Nonstandard Lagrangian in Cosmology - Constrained System

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Bucharest
23 April, 2018
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Introduction

- We study real scalar field with non-standard Lagrangian density, important (at least for us) in Cosmological context.

- General space-time metric:
  \[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3 \]
- Metric tensor: \( \hat{g} \)
- Components of metric tensor: \( g_{\mu \nu} \)
- Determinant of metric tensor: \( g \equiv \det \hat{g} \)
Introduction

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0,1,2,3 \]

- Mostly used:
- Minkowski space-time (Special Relativity):
  \[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + d\vec{x}^2 \]
- Friedmann-Robertson-Walker-Lemaitre (FRWL) flat space-time (General Relativity/Cosmology):
  \[ ds^2 = -N(t)^2 dt^2 + a^2(t) d\vec{x}^2 \]
- Scale factor (how much the Universe changes its size): \( a(t) \)
- Lapse function: \( N(t) \)
Cosmology

• Real scalar field with non-standard Lagrangian density.
• General Lagrangian action:
  \[ S = \int d^4 x \sqrt{-g} \mathcal{L}(X(\partial \phi), \phi) \]
• Lagrangian (Lagrangian density) of the standard form:
  \[ L(\phi, \partial \phi) = X(\partial \phi) - V(\phi) \]
• Non-standard Lagrangian:
  \[ L_{tach}(T, X) = -V(T)\sqrt{1 + 2X(\partial T)} \]
  \[ X = \frac{1}{2} g^{\mu \nu} \partial_\mu T \partial_\nu T \]
Cosmology

- The action:

\[ S = \int d^4x \sqrt{-g} \mathcal{L}(X,T) \]

- In cosmology, scalar fields can be connected with a perfect fluid which describes (dominant) matter in the Universe.

- Components of the energy-momentum tensor:

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \]

\[ T_{\mu\nu} = (P + \rho)u_{\mu} u_{\nu} - P g_{\mu\nu} \]
Cosmology

\[ T_{\mu\nu} = (P + \rho)u_\mu u_\nu - Pg_{\mu\nu} \]

- Pressure, matter density and velocity 4-vector, respectively:

\[ P(X,T) \equiv L(X,T) \]

\[ \rho(X,T) \equiv 2X \frac{\partial L}{\partial X} - L(X,T) \]

\[ u_\mu \equiv \frac{\partial \mu T}{\sqrt{2X}} \]
Cosmology

- Total action: term which describes gravity (Ricci scalar, Einstein-Hilbert action) plus term that describes cosmological fluid (scalar field minimally coupled to gravity):

\[ S = \int d^4x \sqrt{-g} \left( R + L(X,T) \right) \]

- Einstein equations:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \]
Cosmological Perturbations

- Very important: cosmological perturbations around the classical background (give us information about Universe at the very beginning, i.e. CMB and cosmological structure formation).

- ADM decomposition of the metric:

\[
 ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \\
 S = \int d^4x \sqrt{-g} \left( \pi^{ij} \dot{\gamma}_{ij} - NH_G - N_i H_G^i + L(X,T) \right)
\]
Cosmological Perturbations

- Perturb everything:
  \[ \pi^{ij} = \pi^{ij}(t) + \delta \pi^{ij} \]
  \[ \gamma_{ij} = \gamma_{ij}(t) + \delta \gamma_{ij} \]
  \[ T = T(t) + \delta T \ldots \]
  \[ S = S^{(0)} + S^{(1)} + S^{(2)} + \ldots \]

- Action for classical background: \( S^{(0)} \)
  \[ S^{(1)} = 0 \]
  \[ S^{(2)} = S^{(2)}_{QUAD} + S^{(2)}_{INT} \]
Cosmological Perturbations

• Path integral quantization of cosmological perturbations:

\[ K \propto \int D\mu \exp[iS^{(2)}_{\text{QUAD}} + iS^{(2)}_{\text{INT}}] \]

• Transition amplitude (tells us how the system evolve quantum-mechanically): \( K \)
• Path integral measure: \( D\mu \)
• If you are lucky, „standard“ Feynman measure (no constraints)
• If you are less lucky, Faddeev-Popov measure (first class constraints)
• No luck at all, Senjanovic measure (both first and second class constraints)
Cosmological Perturbations

- **Background FLRW space-time:**
  \[
  ds_{FLRW}^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2
  \]

- **Friedmann equation** (*H*-Hubble parameter):
  \[
  H^2 = \frac{1}{3} \rho(X,T), \quad H = \frac{\dot{a}}{a}
  \]

- **Acceleration equation:**
  \[
  \ddot{a} = -\frac{1}{6} \left( \rho(X,T) + 3P(X,T) \right)
  \]

- **Continuity equation:**
  \[
  \dot{\rho} = -3H \left( \rho(X,T) + P(X,T) \right)
  \]
Non-standard Lagrangian

• Effective field theory of non-standard lagrangian of DBI-type (tachyon scalar field):

\[ L_{tach} = -V(T)\sqrt{1 + g_{\mu\nu} \partial^\mu T \partial^\nu T} \]

• EoM:

\[
\left( g^{\mu\nu} - \frac{\partial^\mu T \partial^\nu T}{1 - (\partial T)^2} \right) \partial_\mu T \partial_\nu T = - \frac{1}{V(T)} \frac{dV}{dT} (1 - (\partial T)^2)
\]
Non-standard Lagrangian

- Minisuperspace model for the background:
  \[ ds^2_{FLRW} = -N^2(t)dt^2 + a^2(t)d\vec{x}^2 \]

- Total Lagrangian:
  \[
  L = L_g(N, a, \dot{a}) + L_{tach}(T, \dot{T}, N, a)
  \]
  \[
  L_g(N, a, \dot{a}) = -6 \frac{a\dot{a}^2}{N} + 6kNa
  \]
  \[
  L_{tach}(T, \dot{T}, N, a) = -Na^3V(T) \sqrt{1 - N^{-2}\dot{T}^2}
  \]
Extension

- Minisuperspace model for the background:
  \[ ds_{FLRW}^2 = -N^2(t)dt^2 + a^2(t)d\bar{x}^2 \]

- Auxiliary field: \( \sigma \)

- Total Lagrangian:

\[
L = L_g(N, a, \dot{a}) + L_{ext}(T, \dot{T}, N, a, \sigma)
\]

\[
L_g(N, a, \dot{a}) = -6\frac{a\dot{a}^2}{N} + 6kNa
\]

\[
L_{ext}(T, \dot{T}, N, a, \sigma) = +Na^3\left[\frac{1}{2N^2}\frac{\dot{T}^2}{\sigma} - \frac{1}{2}\sigma V^2(T) - \frac{1}{2\sigma}\right]
\]
Extension

• From the EoM for $\sigma$ :

$$\sigma = f(T,\dot{T},N) = \frac{1}{V(T)} \sqrt{1 - N^{-2}\dot{T}^2}$$

$$L_{\text{ext}}(T,\dot{T},N,a,\sigma)|_{\sigma = f(T,\dot{T},N)} = L_{\text{tach}}(T,\dot{T},N,a)$$

$$L = -6\frac{a\dot{a}^2}{N} + 6kNa + \frac{a^3}{2N} \frac{\dot{T}^2}{\sigma} - \frac{1}{2} \sigma N a^3 V^2(T) - \frac{Na^3}{2\sigma}$$

$$S^{(0)} = \int dtL$$
Constraints

- We need to know what are the constraints in order to proceed with $S^{(2)}$.

- Conjugated momenta:

  \[ p_a = \frac{\partial L}{\partial \dot{a}} = -12 \frac{a \dot{a}}{N} \implies \dot{a} = -\frac{N}{12a} p_a \]

  \[ p_T = \frac{\partial L}{\partial \dot{T}} = \frac{a^3 \dot{T}}{N \sigma} \implies \dot{T} = \frac{N \sigma}{a^3} p_T \]

  \[ p_N = \frac{\partial L}{\partial \dot{N}} = 0 \]

  \[ p_\sigma = \frac{\partial L}{\partial \dot{\sigma}} = 0 \]
Constraints

• Primary constraints:
  \[ \phi_1 \equiv p_N - 0 \approx 0 \]
  \[ \phi_2 \equiv p_\sigma - 0 \approx 0 \]

• Canonical Hamiltonian:
  \[ H_c \equiv NH_0 \]
  \[ H_0 \equiv -\frac{1}{24} \frac{p_a^2}{a} - 6ka + \frac{\sigma}{2a^3} p_T^2 + \frac{1}{2} \sigma a^3 V^2(T) + \frac{a^3}{2\sigma} \]

• Total Hamiltonian (add all constraints to canonical Hamiltonian):
  \[ H_T \equiv H_c + \lambda_1 \phi_1 + \lambda_2 \phi_2 \]
Constraints

• Consistency conditions:
  \[ \dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \]
  \[ \dot{\phi}_2 = \{\phi_2, H_T\} \approx 0 \]

• Secondary constraints:
  \[ \chi_1 \equiv -H_0 \approx 0 \]
  \[ \chi_2 \equiv -N[\frac{1}{2a^3} p_T^2 + \frac{1}{2} \frac{a^3 V^2(T)}{\sigma^2} - \frac{a^3}{\sigma^2}] \approx 0 \]

• Consistency conditions on secondary constraints:
  \[ \dot{\chi}_1 = \{\chi_1, H_T\} \approx 0 \]
  \[ \dot{\chi}_2 = \{\chi_2, H_T\} \approx 0 \]
Constraints

- (Reminder) First class constraints: their Poisson brackets with other constraints vanish:
  \[ \{ \phi_{1st}, any\ constr. \} \approx 0 \]
- (Reminder) Constraints that are not of the first class are called second class.
- (Reminder) The number of second class constraints is always even.
- For this extended model:
  - two first class constraints and
  - two second class constraints.
Conclusion

• There are both type of constraints.
• So, Senjanovic method/measure should be used.
• As already mentioned, extended model is classically equivalent to the initial one.
• As already mentioned, extended model is more suitable for path integral quantization.
References

- J.O. Gong, M.S. Seo and G. Shiu, JHEP 07, 099 (2016).
• This work is supported by the SEENET-MTP Network under the ICTP grant NT-03.
• The financial support of the Serbian Ministry for Education and Science, Projects OI 174020 and OI 176021 is also kindly acknowledged.
THANK YOU!

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