B-type Landau-Ginzburg models on open Riemann surfaces

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Outline

Definition

A *Landau-Ginzburg pair* is a pair (X, W) such that:

- X is a *non-compact* Kählerian manifold whose canonical line bundle K_X is holomorphically trivial.
- $W : X \to \mathbb{C}$ is a *non-constant* holomorphic function, called **superpotential**.

Let $d \stackrel{\text{def.}}{=} \dim_C X$.

Two-dimensional open-closed topological B-type Landau-Ginzburg models with D-branes can be associated to any Landau-Ginzburg pair. In general, such models are *not* scale invariant and hence they are B-type twists of two-dimensional $\mathcal{N} = 2$ supersymmetric field theories which are *not* conformally invariant (such supersymmetric field theories have a non-anomalous axial $\mathrm{U}(1)$ R-symmetry but they have no vector $\mathrm{U}(1)$ R-symmetry).

Definition

The *critical set of W* is the set of its critical points:

$$Z_W \stackrel{\text{def.}}{=} \{p \in X | (\partial W)(p) = 0\} .$$

Definition

The *signature* of a Landau-Ginzburg pair (X, W) is defined as the mod 2 reduction of the complex dimension of X:

$$\mu(X, W) \stackrel{\text{def.}}{=} \hat{d} \in \mathbb{Z}_2$$

Technical Assumption

We will assume that the critical set Z_W is compact. This insures finite-dimensionality of the (on-shell) closed and open topological string state spaces.

Remarks

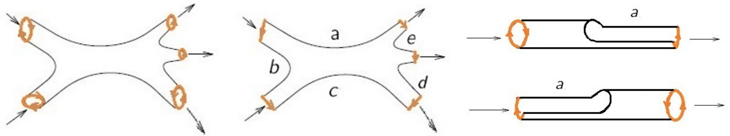
- *The class of all non-compact Kählerian manifolds is extremely large.*
- *A* **very special** *sub-class of Kählerian manifolds is provided by Stein manifolds.*
- *A* **very special** *subclass of Stein manifolds is provided by the analyticizations of non-singular complex affine varieties.*

The monoidal functor formalism

A non-anomalous oriented quantum 2-dimensional open-closed TFT (with finite-dimensional open and closed string state spaces) can be described axiomatically as a symmetric monoidal functor from a certain symmetric monoidal category Cob_2 of oriented cobordisms with corners to the symmetric monoidal category $\mathrm{vect}^s_{\mathbb{C}}$ of finite-dimensional supervector spaces over $\mathbb{C}$:

$$Z : (\mathrm{Cob}_2, \sqcup, \emptyset) \longrightarrow (\mathrm{vect}^s_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$$

The objects of Cob_2 are finite disjoint unions of oriented circles and oriented closed intervals. The morphisms are compact oriented 2-manifolds with corners (corresponding to the worldsheets of open and closed strings). The corners coincide with the boundary points of the intervals. The monoidal structure is given by the disjoint union, while the composition is the sewing of cobordisms.



The labels associated to the ends of the intervals indicate the corresponding boundary conditions (or the corresponding D-branes).

Theorem (C.I.L. 2000)

A (non-anomalous) oriented 2-dimensional open-closed TFT is equivalent to an algebraic structure known as a **TFT datum**.

To define a TFT datum, we first define the notions of **pre-TFT datum** and of **Calabi-Yau supercategory**.

Definition

A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

- The **bulk algebra** $\mathcal{H}$, which is a finite-dimensional supercommutative $\mathbb{C}$-superalgebra with unit $1_{\mathcal{H}}$. This describes the algebra of on-shell states of the closed oriented topological string.
- The **category of topological D-branes**, which is a Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category, with composition of morphisms denoted by $\circ$ and units:

  $$1_a \in \mathrm{Hom}_{\mathcal{T}}(a, a) \ , \ \forall a \in \mathrm{Ob}\mathcal{T}$$

  The objects of $\mathcal{T}$ describe the topological D-branes, while $\mathrm{Hom}_{\mathcal{T}}(a, b)$ describes the space of on-shell boundary states of the *open* oriented topological string stretching from the D-brane a to the D-brane b.
- A family $e = (e_a)_{a\in\mathrm{Ob}\mathcal{T}}$ of $\mathbb{C}$-linear **bulk-boundary maps** $e_a : \mathcal{H} \to \mathrm{Hom}_{\mathcal{T}}(a, a)$ such that the following conditions are satisfied:
  - For any $a \in \mathrm{Ob}\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the algebra $(\mathrm{End}_{\mathcal{T}}(a), \circ)$, where $\mathrm{End}_{\mathcal{T}}(a) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathcal{T}}(a, a)$.
  - For any $a, b \in \mathrm{Ob}\mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous bulk state $h \in \mathcal{H}$ and any $\mathbb{Z}_2$-homogeneous elements $t \in \mathrm{Hom}_{\mathcal{T}}(a, b)$, we have:

  $$e_b(h) \circ t = (-1)^{\deg h \deg t}\, t \circ e_a(h).$$

Definition

A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in Ob\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-degree μ

$$\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$$

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$, the $\mathbb{C}$-bilinear pairing:

$$\langle \cdot, \cdot \rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a, b) \times \mathrm{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$$

  defined through:

$$\langle t_1, t_2 \rangle_{a,b} = \mathrm{tr}_b(t_1 \circ t_2), \ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b), \ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$$

  is non-degenerate.

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$, we have:

$$\langle t_1, t_2 \rangle_{a,b} = (-1)^{\mathrm{deg} t_1 \mathrm{deg} t_2} \langle t_2, t_1 \rangle_{b,a}$$

Definition

A TFT datum of parity $\mu \in \mathbb{Z}_2$ is a system $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a **pre-TFT datum**
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the **bulk trace**)
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called **boundary traces**)

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra, meaning that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate (i.e. the condition $\mathrm{Tr}(hh') = 0$ for all $h' \in \mathcal{H}$ implies $h = 0$)
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi-Yau supercategory of parity μ.
- The so-called *topological Cardy constraint* holds for all $a, b \in \mathrm{Ob}\mathcal{T}$.

The topological Cardy constraint has the form:

$$\boxed{\mathrm{Tr}(f_a(t_a)f_b(t_b)) = \mathrm{str}(\Phi_{ab}(t_a,t_b)) \ \ , \ \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)}$$

where:

- "str" is the supertrace on the $\mathbb{Z}_2$-graded vector space $End_{\mathbb{C}}(Hom_{\mathcal{T}}(a,b))$
- $f_a : \mathrm{Hom}_{\mathcal{T}}(a,a) \to \mathcal{H}$ is the *boundary-bulk map of a*, which is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to Hom_{\mathcal{T}}(a,a)$ with respect to Tr and tr:

$$\mathrm{Tr}(hf_a(t_a)) = \mathrm{tr}_a(e_a(h)\circ t_a), \ \forall h \in \mathcal{H}, \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a)$$

- $\Phi_{ab}(t_a,t_b) : \mathrm{Hom}_{\mathcal{T}}(a,b) \to \mathrm{Hom}_{\mathcal{T}}(a,b)$ is the $\mathbb{C}$-linear map defined through:

$$\Phi_{ab}(t_a,t_b)(t) = t_b \circ t \circ t_a \ , \ \forall t \in \mathrm{Hom}_{\mathcal{T}}(a,b) \ , \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)$$

In our previous work, we made a mathematically rigorous proposal for the TFT datum associated to a general Landau-Ginzburg pair. The proposal is inspired by path integral arguments.

For any Landau-Ginzburg pair (X, W), we proved that our proposal satisfies all axioms of a TFT datum (including the non-degeneracy of bulk and boundary traces), except for the topological Cardy constraint (whose proof in full generality is work in progress).

Modulo the proof of the topological Cardy constraint, our proposal amounts to a rigorous construction of the quantum oriented open-closed B-type Landau-Ginzburg 2-dimensional topological field theory in such extreme generality.

Remark

According to our proposal, the TFT datum of the B-type Landau-Ginzburg theory associated to a Landau-Ginzburg pair (X, W) is not only $\mathbb{C}$-linear but also "partially $\mathrm{O}(X)$-linear", where $\mathrm{O}(X)$ is the unital commutative ring of complex-valued holomorphic functions defined on X. Namely:

- *$\mathcal{H}$ is an $\mathrm{O}(X)$-module (which is finite-dimensional over $\mathbb{C} \subset \mathrm{O}(X)$).*
- *$\mathcal{T}$ is an $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded category (which is Hom-finite over $\mathbb{C}$).*
- *The maps e_a and f_a are $\mathrm{O}(X)$-linear.*
- *However, the bulk and boundary traces Tr and tr_a are only $\mathbb{C}$-linear.*

Stein manifolds

Definition

Let X be a complex manifold. We say that X is a **Stein** if it admits a holomorphic embedding as a closed complex submanifold of $\mathbb{C}^N$ for some N.

Remarks

- *There exist numerous equivalent definitions of Stein manifolds.*
- *Any Stein manifold is Kählerian.*
- *The analyticization of any non-singular complex affine variety is Stein, but the vast majority of Stein manifolds are* **not** *of this type.*

Examples

- $\mathbb{C}^d$ is a Stein manifold.
- Every domain of holomorphy in $\mathbb{C}^d$ is a Stein manifold
- Every closed complex submanifold of a Stein manifold is a Stein manifold.
- Any non-singular analytic complete intersection in $\mathbb{C}^N$ is a Stein manifold.
- Any (non-singular) open Riemann surface without border is a Stein manifold.

Theorem (Cartan's theorem B)

For every coherent analytic sheaf $\mathcal{F}$ on a Stein manifold X, the sheaf cohomology $\mathrm{H}^i(X, \mathcal{F})$ vanishes for all $i > 0$.

Since Stein manifolds are Kählerian, we can consider B-type Landau-Ginzburg models with Stein manifold target X. In this case, our model for the TFT datum simplifies.

Let (X, W) be a Landau-Ginzburg pair and:

- $\mathcal{O}_X$ be the sheaf of complex-valued holomorphic functions on X and
- $\mathrm{O}(X) \stackrel{\text{def.}}{=} \Gamma(X, \mathcal{O}_X)$ be the ring of globally-defined holomorphic functions.
- $\boldsymbol{\iota}_W \stackrel{\text{def.}}{=} -\mathrm{i}(\partial W)\lrcorner : TX \to \mathcal{O}_X$ the morphism of sheaves of $\mathcal{O}_X$-modules given by left contraction with $-\mathrm{i}\partial W$.

Definition

- The critical sheaf $\mathcal{J}_W \stackrel{\text{def.}}{=} \mathrm{im}(\boldsymbol{\iota}_W : TX \to \mathcal{O}_X)$.
- The Jacobi sheaf $Jac_W \stackrel{\text{def.}}{=} \mathcal{O}_X/\mathcal{J}_W$.
- The Jacobi algebra $\mathrm{Jac}(X, W) \stackrel{\text{def.}}{=} \Gamma(X, Jac_W)$, which is a unital and commutative $\mathrm{O}(X)$-algebra.
- The critical ideal $\mathrm{J}(X, W) \stackrel{\text{def.}}{=} \mathcal{J}_W(X) = \boldsymbol{\iota}_W(\Gamma(X, TX)) \subset \mathrm{O}(X)$.

Theorem

Let (X, W) be a Landau-Ginzburg pair such that X is a Stein manifold of complex dimension d and suppose that Z_W is compact. Then the following statements hold:

1. *The critical locus Z_W is necessarily finite.*
2. *The bulk algebra is concentrated in even degree and can be identified with the Jacobi algebra:*
$$\mathcal{H} \equiv_{\mathrm{O}(X)} \mathrm{Jac}(X, W) \quad .$$
Moreover, we have an isomorphism of $\mathrm{O}(X)$-algebras:
$$\mathrm{Jac}(X, W) \simeq \mathrm{O}(X)/\mathrm{J}(X, W) \quad .$$
3. *If X is holomorphically parallelizable (i.e. if TX is holomorphically trivial), then:*
$$\mathrm{J}(X, W) = \langle u_1(W), \ldots, u_d(W) \rangle \quad ,$$
where $u_1, \ldots, u_d$ is any global holomorphic frame of TX.

Let (X, W) be a Stein LG pair with finite critical set Z_W. For any $p \in Z_W$, let:

$$\mathrm{M}(\hat{W}_p) \stackrel{\text{def.}}{=} \frac{\mathcal{O}_{X,p}}{\langle \partial_1 \hat{W}_p, \ldots, \partial_d \hat{W}_p \rangle}$$

denote the **analytic** Milnor algebra of the analytic function germ $\hat{W}_p$ of W at p. The map $\mathrm{germ}_p : \mathrm{O}(X) \to \mathcal{O}_{X,p}$ induces a well-defined morphism of $\mathrm{O}(X)$-algebras:

$$\Lambda_p : \mathrm{Jac}(X, W) \to \mathrm{M}(\hat{W}_p) \quad .$$

Proposition

The map:

$$\Lambda \stackrel{\text{def.}}{=} \oplus_{p \in Z_W} \Lambda_p : \mathrm{Jac}(X, W) \to \oplus_{p \in Z_W} \mathrm{M}(\hat{W}_p)$$

is an isomorphism of $\mathrm{O}(X)$*-algebras.*

This allows us to identify the bulk algebra with the direct sum of analytic Milnor algebras of W at its critical points.

Let (X, W) be a Landau-Ginzburg pair.

Definition

The holomorphic dg-category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category $\mathrm{F}(X, W)$ defined as follows:

- The objects are the holomorphic factorizations of W.
- $\mathrm{Hom}_{\mathrm{F}(X,W)}(a_1, a_2) \stackrel{\text{def.}}{=} \Gamma(X, Hom(E_1, E_2))$, endowed with the $\mathbb{Z}_2$-grading:

$$\mathrm{Hom}^{\kappa}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom^{\kappa}(E_1, E_2)) \ , \ \forall \kappa \in \mathbb{Z}_2$$

  and with the differentials $\mathfrak{d}_{a_1,a_2}$ determined uniquely by the condition:

$$\mathfrak{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^{\kappa} f \circ D_1 \ , \ \forall f \in \Gamma(X, Hom^{\kappa}(E_1, E_2)) \ , \ \forall \kappa \in \mathbb{Z}_2$$

- The composition of morphisms is the obvious one.

Let $\mathrm{HF}(X, W) \stackrel{\text{def.}}{=} \mathrm{H}(F(X, W))$ be the total cohomology category of the dg category $\mathrm{F}(X, W)$.

The topological D-brane category in the Stein case

Definition

A projective analytic factorization of W is a pair (P, D), where P is a finitely-generated projective $\mathrm{O}(X)$-supermodule and $D \in \mathrm{End}^1_{\mathrm{O}(X)}(P)$ is an odd endomorphism of P such that $D^2 = W \mathrm{id}_P$.

Definition

The dg-category $\mathrm{PF}(X, W)$ of projective analytic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category defined as follows:

- The objects are the projective analytic factorizations of W.
- $\mathrm{Hom}_{\mathrm{PF}(X,W)}((P_1, D_1), (P_2, D_2)) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$, endowed with the obvious $\mathbb{Z}_2$-grading and with the $\mathrm{O}(X)$-linear odd differential $\mathfrak{d} := \mathfrak{d}_{(P_1,D_1),(P_2,D_2)}$ determined uniquely by the condition:

$$\mathfrak{d}(f) = D_2 \circ f - (-1)^{\mathrm{deg} f} f \circ D_1 f$$

  for all $\mathbb{Z}_2$-homogeneous $f \in \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$.
- The composition of morphisms is the obvious one.

Let $\mathrm{HPF}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{PF}(X, W))$ be the total cohomology category of the dg category $\mathrm{PF}(X, W)$.

For any unital commutative ring R, let $\mathrm{MF}(R, W)$ denote category of finite rank matrix factorizations of W over R and $\mathrm{HMF}(R, W)$ denote its total cohomology category (which is $\mathbb{Z}_2$-graded and R-linear).

Theorem

Let (X, W) be a Landau-Ginzburg pair such that X is a Stein manifold. Then:

1. *There exists a natural equivalence of $\mathrm{O}(X)$-linear and $\mathbb{Z}_2$-graded dg categories:*
$$\mathrm{F}(X, W) \simeq_{\mathrm{O}(X)} \mathrm{PF}(X, W)$$
2. *If the critical locus Z_W is finite, then the topological D-brane category $\mathcal{T}$ is given by:*
$$\mathcal{T} \equiv \mathrm{HF}(X, W) \simeq_{\mathrm{O}(X)} \mathrm{HPF}(X, W) \quad .$$
3. *The critical ideal $\mathrm{J}(X, W)$ acts trivially on $\mathrm{HF}(X, W)$, so $\mathcal{T}$ can be viewed as a $\mathbb{Z}_2$-graded $\mathrm{Jac}(X, W)$-linear category.*
4. *The even subcategory $\mathcal{T}^{\hat{0}}$ has a natural triangulated structure.*
5. *There exists a natural $\mathrm{O}(X)$-linear dg-functor $\Xi : \mathrm{F}(X, W) \to \oplus_{p \in Z_W} \mathrm{MF}(\mathcal{O}_{X,p}, \hat{W}_p)$ which induces a full and faithful $\mathrm{Jac}(X, W)$-linear functor $\Xi_* : \mathrm{HF}(X, W) \to \oplus_{p \in Z_W} \mathrm{HMF}(\mathcal{O}_{X,p}, \hat{W}_p)$.*

The remaining objects of the TFT datum are as follows, where $d = \dim_{\mathbb{C}} X$:

- The bulk trace is given by:

$$\mathrm{Tr}(f) = \sum_{p \in Z_W} A_p \, \mathrm{Res}_p \left[\frac{\hat{f}_p \hat{\Omega}_p}{\det_{\hat{\Omega}_p}(\partial W)} \right] \quad ,$$

- The boundary trace of the D-brane (holomorphic factorization) $a = (E, D)$ is given by the sum of *generalized Kapustin-Li traces*:

$$\mathrm{tr}_a(s) = \frac{(-1)^{\frac{d(d-1)}{2}}}{d!} \sum_{p \in Z_W} A_p \, \mathrm{Res}_p \left[\frac{\mathrm{str}\left(\det_{\hat{\Omega}_p}(\partial \hat{D}_p) \hat{s}_p \right) \hat{\Omega}_p}{\det_{\hat{\Omega}_p}(\partial \hat{W}_p)} \right] \quad .$$

  Here Ω is a holomorphic volume form on X, A_p are normalization constants and Res_p denotes the Grothendieck residue on $\mathcal{O}_{X,p}$.

- The bulk-boundary and boundary-bulk maps of $a = (E, D)$ are given by:

$$e_a(f) \equiv \mathrm{i}^d (-1)^{\frac{d(d-1)}{2}} \bigoplus_{p \in Z_W} \hat{f}_p \mathrm{id}_{E_p} \quad , \forall f \in \mathcal{H} \equiv \mathrm{Jac}(X, W)$$

$$f_a(s) \equiv \frac{\mathrm{i}^d}{d!} \bigoplus_{p \in Z_W} \mathrm{str}\left(\det_{\hat{\Omega}_p}(\partial \hat{D}_p) \, \hat{s}_p \right) \quad , \quad \forall s \in \mathrm{End}_{\mathcal{T}}(a) \equiv \Gamma(X, End(E)) \quad .$$

Definition

A *Riemann surface* (in the sense of Weyl-Radó) is a borderless connected complex (Haussdorff) manifold Σ such that $\dim_{\mathbb{C}} \Sigma = 1$.

Theorem (Radó)

The following mutually-equivalent statements hold for any Riemann surface Σ:

- Σ *is paracompact.*
- Σ *has a countable basis.*
- Σ *is countable at infinity.*

Definition

A Riemann surface is called *open* if it is non-compact.

Unlike the compact case, the topological classification of open Riemann surfaces is quite involved, since such a surface can have infinite genus as well as an infinity of Freudenthal ends (a.k.a. "ideal points").

The topological type of open Riemann surfaces

Let Σ be an open Riemann surface.

Definition

- The *transfinite genus* of Σ is the cardinal number:

$$g(\Sigma) = \frac{1}{2}\mathrm{rk}_{\mathbb{Z}}\mathrm{H}^1(\Sigma, \mathbb{Z}) \quad .$$

- The *ideal boundary* $\partial_\infty\Sigma$ of Σ is the set of Freudenthal ends of Σ, endowed with its natural topology.

Proposition (Kerékjártó)

The following statements hold for any open Riemann surface Σ:

- $g(\Sigma)$ *is finite or countable.*
- $\partial_\infty\Sigma$ *is a finite or countable compact Haussdorff topological space which is totally disconnected.*

Remarks

- $\partial_\infty\Sigma$ *can be Cantor space.*
- *'Adding'* $\partial_\infty\Sigma$ *to* Σ *produces the so-called* Kerékjártó-Stoilow *compactification* $\hat{\Sigma}$ *of* Σ*.*

There exists a natural disjoint union decomposition $\partial_\infty \Sigma = \partial^1_\infty \Sigma \sqcup \partial^2_\infty \Sigma$ where the ends belonging to $\partial^1_\infty \Sigma$ and $\partial^2_\infty \Sigma$ are called of the *first* and *second* kind.

Theorem (Kerékjártó)

Let Σ and Σ' be two open Riemann surfaces. Then the following statements are equivalent:

(a) *Σ and Σ' are homeomorphic.*

(b) *There exists a homeomorphism $\psi : \partial_\infty \mathcal{S} \to \partial_\infty \mathcal{S}'$ such that $\psi(\partial^1_\infty \mathcal{S}) = \partial^1_\infty \mathcal{S}'$.*

Theorem (Richards-Stoilow)

The following statements hold:

1. *The (unoriented) homeomorphism type of an open Riemann surface Σ is entirely determined by the triplet $(g(\Sigma), \partial^1_\infty \Sigma, \partial_\infty \Sigma)$.*
2. *Consider any triple (g, F, F^1) with $g \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, F a non-empty compact, Haussdorff and totally disconnected countable topological space and $F^1 \subset F$ a (possibly empty) subset of F (endowed with the induced topology). Then there exists an open Riemann surface Σ with $g(\Sigma) = g$, $\partial_\infty \Sigma = F$ and $\partial^1_\infty \Sigma = F^1$.*

Theorem (Behnke-Stein)

Every open Riemann surface is Stein.

Theorem (Grauert-Röhrl)

Any holomorphic vector bundle on an open Riemann surface Σ is holomorphically trivial. In particular, the analytic Picard group $\mathrm{Pic}(\Sigma)$ vanishes and hence K_Σ is holomorphically trivial.

Proposition

Every non-compact Riemann surface Σ admits a holomorphic embedding in $\mathbb{C}^3$ and a holomorphic immersion in $\mathbb{C}^2$.

Remark

It is not known if any open Riemann surface admits a proper holomorphic embedding in $\mathbb{C}^2$.

The ring $\mathrm{O}(\Sigma)$

Let Σ be an open Riemann surface.

Theorem (Bers)

Let Σ_1 and Σ_2 be two connected non-compact Riemann surfaces. Then Σ_1 and Σ_2 are biholomorphic iff their rings of holomorphic functions $\mathrm{O}(\Sigma_1)$ and $\mathrm{O}(\Sigma_2)$ are isomorphic **as unital $\mathbb{C}$-algebras**.

Theorem (Iss'sa)

Let Σ_1 and Σ_2 be two connected non-compact Riemann surfaces. Then Σ_1 and Σ_2 are biholomorphic iff their fields of meromorphic functions $\mathcal{M}(\Sigma_1)$ and $\mathcal{M}(\Sigma_2)$ are isomorphic **as unital $\mathbb{C}$-algebras**.

Proposition (Henriksen-Alling)

The following statements hold:

1. *The cardinal Krull dimensions of all open Riemann surfaces are equal to each other (denote this cardinal number by* $\mathbf{k}$*).*
2. *We have* $\mathbf{k} \geq 2^{\aleph_1}$.

Definition

An *elementary divisor domain* (EDD) is an integral domain R such that **any** matrix with coefficients from R admits a Smith normal form.

Theorem (Helmer-Henriksen-Alling)

For any open Riemann surface Σ, the ring $\mathrm{O}(\Sigma)$ is an elementary divisor domain.

Let Σ be an open Riemann surface and $\mathrm{ord}_x : \mathcal{M}(\Sigma) \to \mathbb{Z}$ be the map which assigns to a meromorphic function $f \in \mathcal{M}(\Sigma)$ its order at the point $x \in \Sigma$.

Definition

A *special local uniformizer* for Σ at the point $x \in \Sigma$ is a meromorphic function $t_x \in \mathcal{M}(\Sigma)$ such that $\mathrm{ord}_x(t_x) = 1$ and $\mathrm{ord}_y(t_x) = 0$ for all $x \neq y$.

Proposition

The special local uniformizers of Σ coincide with the prime elements of $\mathrm{O}(\Sigma)$. In particular, a special local uniformizer at x is determined up to multiplication by a unit of $\mathrm{O}(\Sigma)$.

Critically-finite elements of $\mathrm{O}(\Sigma)$

Definition

A divisor $d \in \mathrm{O}(\Sigma)$ of $f \in \mathrm{O}(\Sigma)$ is called *critical* if $d^2 | f$.

Definition

A non-zero non-unit $f \in \mathrm{O}(\Sigma)$ is called:

- *non-critical*, if f has no critical divisors;
- *critically-finite*, f has a factorization of the form:

$$f = f_0 f_c \quad \text{with} \quad f_c = p_1^{n_1} \dots p_N^{n_N} \quad ,$$

  where $N \geq 1$, $n_i \geq 2$, $p_1, \dots, p_N \in \mathrm{O}(\Sigma)$ are critical prime divisors of f with $(p_i) \neq (p_j)$ for $i \neq j$ and $f_0 \in \mathrm{O}(\Sigma)$ is non-critical and coprime with f_c.

Lemma

Let $f \in \mathrm{O}(\Sigma)$ be a non-zero element and $D(f)$ be the divisor of f. Then the following statements hold:

1. *f is a unit of $\mathrm{O}(\Sigma)$ iff $D(f) = 0$.*
2. *f is a prime element of $\mathrm{O}(\Sigma)$ (i.e. a special uniformizer) iff $D(f) = x$ for some $x \in \Sigma$.*
3. *f is non-critical iff $D(f)$ is multiplicity-free at any point in its support, i.e. iff it has the form:*

$$D(f) = \sum_{x \in \mathcal{Z}(f)} x \quad .$$

4. *If $g \in \mathrm{O}(\Sigma)$ is another holomorphic function, then $f|g$ iff $D(f) \leq D(g)$.*
5. *f is critically-finite iff $f = gh$, where g is non-critical and:*

$$D(h) = \sum_{x \in \mathcal{Z}(h)} n_x \cdot x$$

with $\mathcal{Z}(h)$ a non-empty finite set and $n_x > 1$ for every $x \in \mathcal{Z}(h)$.

Definition

A *one-dimensional LG pair* is a Landau-Ginzburg pair (X, W) such that $\dim_{\mathbb{C}} X = 1$, i.e. such that X is an open Riemann surface.

Such an LG pair is of Stein type. Let (Σ, W) be a one-dimensional LG pair with finite critical set Z_W. For any $p \in Z_W$, let $t_p \in \mathcal{M}(\Sigma)$ be a special local uniformizer of Σ at p.

Proposition

The analytic Milnor algebra of W at $p \in Z_W$ is given by:

$$\mathrm{M}(\hat{W}_p) \simeq_{\mathrm{O}(\Sigma)} \oplus_{i=1}^{N} \mathcal{O}_{\Sigma,p}/\langle \hat{t}_p^{\nu_p} \rangle \quad ,$$

where $\nu_p \stackrel{\mathrm{def.}}{=} \dim_{\mathbb{C}} \mathrm{M}(\hat{W}_p)$ *is the Milnor number at p. Moreover, the bulk state space has dimension:*

$$\dim_{\mathbb{C}} \mathcal{H} = \sum_{p \in Z_W} \nu_p \quad .$$

Proposition

Let (Σ, W) be a one-dimensional Landau-Ginzburg pair. Then the following statements are equivalent:

(a) *W is a critically-finite element of the Bézout ring $\mathrm{O}(\Sigma)$.*

(b) *The intersection $\mathcal{Z}(W) \cap Z_W$ is finite.*

(c) *The divisor of W has the form:*

$$D(W) = D_0 + \sum_{i=1}^{N} n_i x_i \quad ,$$

where D_0 is an effective divisor whose multiplicity at every point of its support is one, $x_1, \ldots, x_N$ (with $N \geq 1$) is a finite set of points of Σ which do not belong to the support of D_0 and $n_i \geq 2$ for all $i = 1, \ldots, N$.

(d) *We have $W = W_0 W_c$, where W_0 has only simple zeros, W_c has a finite number of zeros, each of which has multiplicity at least two and W_0, W_c have no common zero.*

In particular, any holomorphic function W with finite critical set is a critically-finite element of $\mathrm{O}(\Sigma)$.

Remark

Any critically-finite superpotential $W \in \mathrm{O}(\Sigma)$ can be written as:

$$W = W_0 W_c \quad , \text{ with } \quad W_c = \prod_{i=1}^{N} t_{x_i}^{n_i} \quad , \tag{1}$$

where:

1. $W_0 \in \mathrm{O}(\Sigma)$ *has a finite or countable number of zeros, all of which are simple.*
2. $x_1, \ldots, x_N$ *(with* $N \geq 1$*) are distinct points of* Σ *and* t_{x_i} *are special uniformizers at these points.*
3. *None of the points* x_i *is a zero of* W_0.

In this case, we have $D_c = \sum_{i=1}^{N} n_i x_i$ *and* $D_0 = D(W_0)$.

Proposition

Let (Σ, W) be a one-dimensional LG pair. Then there exists a natural equivalence of $\mathbb{Z}_2$-graded $\mathrm{O}(\Sigma)$-linear dg categories:

$$\mathrm{PF}(\Sigma, W) \simeq_{\mathrm{O}(\Sigma)} \mathrm{MF}(\mathrm{O}(\Sigma), W)$$

Definition

A matrix factorization (M, D) of W over $\mathrm{O}(\Sigma)$ is called *elementary* if $\mathrm{rk} M^{\hat{0}} = \mathrm{rk} M^{\hat{1}} = 1$.

Definition

The $\mathbb{Z}_2$-graded *cocycle category* $\mathrm{ZMF}(\mathrm{O}(\Sigma), W)$ is defined to be the subcategory of $\mathrm{MF}(\mathrm{O}(\Sigma), W)$ having the same objects, but whose morphisms consist of those morphisms of $\mathrm{MF}(\mathrm{O}(\Sigma), W)$ which are closed with respect to the differential.

Let $\mathrm{zmf}(\mathrm{O}(\Sigma), W)$ denote the even subcategory of $\mathrm{ZMF}(\mathrm{O}(\Sigma), W)$. Two matrix factorizations of W are called *strongly isomorphic* if they are isomorphic in $\mathrm{zmf}(\mathrm{O}(\Sigma), W)$.

Proposition

Any elementary factorization of W over $\mathrm{O}(\Sigma)$ is strongly isomorphic to an elementary factorization of the form $e_v \stackrel{\mathrm{def.}}{=} (\mathrm{O}(\Sigma)^{1|1}, D_v)$, where $v \in \mathrm{O}(\Sigma)$ is a divisor of W and $D_v \stackrel{\mathrm{def.}}{=} \left[\begin{array}{cc} 0 & v \\ u & 0 \end{array}\right]$, with $u \stackrel{\mathrm{def.}}{=} W/v \in \mathrm{O}(\Sigma)$.

Definition

An element $f \in \mathrm{O}(\Sigma)$ is called *primary* if it is a power of a prime element of $\mathrm{O}(\Sigma)$. An elementary factorization e_v of W is called *primary* if v is a primary divisor of W.

If e_v is a primary factorization of W over $\mathrm{O}(\Sigma)$, then we have $v = at_x^k$ for some $a \in \mathrm{O}(\Sigma)^\times$, some $k \in \mathbb{Z}_{>0}$ and some $x \in \Sigma$ and W has a zero of order at least k at the point x.

Theorem

Suppose that W is critically-finite. Then the additive category $\mathrm{hmf}(\mathrm{O}(\Sigma), W)$ is a Krull-Schmidt category whose non-zero indecomposables are the nontrivial primary matrix factorizations of W.

Proposition

Suppose that W is critically finite of the form (1). The number of isomorphism classes of indecomposable non-zero objects of the category $\mathrm{hmf}(\mathrm{O}(\Sigma), W)$ equals $\sum_{i=1}^{N}(n_i - 1) = -N + \sum_{i=1}^{N} n_i$.

The degrees n_i of the prime factors t_{x_i} arising in the decomposition (1) of W_c define a $\mathbb{Z}_2$-grading on the set $I_N \stackrel{\text{def.}}{=} \{1, \ldots, N\}$ whose components are given by:

$$I_N^{\hat{0}} \stackrel{\text{def.}}{=} \{i \in I_N \,|\, n_i \text{ is even}\} \ , \ I_N^{\hat{1}} \stackrel{\text{def.}}{=} \{i \in I_N \,|\, n_i \text{ is odd}\} \ .$$

Let:

$$N^{\hat{0}} \stackrel{\text{def.}}{=} |I_N^{\hat{0}}| \ \text{ and } \ N^{\hat{1}} \stackrel{\text{def.}}{=} |I_N^{\hat{1}}|$$

denote the cardinalities of these subsets of I, which satisfy $N^{\hat{0}} + N^{\hat{1}} = N$. Any non-empty subset $K \subset I_N$ is endowed with the $\mathbb{Z}_2$-grading induced from I_N, which has components:

$$K^{\hat{0}} \stackrel{\text{def.}}{=} K \cap I_N^{\hat{0}} \ , \ K^{\hat{1}} \stackrel{\text{def.}}{=} K \cap I_N^{\hat{1}} \ .$$

Theorem

Suppose that W is a critically-finite element of $\mathrm{O}(\Sigma)$ with the decomposition (1). Then:

1. *The number of isomorphism classes of elementary factorizations in the category $\mathrm{hmf}(\mathrm{O}(\Sigma), W)$ is given by:*

$$\widehat{\mathcal{N}}_W = \sum_{k=0}^{N^{\hat{1}}} \sum_{\substack{K \subsetneq I_N, \\ |K^{\hat{1}}|=k}} 2^{N^{\hat{0}}+k} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \ .$$

2. *The number of isomorphism classes of elementary matrix factorizations in the category $\mathrm{HMF}(\mathrm{O}(\Sigma), W)$ is given by:*

$$\mathcal{N}_W = 2^{r^{\hat{0}}} + \sum_{k=0}^{N^{\hat{1}}} 2^{N^{\hat{0}}+k-1} \sum_{\substack{K \subsetneq I_N \\ |K^{\hat{1}}|=k}} \prod_{i \in K} \lfloor \frac{n_i - 1}{2} \rfloor \ .$$

Let R be a unital commutative ring. Let mod_R be the Abelian category of finitely-generated R-modules.

Definition

1. The *projectively stable category* $\underline{\mathrm{mod}}_R$ has the same objects as mod_R and modules of morphisms given by:

$$\underline{\mathrm{Hom}}_R(M, N) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_R(M, N)/\mathcal{P}_R(M, N) \quad \forall M, N \in \mathrm{Ob}(\mathrm{mod}_R) ,$$

   where $\mathcal{P}_R(M, N) \subset \mathrm{Hom}_R(M, N)$ consists of those morphisms of mod_R which factor through a projective module of finite rank.

2. The *injectively stable category* $\overline{\mathrm{mod}}_R$ has the same objects as mod_R and modules of morphisms given by:

$$\overline{\mathrm{Hom}}_R(M, N) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_R(M, N)/\mathcal{I}_R(M, N) \quad \forall M, N \in \mathrm{Ob}(\mathrm{mod}_R) ,$$

   where $\mathcal{I}_R(M, N) \subset \mathrm{Hom}_R(M, N)$ consists of those morphisms of mod_R which factor through an injective module of finite rank.

Fact When R is a self-injective ring, the categories $\underline{\mathrm{mod}}_R$ and $\overline{\mathrm{mod}}_R$ are equivalent to each other and the category $\underline{\mathrm{mod}}_R$ is naturally triangulated since in this case mod_R is a Frobenius category (i.e. an exact category whose projective and injective objects coincide).

Theorem

Let $W \in \mathrm{O}(\Sigma)$ be a critically-finite superpotential of the form (1). Then:

1. *The ring $A_i \stackrel{\mathrm{def.}}{=} \mathrm{O}(\Sigma)/\langle t_{x_i}^{n_i}\rangle$ is Artinian and Frobenius (hence also self-injective) for all $i = 1, \ldots, N$.*
2. *For each $i = 1, \ldots, N$, there exist equivalences of triangulated categories:*

$$\mathrm{D}_{\mathrm{Sing}}(A_i) \simeq \underline{\mathrm{mod}}_{A_i} \quad ,$$

where $\underline{\mathrm{mod}}_{A_i}$ denotes the projectively stable category of finitely-generated A_i-modules.

3. *The triangulated category $\underline{\mathrm{mod}}_{A_i}$ is Krull-Schmidt with non-zero indecomposable objects given by the A_i-modules:*

$$V_i^{(k)} \stackrel{\mathrm{def.}}{=} \mathrm{O}(\Sigma)/\langle t_{x_i}^{k}\rangle \simeq \langle t_{x_i}^{n_i-k}\rangle/\langle t_{x_i}^{n_i}\rangle \text{ where } k = 1, \ldots, n_i - 1 \quad .$$

This category admits Auslander-Reiten triangles, having the Auslander-Reiten quiver shown the Figure. Moreover, it is classically generated by the residue field $V_i^{(1)} = \mathrm{O}(\Sigma)/\langle t_{x_i}\rangle$ of A_i.

4. *The triangulated category $\mathrm{D}_{\mathrm{Sing}}(A_i) \simeq \underline{\mathrm{mod}}_{A_i}$ is 1-Calabi-Yau for all $i = 1, \ldots, N$, with involutive shift functor given by:*

$$\Omega(V_i^{(k)}) = V_i^{(n_i-k)} \quad \forall k = 1, \ldots, n_i - 1 \quad .$$

5. *There exist equivalences of triangulated categories:*

$$\mathrm{hmf}(\mathrm{O}(\Sigma), W) \simeq \vee_{i=1}^{N} \mathrm{D}_{\mathrm{Sing}}(A_i) \simeq \vee_{i=1}^{N} \underline{\mathrm{mod}}_{A_i} \quad .$$

6. *The triangulated category $\mathrm{hmf}(\mathrm{O}(\Sigma), W)$ is 1-Calabi-Yau, Krull-Schmidt and admits Auslander-Reiten triangles. Its Auslander-Reiten quiver is disconnected, with connected components given by the Auslander-Reiten quivers of the categories $\underline{\mathrm{mod}}_{A_i}$.*

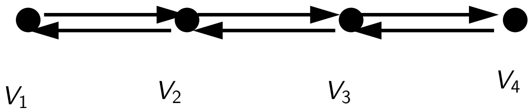


Figure: Auslander-Reiten quiver for $\underline{\mathrm{mod}}_{A_i}$ when $n_i = 5$. The Auslander-Reiten translation fixes all vertices and the multiplicities of all arrows are trivial.

Proposition

Let Σ be an open Riemann surface. Then the following statements are equivalent:

(a) *There exists a non-singular complex affine curve C such that Σ coincides with the analyticization C^{an} of C.*

(b) *There exists a compact Riemann surface $\hat{\Sigma}$ a non-empty finite set $S \subset \Sigma$ such that $\Sigma = \hat{\Sigma} \setminus S$.*

Definition

An *one-dimensional affine LG pair* is a pair (C, W) such that:

1. C is a non-singular complex affine curve whose canonical line bundle is *algebraically* trivial.
2. W is regular complex-valued function defined on C.

Definition

The *analyticization* of a one-dimensional affine LG pair (C, W) is the one-dimensional LG pair (Σ, W), where $\Sigma = C^{\mathrm{an}}$ and W is viewed as a holomorphic function defined on Σ. Notice that W has finite critical set and hence is critically-finite.

Let (C, W) be a one-dimensional affine LG pair. Then Serre's GAGA functor $G : \mathrm{Coh}(C) \to \mathrm{Coh}(\Sigma)$ induces a *dg GAGA functor on factorizations*:

$$\mathcal{G} : \mathrm{PF}(C, W) \to \mathrm{MF}(\mathrm{O}(\Sigma), W) \quad .$$

In turn, this induces a *cohomological GAGA functor on factorizations*:

$$\mathrm{H}\mathcal{G} : \mathrm{HPF}(C, W) \to \mathrm{HMF}(\mathrm{O}(\Sigma), W) \quad ,$$

whose even part we denote by $\mathrm{H}\mathcal{G}^{\hat{0}} : \mathrm{hpf}(C, W) \to \mathrm{hmf}(\mathrm{O}(\Sigma), W)$.

Theorem

The functor $\mathrm{H}\mathcal{G}^{\hat{0}}$ is an equivalence of triangulated categories.