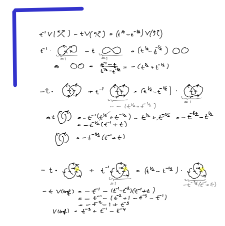
Mathematical and physical aspects of field theories with defects

(joint w. Nils Carqueville and Gregor Schaumann)

↪ TFT (mostly in 3d) on stratified manifolds and appl.

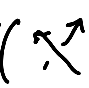
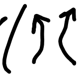
1) TFTs and ribbon graphs

The Jones polynomial (link) $\overset{V}{\longmapsto}$ (Laurent poly in $t^{1/2}$)

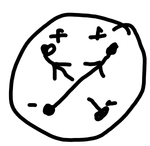
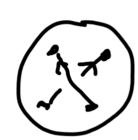
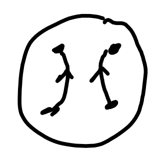


e.g. V() = 1

V() $= t^{-1} + t^{-3} - t^{-4}$

Defining property t^{-1} V() $- t$ V() $= (t^{1/2} - t^{-1/2})$ V()

"Cut out 3-ball"

" t^{-1}  $- t$  $= (t^{1/2} - t^{-1/2})$  "

Use 3d TFT to make sense of this

Bord_n : <u>obj</u> $n-1$ dim. compact closed oriented mf $U, V, \ldots$
<u>morph</u> n dim mf. with param. bnd

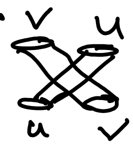
$$U \xrightarrow{M} V \quad : \quad \overline{U} \sqcup V \xrightarrow{\sim} \partial M$$

up to diffeo.

Def (Atiyah '88)

An n-dim topological quantum field theory is a symmetric monoidal functor $Z : \mathrm{Bord}_n \longrightarrow \mathrm{Vec}_k$

$Z(U \sqcup V) \simeq Z(U) \otimes Z(V)$ & dito for bord.



$\longmapsto$ flip $Z(U) \otimes Z(V) \longrightarrow Z(V) \otimes Z(U)$

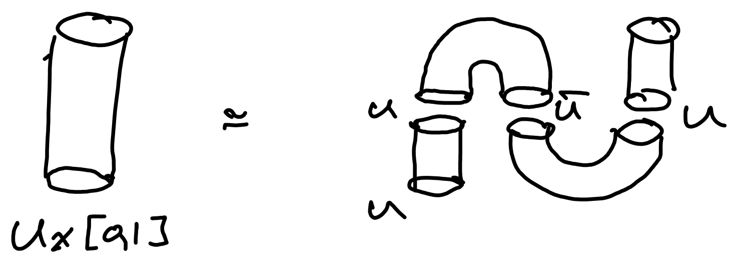
functor: $W \xleftarrow{N} V \xleftarrow{M} U$

$$Z(N \circ M) = Z(N) \circ Z(M)$$

glue along V — compose lin. maps

Lem: Z: n-dim TFT, $U \in \mathrm{Bord}_n$. Then $Z(U)$ is fin. dim.

Pf-sketch:



apply Z ; $Z(U)$ has non-deg. pairing & copairing. $\square$

Main examples in 3d: Reshetikhin-Turaev constr. '91, '94

Input: modular tensor cat. $\mathcal{C}$

e.g. • H: fin.-dim. quasi-triang. Hopf alg., which is

- s.si
- ribbon
- factorisable ($R_{21} R$ non-deg)

take $\mathcal{C} = \mathrm{Rep}\, H$.

• reps of vertex op. alg.

simple objects $U_1, \dots, U_N$

Output $Z_{\mathcal{C}} : \mathrm{Bord}_3 \longrightarrow \mathrm{Vec}_k$

e.g. $S^2 \longmapsto k$

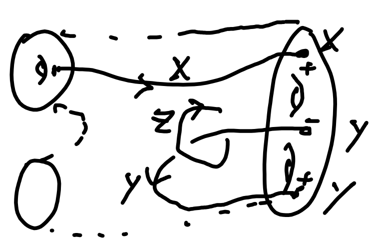
$T \longmapsto$ v.sp. of dim $= n$

but: can decorate obj by pts and bord by 1-dim submf.

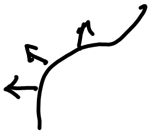
$$Z_{\mathcal{C}} : \mathrm{Bord}_3(\mathcal{C}) \longrightarrow \mathrm{Vec}_k$$

obj 

$X, Y, Z \in \mathcal{C}$

morph 

$\hookrightarrow$ invariants of tangles$^{(*)}$ in 3d bordisms$^{(**)}$

(*) framed tangles ←  "ribbons"

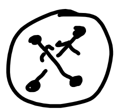
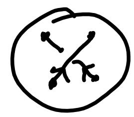
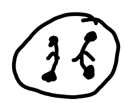
(**) extra data (gluing anomaly)

E.g. $Z_{\mathcal{C}}\left(S^2, (U+)(U+)(U-)(U-) \right)$

$= \mathrm{Hom}_{\mathcal{C}}(1, U \otimes U \otimes U^* \otimes U^*)$ ← dual obj

For fund. repⁿ $U = V_{1/2}$ of $U_q sl_2$: ↺ is 2-dim.

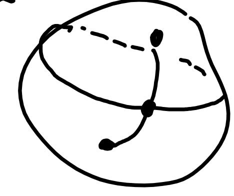
Thus

$Z_{\mathcal{C}}$ of  ,  ,  is lin. dep. in

Q: Include also coloured surfaces?

2) TFTs on stratified mflds

stratified mf:



$M = F_n \supset F_{n-1} \supset \cdots \supset F_0$ filtration

$M_i := F_i \setminus F_{i-1}$: i-dim submf, conn. comp. M_i^α

for $i<j$: either $M_i^\alpha \cap \overline{M_j^\beta} = \emptyset$ or $M_i^\alpha \subset \overline{M_j^\beta}$

$Bord_n^{strat}(\underbrace{D_n, \dots, D_0}_{\text{label sets}})$ morph: as in Bord + stratified, i-strata oriented, labeled by el. of D_i

(skip: compat. cond.) obj: dito, but i-strata labelled by D_{i+1}

Eg in RT above

$D_3 = \{\bullet\}$, $D_2 = \emptyset$, $D_1 = \{\text{obj of } \mathcal{C}\}$, $D_0 = \emptyset$

(skip: framing)

Extend RT to surfaces: Carqueville - Schaumann - IR '17

Kapustin - Saulina '10, Fuchs - Valentino - Schweigert '12

$D_2 = \{ \Delta\text{-separable Frobenius algebras in } \mathcal{C} \}$

- Δ-separable: $\mu \circ \Delta = id_A$
- Frobenius: coalg. $\Delta: A \to A \otimes A$, $\varepsilon: A \to \mathbb{1}$ and Δ is a A-bimodule hom.
- algebras: $A \in \mathcal{C}$, $\mu: A \otimes A \to A$, $\eta: \mathbb{1} \to A$ s.th ...

E.g. $\mathbb{1}$ is Δ-separable Frobenius, surface labelled by $\mathbb{1}$ can be omitted.

<u>Thm</u> There is an extension of RT–TFT $Z_{\mathcal{C}}$ to a TFT on

$$\mathrm{Bord}_3^{\mathrm{strat}}\left(D_3 = \{\bullet\},\ D_2 = \{\Delta\text{-sep}\},\ D_1 \supset \{\text{obj of } \mathcal{C}\},\ D_0 = \emptyset\right)$$

Comments

1) $\exists$ examples which distinguish 3 emb. $S^1 \times S^1 \longrightarrow S^1 \times S^1 \times S^1$

2) Only Morita-cl. of algebras relevant.

Difficult : classify Morita-cl. of Δ-sep. F.a. in $\mathcal{C}$

$\updownarrow$

classify $\mathcal{C}$-module categories $\mathcal{M}$