the bottom right. A red dashed line labeled } \varepsilon \text{ connects the two dots.} \end{array}$$

... application: Kramers-Wannier duality

Fröhlich, Fuchs,
Schweigert, IR '04

In critical Ising model CFT one can show

$$\begin{array}{c} \text{---} \\ \text{---} \\ \ddot{\sigma}(x) \\ \sigma \end{array} = \mu(\ddot{x}) \frac{\varepsilon}{\sigma}$$

$$\dots \text{---} \ddot{\sigma}(x) = \frac{\varepsilon}{\sigma} \mu(x)$$

and compute
e.g. for a
4-pt. correlator
of spin fields
on the Sphere;

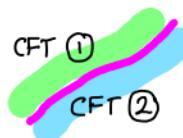
$$\begin{array}{c} \text{---} \\ \text{---} \\ \times_\sigma \quad \times_\sigma \\ \times_\sigma \quad \times_\sigma \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} \text{---} \\ \text{---} \\ \times_\sigma \quad \times_\sigma \\ \times_\sigma \quad \times_\sigma \end{array}$$

$$= \frac{1}{\sqrt{2}} \begin{array}{c} \text{---} \\ \text{---} \\ \mu^x \quad \times_\sigma \\ \mu^x \quad \times_\sigma \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \mu^x \quad \times_\mu \\ \mu^x \quad \times_\mu \end{array}$$

Application : generalised orbifolds

Fröhlich, Fuchs, Schweigert, IR '09
Carqueville, IR '12

Consider a topol. defect



s.th.

$$\langle \dots \rangle = \lambda \cdot \langle \dots \rangle, \lambda \in \mathbb{C}^*$$

The equation shows two configurations of regions. On the left, a green irregular shape contains a light blue circle labeled "CFT ②", which itself contains a smaller light blue circle labeled "CFT ①". On the right, a single green irregular shape contains both "CFT ①" and "CFT ②" regions. Brackets indicate that the correlator of the first configuration is proportional to the correlator of the second, with a factor λ .

Then correlators of CFT ① can be written as correlators of CFT ② with defects ...

... application : generalised orbifolds

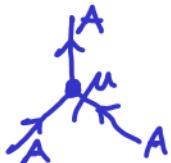
Then correlators of CFT ① can be written as correlators of CFT ② with defects:

$$\langle \text{CFT ①} \rangle = \frac{1}{\lambda^N} \langle \text{CFT ②} \text{ with defects} \rangle$$
$$= \frac{1}{\lambda^N} \langle \text{CFT ② with defects} \rangle = \langle \text{CFT ②} \text{ with defects} \rangle$$

Idea : turn this construction around

... application : generalised orbifolds

Given a CFT C , find a topol. defect A in C together with defect junctions



s.th.

1) crossing

An equation showing two configurations of dashed blue lines crossing each other. The left side shows a standard crossing. The right side shows a crossing where the lines are swapped. An equals sign connects the two sides.

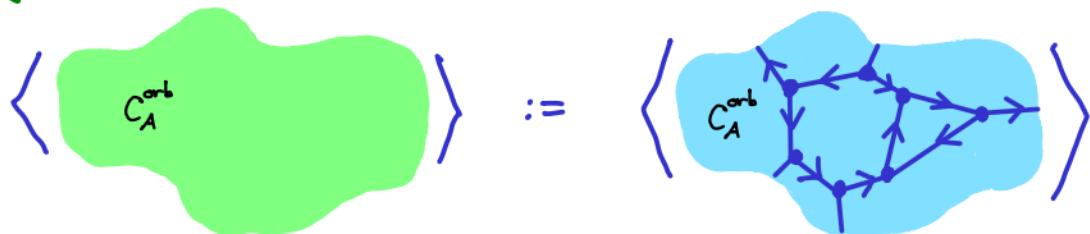
holds for any allowed way to add orientations

2) bubble

An equation showing a closed loop of dashed blue lines with an arrow pointing upwards, labeled 'u'. This is followed by an equals sign and a single vertical dashed blue line with an arrow pointing upwards.

... application : generalised orbifolds

Given A, μ, Δ in a CFT C as above,
get a new new CFT C_A^{orb}



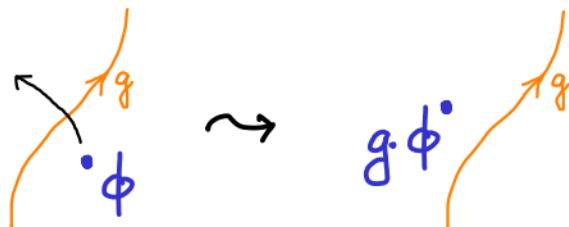
Using the 2dCFT/3dTFT formalism, one proves

Thm Any two rational CFTs with rational symmetry $V \otimes V$ and unique vacuum and non-deg. two-point corr. are generalised orbifolds of each other.

... application : generalised orbifolds

Relation to orbifold by group-symmetry ?

Topol. defects implementing a symmetry group G : for $g \in G$,



Take

A diagram showing a blue curved line with an arrow labeled A . To its right is an equals sign followed by a summation symbol \sum with $g \in G$ as the index. To the right of the summation symbol is another curved orange line with an arrow labeled g . This indicates that the total defect A is the sum of all defects $g \cdot A$ for $g \in G$.

Summary

- k -dim. defects = k -dim. observables
- universality classes : conformal defects in CFT
- reflection & transmission coefficients,
topological defects as totally transmissive defects
- Appl.: - Kramers-Wannier duality of critical Ising
- Generalised orbifolds via defects