Orbifolds of defect TQFTs

Nils Carqueville

Universität Wien & Erwin Schrödinger Institute

Motivation and outline

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$\Longrightarrow$ categories, 2-categories, . . . , n-categories

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$\Longrightarrow$ **orbifold TQFTs**

$\Longrightarrow$ **new equivalences**

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A **2-dimensional closed TQFT** is a symmetric monoidal functor

$$\mathcal{Z} \colon \Big(\mathrm{Bord}_2, \sqcup, \emptyset\Big) \longrightarrow \Big(\mathrm{Vect}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}\Big)$$

Atiyah 1988

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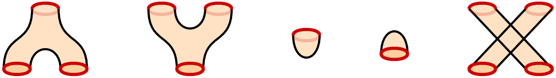

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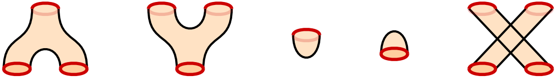

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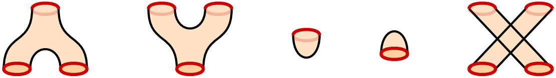

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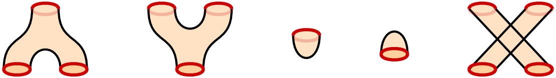

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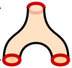
multiplication $\mathcal{Z}\Big($  $\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$

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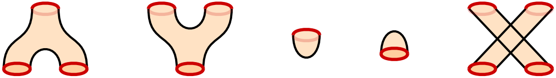

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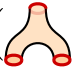
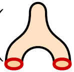
multiplication $\mathcal{Z}\Big($$\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$, pairing $\mathcal{Z}\Big($$\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$

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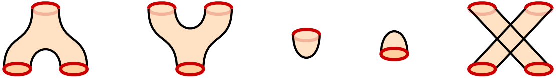

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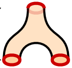
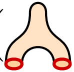
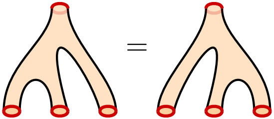
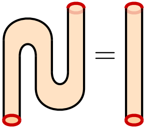
multiplication $\mathcal{Z}\Big($$\Big)\colon \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}$, pairing $\mathcal{Z}\Big($$\Big)\colon \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathbb{C}$





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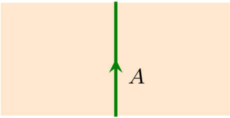
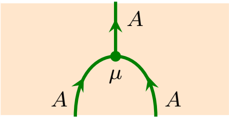
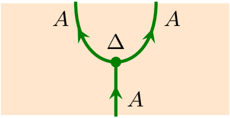
State sum models:

- input: separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)
- choose oriented **triangulation** for every bordism Σ
- on **Poincaré dual** graph, associate A to edges, (co)multiplication μ, Δ to vertices:







Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models in 2d

- input: separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)
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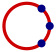
- associate $A^{\otimes k}$ to dual triangulation of circle with k points:

 $\longmapsto A^{\otimes 3}$

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  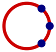 $\longmapsto A^{\otimes 3}$
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  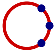 $\longmapsto A^{\otimes 3}$
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  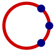 $\longmapsto A^{\otimes 3}$
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- define **state sum model**

$$\begin{aligned} \mathcal{Z}^{\mathrm{ss}}_A \colon \ \mathrm{Bord}_2 &\longrightarrow \mathrm{Vect}_{\mathbb{C}} \\ S^1 &\longmapsto \mathrm{Im}\Big(\pi_{S^1\times[0,1]} \colon A^{\otimes k} \longrightarrow A^{\otimes k}\Big) \cong Z(A) \\ \Big(\Sigma \colon (S^1)^{\times m} \longrightarrow (S^1)^{\times n}\Big) &\longmapsto \Big(\text{induced linear map } Z(A)^{\otimes m} \longrightarrow Z(A)^{\otimes n}\Big) \end{aligned}$$

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Theorem.
State sum model for A is independent of choice of triangulation, and $\mathcal{Z}^{\mathrm{ss}}_A(S^1) \cong Z(A)$.

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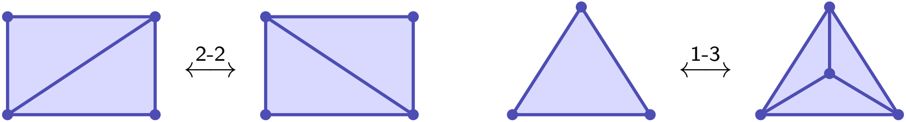

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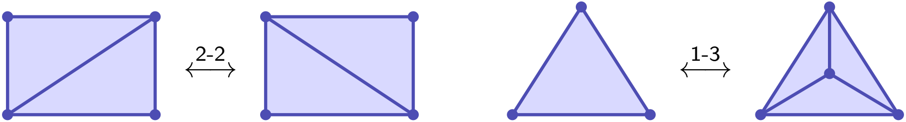

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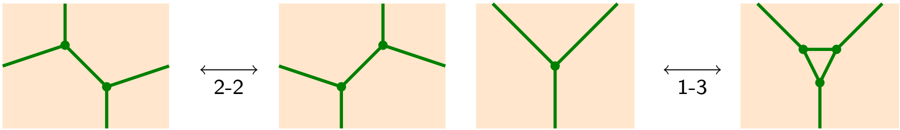

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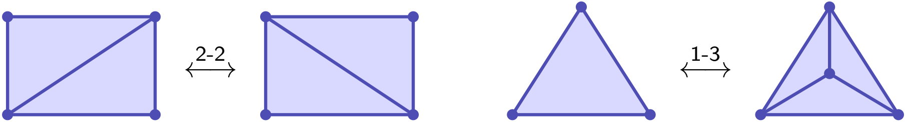

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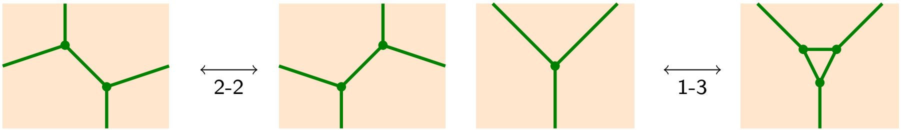

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Davydov/Kong/Runkel 2011

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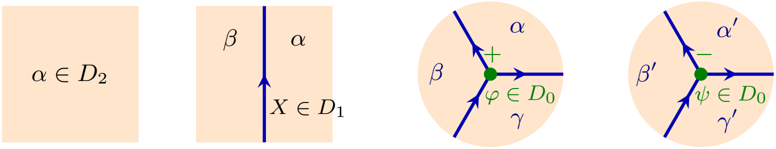

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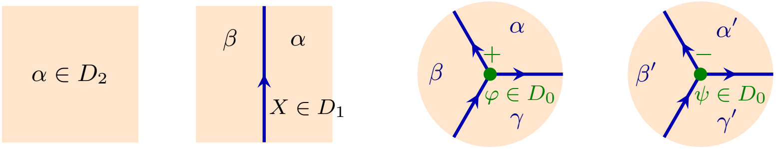

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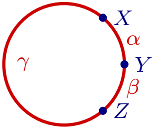

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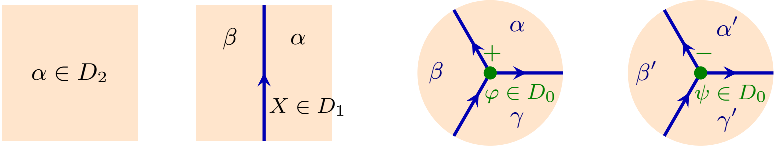
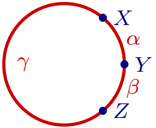
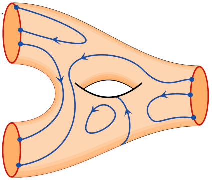


objects: 

morphisms: 

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  - $D_1^{\text{triv}} = \{\mathbb{C}\text{-bimodules}\} = \{\mathbb{C}\text{-vector spaces}\}$
  - $D_0^{\text{triv}} = \{\text{bimodule maps}\} = \{\text{linear maps}\}$

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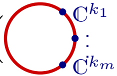
  - $\mathcal{Z}^{\text{triv}}\Big($ 

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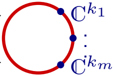
  - $\mathcal{Z}^{\text{triv}}\Bigg($ 

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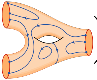
  - $\mathcal{Z}^{\text{triv}}\Bigg($  $\Bigg) \overset{\text{def}}{=}$ (evaluate line and point defect in $\text{Vect}_{\mathbb{C}}$)

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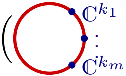
  - $\mathcal{Z}^{\mathrm{triv}}\Bigg($ 

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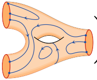
  - $\mathcal{Z}^{\mathrm{triv}}\Bigg($  $\Bigg) \overset{\mathrm{def}}{=}$ (evaluate line and point defect in $\mathrm{Vect}_{\mathbb{C}}$)
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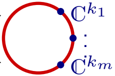
  - $\mathcal{Z}^{\text{triv}}\Bigg($ 

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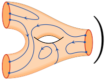
  - $\mathcal{Z}^{\text{triv}}\Bigg($  $\Bigg) \stackrel{\text{def}}{=}$ (evaluate line and point defect in $\operatorname{Vect}_{\mathbb{C}}$)
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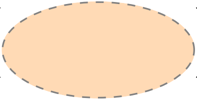
$\mathcal{Z}^G\Bigg($  $\Bigg)$

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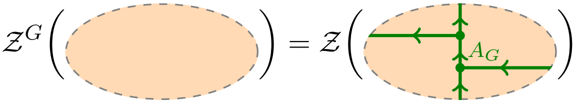

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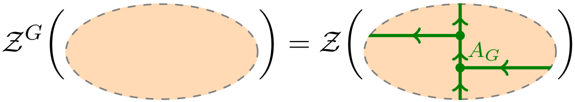

- consistent if A_G is **separable symmetric Frobenius algebra** internal to 2-category associated to $\mathcal{Z}$

$\Longrightarrow$ **group orbifolds from special types of algebras**

Fröhlich/Fuchs/Runkel/Schweigert 2009, Carqueville/Runkel 2012

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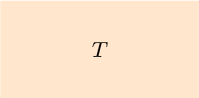
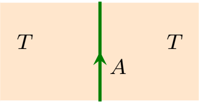
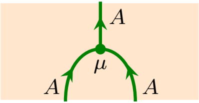


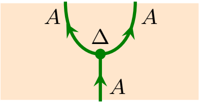
$T \in D_2$



$A \in D_1$



$\mu \in D_0$



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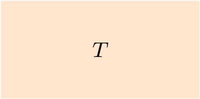
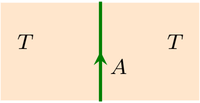
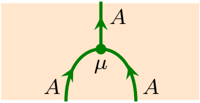


$T \in D_2$



$A \in D_1$



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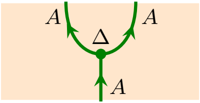

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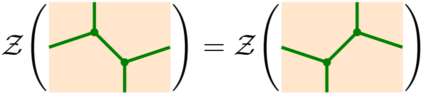

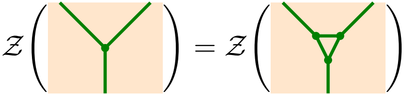

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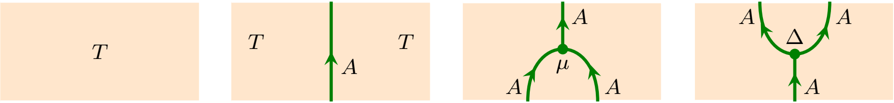

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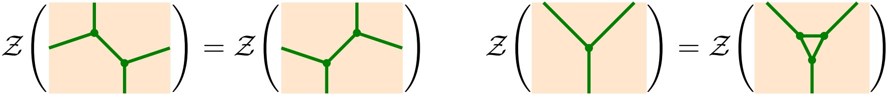

Definition & Theorem.
Applying $\mathcal{Z}$ to $\mathcal{A}$-decorated dual triangulations gives **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \operatorname{Bord}_2 \longrightarrow \operatorname{Vect}_{\mathbb{C}}$$

Carqueville/Runkel 2012

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(also $\mathrm{S}_{11}/\mathrm{W}_{13}$ and $\mathrm{Z}_{13}/\mathrm{Q}_{11}$ and $\mathrm{E}_{13}/\mathrm{Z}_{11}$)

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Newton/Ros Camacho 2015, Recknagel/Weinreb 2015–2017

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Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Newton/Ros Camacho 2015, Recknagel/Weinreb 2015–2017

In any dimension $n \geqslant 1$, the **generalised orbifold construction** *works for any n-dimensional defect TQFT*

$$\mathcal{Z}\colon\ \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,.$$

Carqueville/Runkel/Schaumann 2017

Triangulations

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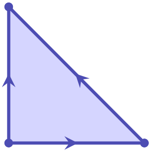

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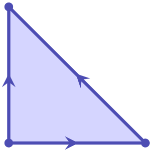

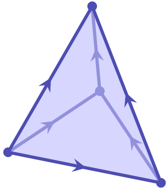

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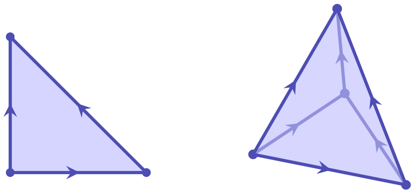

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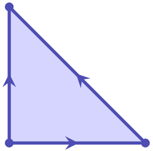

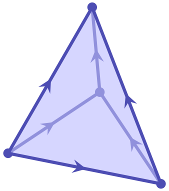

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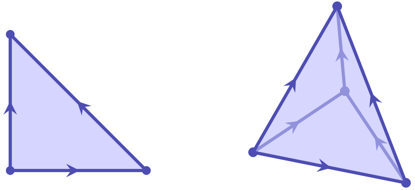

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- (details for smooth, oriented, . . .)

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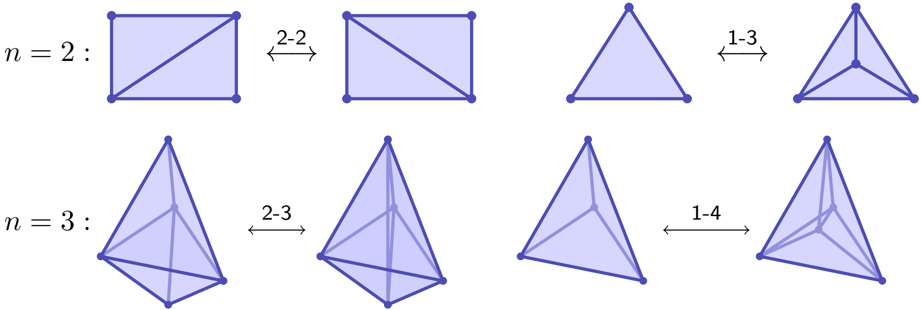

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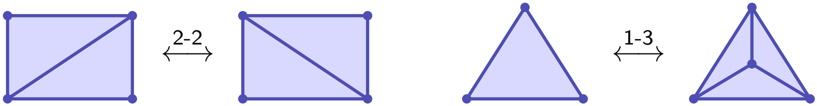

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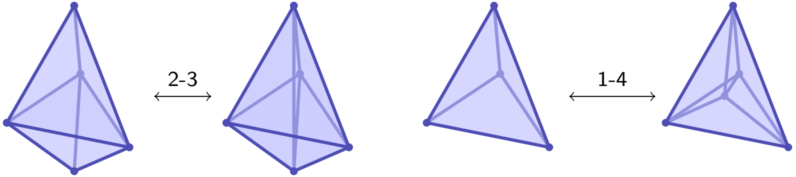

Theorem.
If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.

Pachner 1991

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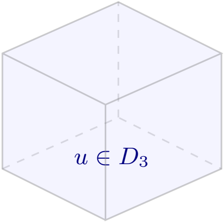

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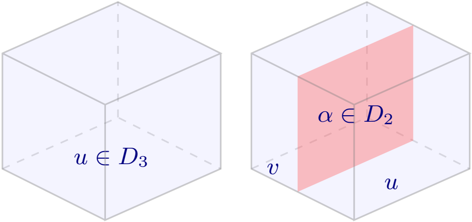

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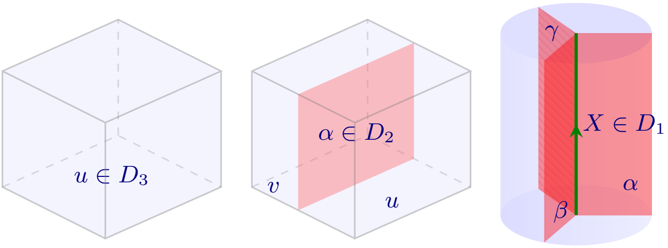

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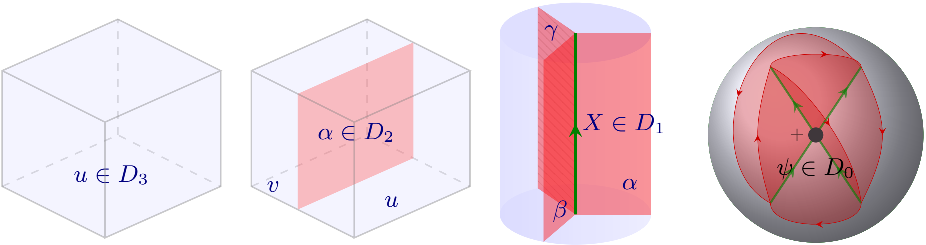

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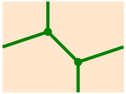
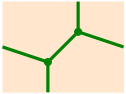
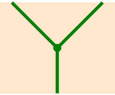
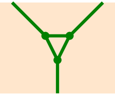
$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$ $\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

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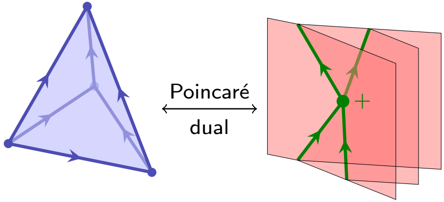

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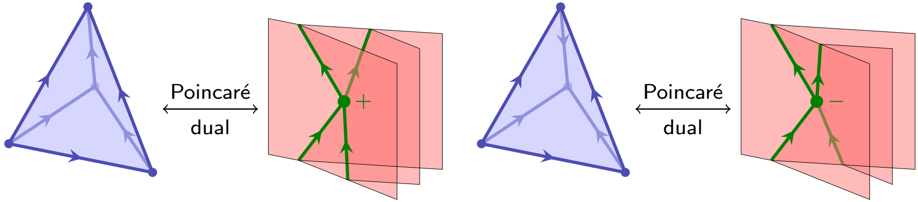

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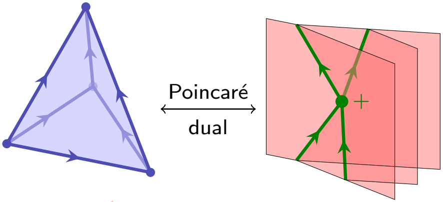

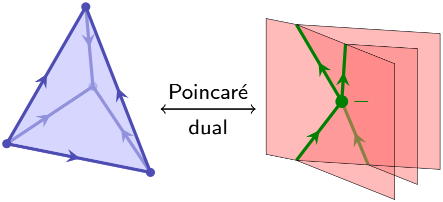

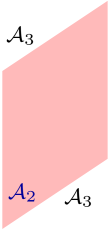

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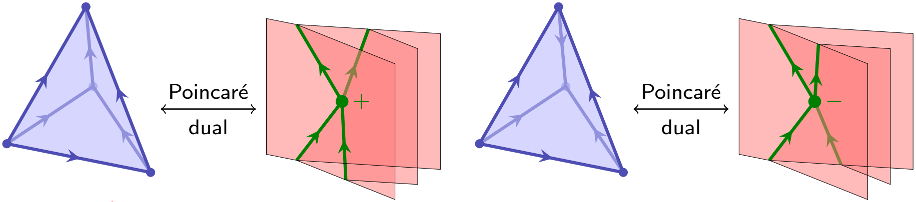

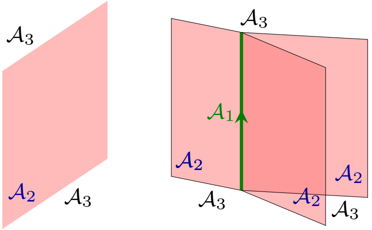

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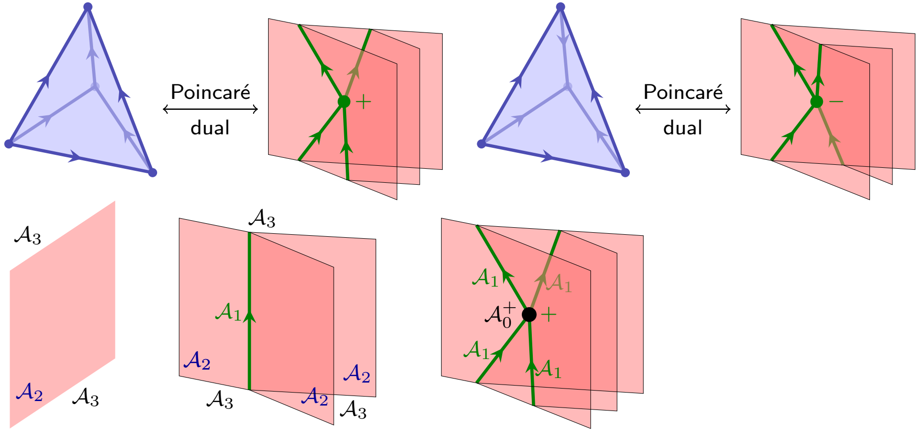

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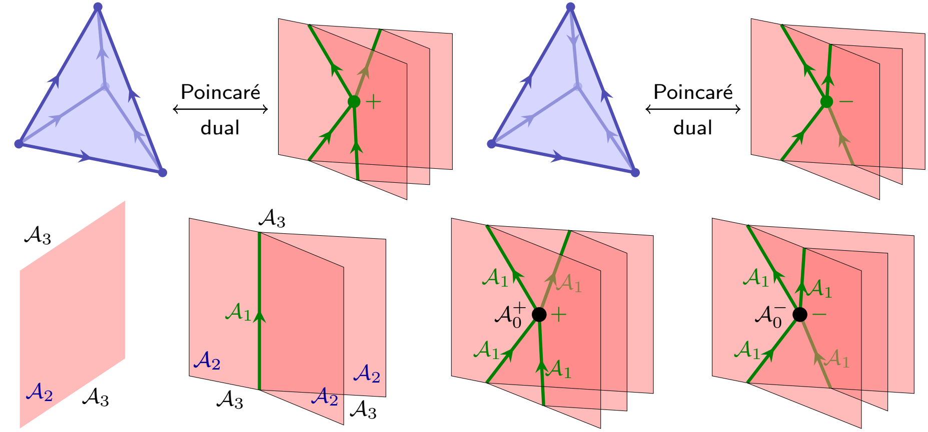

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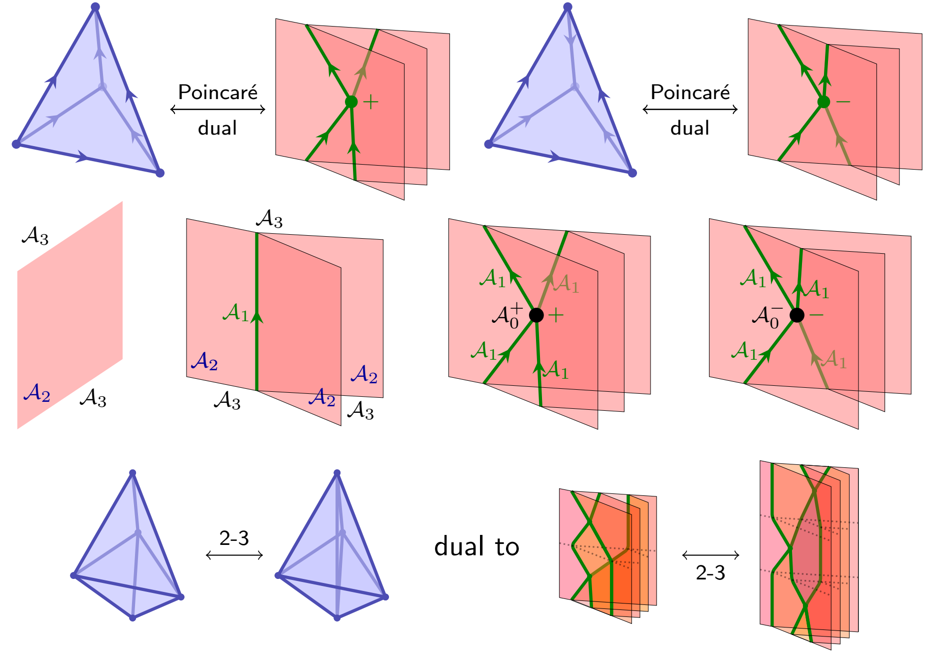

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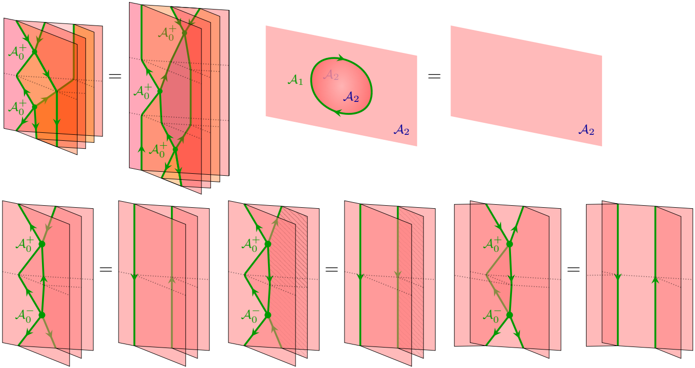
Theorem.
For $n = 3$, it is sufficient that under $\mathcal{Z}$:



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- G-crossed modular tensor categories $\mathcal{M}_G^{\times} = \bigoplus_{g \in G} \mathcal{M}_g$ (work in progress)

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  - $\mathcal{A}_2 = \mathcal{A}$
  - $\mathcal{A}_1 = \otimes\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$
  - $\mathcal{A}_0^{\pm}$ = associator (+ details. . .)
- **Reshetikhin-Turaev theory** (= quantised Chern-Simons theory) from modular tensor category $\mathcal{M}$ (e. g. $\mathcal{M} = \widehat{\mathfrak{sl}}(2)_k$):
  - $D_3 = \{\mathcal{M}\}$
  - $D_2 = \{$separable symmetric Frobenius algebras $A \in \mathcal{M}\}$
  - $D_1 = \{$cyclic modules$\}$
  - $D_0 = \{$bimodule maps$\}$
  - orbifold datum for $\widehat{\mathfrak{sl}}(2)_k$: $A = (0) \oplus (k)$ for $k = 0 \bmod 4$
    $A = (0) \oplus A_{\mathbb{Z}_2}$ for $k = 2 \bmod 4$
- G-crossed modular tensor categories $\mathcal{M}_G^{\times} = \bigoplus_{g\in G} \mathcal{M}_g$ (work in progress)
- **topological quantum computation**: $\mathcal{M} = \mathcal{C}^{\boxtimes n}$ (work in progress)

Carqueville/Runkel/Schaumann 2017–201?