A differential model for B-type Landau-Ginzburg theories

Mirela Babalic, Dmitry Doryn, Calin Lazaroiu, Mehdi Tavakol

IBS – Center for Geometry and Physics, Pohang

Seminar 'Geometry & Physics @ DFT'
http://events.theory.nipne.ro/gap/index.php/seminar

21.07.2017

We study general open-closed B-type Landau-Ginzburg models (including their coupling to topological D-branes), without making unnecessary assumptions.

Classical oriented open-closed topological Landau-Ginzburg theories of type B are classical field theories, defined on compact oriented Riemann surfaces Σ with corners and parameterized by Landau-Ginzburg pairs (X, W).
It is expected that such theories admit a non-anomalous quantization when X is a Calabi-Yau manifold.

Definition

A *Landau-Ginzburg pair* is a pair (X, W) such that:

- X is a non-compact Kählerian manifold whose canonical line bundle is holomorphically trivial.
- $W : X \to \mathbb{C}$ is a non-constant holomorphic function.

Definition

The *critical set of* W is the set of critical points of W:

$$Z_W = \{p \in X | (\partial W)(p) = 0\} .$$

A physically acceptable quantization procedure must produce a quantum oriented open-closed topological field theory which can be described equivalently by an algebraic structure called a *TFT datum*.

We want to find the precise realization of the TFT datum for these models.

Limitations of previous work

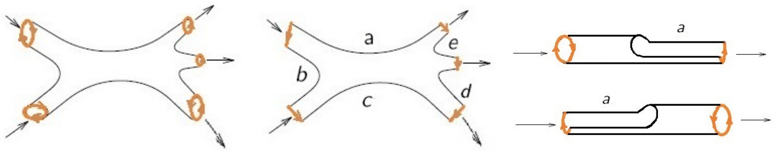
All previous work assumed algebraicity of X and W and most of it was limited to very simple examples such as $X = \mathbb{C}^d$. It was also assumed that the critical points of the superpotential W are isolated.

We do not require that X is algebraic, since there is no Physics reason to do so. Moreover, we require only that the critical locus of W is compact.

A non-anomalous oriented 2-dimensional open-closed TFT can be described axiomatically as a monoidal functor from a certain category Cob_2 of oriented cobordisms with corners to the category of finite-dimensional vector spaces over $\mathbb{C}$:

$$Z : (\mathrm{Cob}_2, \sqcup, \emptyset) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$$

The objects of Cob_2 are finite disjoint unions of oriented circles and oriented closed intervals. The morphisms are compact oriented smooth 2-manifolds with corners (corresponding to the worldsheets of open and closed strings). The corners coincide with the boundary points of the intervals.



The labels associated to the ends of the intervals indicate the corresponding boundary conditions (or the corresponding D-branes).

TFT data

Theorem

Any (non-anomalous) oriented 2-dimensional open-closed TFT can be described equivalently by an algebraic structure known as a TFT datum.

Definition

A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

- $\mathcal{H}$ = **bulk algebra**, a finite-dimensional supercommutative $\mathbb{C}$-superalgebra with unit $1_{\mathcal{H}}$ (the space of on-shell states of the closed oriented topological string)
- $\mathcal{T}$ = **category of topological D-branes**, a Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category, with composition of morphisms denoted by $\circ$ and units:

  $$1_a \in \mathrm{Hom}_{\mathcal{T}}(a, a) \ , \ \forall a \in \mathrm{Ob}\mathcal{T}$$

  Here $\mathrm{Hom}_{\mathcal{T}}(a, b)$ is the space of on-shell boundary states (corresponding to the *open* oriented topological string stretching between the D-branes a and b)
- $e = (e_a)_{a \in \mathrm{Ob}\mathcal{T}}$, a family of $\mathbb{C}$-linear **bulk-boundary maps** $e_a : \mathcal{H} \to \mathrm{Hom}_{\mathcal{T}}(a, a)$ such that the following conditions are satisfied:
  - For any $a \in \mathrm{Ob}\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the algebra $(\mathrm{End}_{\mathcal{T}}(a), \circ)$, where $\mathrm{End}_{\mathcal{T}}(a) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathcal{T}}(a, a)$.
  - For any $a, b \in \mathrm{Ob}\mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous bulk state $h \in \mathcal{H}$ and any $\mathbb{Z}_2$-homogeneous elements $t \in \mathrm{Hom}_{\mathcal{T}}(a, b)$, we have:

    $$e_b(h) \circ t = (-1)^{\deg h \deg t} t \circ e_a(h).$$

Definition

A Calabi-Yau supercategory of parity $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in Ob\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-degree μ

$$\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$$

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$, the $\mathbb{C}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a, b) \times \mathrm{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$$

  defined through:

$$\langle t_1, t_2 \rangle_{a,b} = \mathrm{tr}_b(t_1 \circ t_2),\ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b),\ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$$

  is non-degenerate.

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$, we have:

$$\langle t_1, t_2 \rangle_{a,b} = (-1)^{\mathrm{deg} t_1 \mathrm{deg} t_2} \langle t_2, t_1 \rangle_{b,a}$$

Definition

A TFT datum of parity $\mu \in \mathbb{Z}_2$ is a system $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a **pre-TFT datum**
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the **bulk trace**)
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called **boundary traces**)

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra, meaning that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate (i.e. the condition $\mathrm{Tr}(hh') = 0$ for all $h' \in \mathcal{H}$ implies $h = 0$)
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi-Yau supercategory of parity μ.
- The so-called *topological Cardy constraint* holds for all $a, b \in \mathrm{Ob}\mathcal{T}$.

The topological Cardy constraint has the form:

$$\boxed{\mathrm{Tr}(f_a(t_a)f_b(t_b)) = \mathrm{str}(\Phi_{ab}(t_a,t_b)) \ \ , \ \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)}$$

where:

- "str" is the supertrace on the $\mathbb{Z}_2$-graded vector space $End_{\mathbb{C}}(Hom_{\mathcal{T}}(a,b))$
- $f_a : \mathrm{Hom}_{\mathcal{T}}(a,a) \to \mathcal{H}$ is the *boundary-bulk map of a*, which is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to Hom_{\mathcal{T}}(a,a)$ with respect to Tr and tr:

$$\mathrm{Tr}(hf_a(t_a)) = \mathrm{tr}_a(e_a(h) \circ t_a), \ \forall h \in \mathcal{H}, \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a)$$

- $\Phi_{ab}(t_a,t_b) : \mathrm{Hom}_{\mathcal{T}}(a,b) \to \mathrm{Hom}_{\mathcal{T}}(a,b)$ is the $\mathbb{C}$-linear map defined through:

$$\Phi_{ab}(t_a,t_b)(t) = t_b \circ t \circ t_a \ , \ \forall t \in \mathrm{Hom}_{\mathcal{T}}(a,b) \ , \ \forall t_a \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_b \in \mathrm{Hom}_{\mathcal{T}}(b,b)$$

Definition

A Landau-Ginzburg (LG) pair of dimension d is a pair (X, W), where:

1. X is a non-compact Kählerian manifold of complex dimension d which is *weakly Calabi-Yau* in the sense that the canonical line bundle $K_X = \wedge^d T^*X$ is holomorphically trivial.
2. $W : X \to \mathbb{C}$ is a *non-constant* complex-valued holomorphic function defined on X.

The *signature* $\mu(X, W)$ of a Landau-Ginzburg pair (X, W) is defined as the mod 2 reduction of the complex dimension of X:

$$\mu(X, W) = \hat{d} \in \mathbb{Z}_2$$

Definition

The *critical set of W* is the set:

$$Z_W = \{p \in X | (\partial W)(p) = 0\}$$

of critical points of W.

Definition

Let (X, W) be a Landau-Ginzburg pair with $\dim_{\mathbb{C}} X = d$. The space of polyvector valued forms is defined through:

$$\mathrm{PV}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathrm{PV}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathcal{A}^j(X, \wedge^{|i|} TX)$$

where $\mathcal{A}^j(X, \wedge^{|i|} TX) \equiv \Omega^{0,j}(X)$.

We denote by TX and $\bar{T}X$ are the holomorphic and antiholomorphic tangent bundles of X and by T^*X and $\bar{T}^*X$ the corresponding cotangent bundles. Let $z = (z_1, \ldots, z_d)$ be local holomorphic coordinates defined on $U \subset X$ and $\partial_k := \frac{\partial}{\partial z_k}$, $\overline{\partial}_k := \frac{\partial}{\partial \bar{z}_k}$, then:

$$TX|_U = \mathrm{Span}_{\mathbb{C}}\left\{\partial_1, \ldots, \partial_d\right\} \quad , \quad \overline{T}X|_U = \mathrm{Span}_{\mathbb{C}}\left\{\overline{\partial}_1, \ldots, \overline{\partial}_d\right\} \quad ,$$
$$T^*X|_U = \mathrm{Span}_{\mathbb{C}}\{\mathrm{d}z_1, \ldots, \mathrm{d}z_d\} \quad , \quad \overline{T}^*X|_U = \mathrm{Span}_{\mathbb{C}}\{\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_d\} \quad .$$

A polyvector valued form $\omega \in \mathrm{PV}^{i,j}(X)$ expands as:

$$\omega =_U \sum_{|I|=-i, |J|=j} \omega^I{}_J \mathrm{d}\bar{z}_J \otimes \partial_I \quad , \quad \omega^I{}_J \in C^\infty(X)$$

$$\mathrm{d}\bar{z}_J \overset{\mathrm{def.}}{=} \mathrm{d}\bar{z}_{t_1} \wedge \mathrm{d}\bar{z}_{t_2} \wedge \cdots \wedge \mathrm{d}\bar{z}_{t_j} \quad , \quad \partial_I \overset{\mathrm{def.}}{=} \partial_{t_1} \wedge \ldots \wedge \partial_{t_{|I|}}$$
$$\omega \wedge \eta =_U \sum_{|I|,|J|,|K|,|L|} (-1)^{il} \omega^I{}_J \eta^K{}_L (\mathrm{d}\bar{z}_J \wedge \mathrm{d}\bar{z}_L) \otimes (\partial_J \wedge \partial_K) \quad .$$

The **twisted Dolbeault differential** determined by W on $\mathrm{PV}(X)$:

$$\delta_W : \mathrm{PV}(X) \to \mathrm{PV}(X)$$

is defined through $\delta_W = \boldsymbol{\overline{\partial}} + \boldsymbol{\iota}_W$, where:

- $\boldsymbol{\overline{\partial}}$ is the (antiholomorphic) Dolbeault operator of $\wedge TX$, which satisfies $\boldsymbol{\overline{\partial}}(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i,j+1}(X)$

$$\overline{\partial}\omega =_U \sum_{|I|=-i,|J|=j} [(\overline{\partial}\omega^I_J) \wedge \mathrm{d}\bar{z}_J] \otimes \partial_I = \sum_{|I|=-i,|J|=j} \sum_{r=1}^{d} (\overline{\partial}_r \omega^I_J)(\mathrm{d}\bar{z}_r \wedge \mathrm{d}\bar{z}_J) \otimes \partial_I$$

- $\boldsymbol{\iota}_W = -\mathrm{i}(\partial W) \lrcorner$, which satisfies $\boldsymbol{\iota}_W(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i+1,j}(X)$

$$\boldsymbol{\iota}_W \omega = -\mathrm{i}\, \iota_{\partial W}\omega =_U -\mathrm{i} \sum_{r=1}^{d} (\partial_r W) \mathrm{d}z^r \lrcorner \omega$$

Notice that $(\mathrm{PV}(X), \boldsymbol{\overline{\partial}}, \boldsymbol{\iota}_W)$ is a bicomplex since:

$$\delta_W^2 = \boldsymbol{\overline{\partial}}^2 = \boldsymbol{\iota}_W^2 = \boldsymbol{\overline{\partial}}\boldsymbol{\iota}_W + \boldsymbol{\iota}_W \boldsymbol{\overline{\partial}} = 0$$

Definition

The twisted Dolbeault algebra of polyvectors of the Landau-Ginzburg pair (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear dG algebra $(\mathrm{PV}(X), \delta_W)$, where $\mathrm{PV}(X)$ is endowed with the canonical $\mathbb{Z}$-grading.

Definition

The cohomological twisted Dolbeault algebra of (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear algebra defined through:

$$\mathrm{HPV}(X, W) = H(\mathrm{PV}(X), \delta_W)$$

We will use the following notations:
$\mathrm{O}(X)=$ the algebra of complex-valued holomorphic functions defined on X,
$O_X=$ the sheaf of holomorphic complex-valued functions defined on X.

Definition

The *sheaf Koszul complex* of W is the following complex of locally-free sheaves of $\mathcal{O}_X$-modules:

$$(\mathcal{Q}_W): \quad 0 \to \wedge^d TX \stackrel{\iota_W}{\to} \wedge^{d-1} TX \stackrel{\iota_W}{\to} \cdots \stackrel{\iota_W}{\to} \mathcal{O}_X \to 0$$

where $\mathcal{O}_X$ sits in degree zero and we identify the exterior power $\wedge^k TX$ with its locally-free sheaf of holomorphic sections.

Proposition

Let $\mathbb{H}(\mathcal{Q}_W)$ denote the hypercohomology of the Koszul complex $\mathcal{Q}_W$. There exists a natural isomorphism of $\mathbb{Z}$-graded $\mathrm{O}(X)$-modules:

$$\mathrm{HPV}(X, W) \cong_{\mathrm{O}(X)} \mathbb{H}(Q_W)$$

where $\mathrm{HPV}(X, W)$ is endowed with the canonical $\mathbb{Z}$-grading. Thus:

$$H^k(\mathrm{PV}(X), \delta_W) \cong_{\mathrm{O}(X)} \mathbb{H}^k(\mathcal{Q}_W), \ \forall k \in \{-d, \ldots, d\}$$

Moreover, we have:

$$\mathbb{H}^k(\mathcal{Q}_W) = \bigoplus_{i+j=k} \mathbf{E}_\infty^{i,j}$$

where $(\mathbf{E}_r^{i,j}, d_r)_{r \geq 0}$ is a spectral sequence which starts with:

$$\mathbf{E}_0^{i,j} := \mathrm{PV}^{i,j}(X) = \mathcal{A}^j(X, \wedge^{|i|} TX), \ \ \mathrm{d}_0 = \overline{\boldsymbol{\partial}} \ \ , \ (i = -d, \ldots, 0, \ j = 0, \ldots, d)$$

$$\begin{array}{ccccccc}
\mathbf{E}_0^{-d,d} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,d} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,d} & \cdots\cdots & \mathbf{E}_0^{0,d} \\
\vdots & & \vdots & & \vdots & \iddots & \vdots \\
\mathbf{E}_0^{-d,2} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,2} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,2} & \cdots\cdots & \mathbf{E}_0^{0,2} \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow \\
\mathbf{E}_0^{-d,1} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,1} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,1} & \cdots\cdots & \mathbf{E}_0^{0,1} \\
\bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow & & \bar{\partial}\uparrow \\
\mathbf{E}_0^{-d,0} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,0} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,0} & \cdots\cdots & \mathbf{E}_0^{0,0}
\end{array}$$

The zeroth page of the spectral sequence.

Definition

A holomorphic vector superbundle on X is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on X, i.e. a complex holomorphic vector bundle E endowed with a direct sum decomposition $E = E^{\hat{0}} \oplus E^{\hat{1}}$, where $E^{\hat{0}}$ and $E^{\hat{1}}$ are holomorphic sub-bundles of E.

Definition

A holomorphic factorization of W is a pair $a = (E, D)$, where $E = E^{\hat{0}} \oplus E^{\hat{1}}$ is a holomorphic vector superbundle on X and $D \in \Gamma(X, End^{\hat{1}}(E))$ is a holomorphic section of the bundle $End^{\hat{1}}(E) = Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \subset End(E)$ which satisfies the condition $D^2 = W\mathrm{id}_E$.

Let $a = (E, D)$ be a holomorphic factorization of W. Decomposing $E = E^{\hat{0}} \oplus E^{\hat{1}}$, the condition that D is odd implies:

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$$

where $u \in \Gamma(X, Hom(E^{\hat{0}}, E^{\hat{1}}))$ and $v \in \Gamma(X, Hom(E^{\hat{1}}, E^{\hat{0}}))$. The condition $D^2 = W\mathrm{id}_E$ amounts to:

$$v \circ u = W\mathrm{id}_{E^{\hat{0}}} \;, \quad u \circ v = W\mathrm{id}_{E^{\hat{1}}}$$

Definition

The twisted Dolbeault category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category $\mathrm{DF}(X, W)$ defined as follows:

- The objects of $\mathrm{DF}(X, W)$ are the holomorphic factorizations of W.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$:

$$\mathrm{Hom}_{\mathrm{DF}(X,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \mathcal{A}(X, Hom(E_1, E_2))$$

  endowed with the total $\mathbb{Z}_2$-grading and with the twisted differentials δ_{a_1,a_2}:

$$\begin{aligned}
&\delta_{a_1,a_2} \stackrel{\mathrm{def.}}{=} \boldsymbol{\overline{\partial}}_{a_1,a_2} + \mathfrak{d}_{a_1,a_2} \quad , \quad \textit{where} \\
&\boldsymbol{\overline{\partial}}_{a_1,a_2} := \boldsymbol{\overline{\partial}}_{Hom(E_1,E_2)} \\
&\mathfrak{d}_{a_1,a_2}(\rho \otimes f) = (-1)^{\mathrm{rk}\rho} \rho \otimes (D_2 \circ f) - (-1)^{\mathrm{rk}\rho + \sigma(f)} \rho \otimes (f \circ D_1)
\end{aligned}$$

- The composition of morphisms
  $\circ : \mathcal{A}(X, Hom(E_2, E_3)) \times \mathcal{A}(X, Hom(E_1, E_2)) \to \mathcal{A}(X, Hom(E_1, E_3))$

$$(\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f)\,\mathrm{rk}\eta} (\rho \wedge \eta) \otimes (f \circ g)$$

  $\forall \rho, \eta \in \mathcal{A}(X)$, $\forall f \in \Gamma_\infty(X, Hom(E_2, E_3))$, $\forall g \in \Gamma_\infty(X, Hom(E_1, E_2))$.

$$\delta^2 = \boldsymbol{\overline{\partial}}^2 = \mathfrak{d}^2 = \boldsymbol{\overline{\partial}} \circ \mathfrak{d} + \mathfrak{d} \circ \boldsymbol{\overline{\partial}} = 0$$

Definition

The cohomological twisted Dolbeault category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear algebra defined through:

$$\mathrm{HDF}(X, W) = H(\mathrm{DF}(X), \delta_{a_1,a_2})$$

Theorem

Suppose that the critical set Z_W is compact. Then the cohomology algebra $\mathrm{HPV}(X, W)$ of $(\mathrm{PV}(X), \delta_W)$ is finite-dimensional over $\mathbb{C}$ while the total cohomology category $\mathrm{HDF}(X, W)$ of $\mathrm{DF}(X, W)$ is Hom-finite over $\mathbb{C}$. Moreover, the system $(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$ obeys the defining properties of a TFT datum (up to non-degeneracy of the bulk and boundary traces and the topological Cardy constraint which remain to be proven).

In our papers we gave explicit formulas for the objects Tr, tr and e of the Landau-Ginzburg TFT datum.

Conjecture

Suppose that Z_W is compact. Then $(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$ is a TFT datum and hence defines a quantum open-closed TFT.

Definition

Let X be a complex manifold with $\dim_{\mathbb{C}} X = d$. We say that X is a **Stein manifold** if the following three conditions are satisfied:

- Holomorphic functions separate points of X.
- X is holomorphically convex.
- For every point $x \in X$ there exist globally-defined holomorphic functions $f_1, \ldots, f_d \in \mathrm{O}(X)$ whose differentials $\mathrm{d}f_j$ are linearly independent at x.

Example

- $\mathbb{C}^d$ is a Stein manifold
- Every domain of holomorphy in $\mathbb{C}^d$ is a Stein manifold
- Every closed complex submanifold of a Stein manifold is a Stein manifold
- Every Stein manifold X of complex dimension d can be embedded in $\mathbb{C}^{2d+1}$ through a biholomorphic proper map
- A complex manifold is Stein iff it is biholomorphic to a closed complex submanifold of $\mathbb{C}^N$ for some N.

Cartan's theorem B

For every coherent analytic sheaf $\mathcal{F}$ on a Stein manifold X, the cohomology $\mathrm{H}^i(X,\mathcal{F}) = 0$ for all $i > 0$.

Theorem

Suppose that X is Stein. Then the spectral sequence defined previously collapses at E_2 and $\mathrm{HPV}(X, W)$ is concentrated in non-positive degrees.
For all $k = -d, \dots, 0$, the $\mathrm{O}(X)$-module $\mathrm{HPV}^k(X)$ is isomorphic with the cohomology at position k of the following sequence of finitely-generated projective $\mathrm{O}(X)$-modules:

$$(\mathcal{P}_W): \quad 0 \to H^0(X, \wedge^d TX) \stackrel{\iota_W}{\to} \cdots \stackrel{\iota_W}{\to} H^0(X, TX) \stackrel{\iota_W}{\to} \mathrm{O}(X) \to 0$$

where $\mathrm{O}(X)$ sits in position zero.

An analytic model for the bulk algebra in the Stein case

Proof: Since X is Stein, Cartan's theorem B implies $\mathbf{E}_1^{i,j} = \mathrm{H}^j_{\boldsymbol{\overline{\partial}}}(\mathcal{A}(X, \wedge^{|i|} TX)) = 0$ for $j > 0$ and all $i = -d, \ldots, 0$. Thus the only non-trivial row of the page $\mathbf{E}_1$ of the spectral sequence is the bottom row $\mathbf{E}_1^{\bullet,0}$, whose nodes are given by:

$$\mathbf{E}_1^{i,0} := \mathrm{H}^0_{\boldsymbol{\overline{\partial}}}(\mathcal{A}(X, \wedge^{|i|} TX)) = \mathrm{H}_{\boldsymbol{\overline{\partial}}}(\mathrm{PV}^{i,0}(X)) = \Gamma(X, \wedge^{|i|} TX) = \mathrm{H}^0(\wedge^{|i|} TX)$$

Thus page $\mathbf{E}_1$ reduces to:

$$\begin{array}{ccccccc}
\mathbf{E}_1^{-d,d} = 0 & \longrightarrow & \mathbf{E}_1^{-d+1,d} = 0 & \longrightarrow & \mathbf{E}_1^{-d+2,d} = 0 & \cdots & \mathbf{E}_1^{0,d} = 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
\mathbf{E}_1^{-d,2} = 0 & \longrightarrow & \mathbf{E}_1^{-d+1,2} = 0 & \longrightarrow & \mathbf{E}_0^{-d+2,2} = 0 & \cdots & \mathbf{E}_1^{0,2} = 0 \\
\uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} \\
\mathbf{E}_1^{-d,1} = 0 & \longrightarrow & \mathbf{E}_1^{-d+1,1} = 0 & \longrightarrow & \mathbf{E}_1^{-d+2,1} = 0 & \cdots & \mathbf{E}_1^{0,1} = 0 \\
\uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} \\
\mathbf{E}_1^{-d,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_1^{-d+1,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_1^{-d+2,0} & \cdots & \mathbf{E}_1^{0,0}
\end{array}$$

Hence the spectral sequence collapses at $\mathbf{E}_2$ and we have
$\mathbf{E}_\infty^k = \mathbf{E}_2^{k,0} = \mathrm{H}^k_{\boldsymbol{\iota}_W}(\mathbf{E}_1^{\bullet,0}) = \mathrm{H}^k(\mathcal{P}_W)$ for all $k = -d, \ldots, 0$.
Since $\wedge^k TX$ are vector bundles, the Serre-Swan theorem for Stein manifolds implies that $(\mathcal{P}_W)$ is a sequence of finitely-generated projective $\mathrm{O}(X)$-modules.

Proposition

Suppose that X is Stein and $\dim_{\mathbb{C}} Z_W = 0$. Then $\mathrm{HPV}^k(X) = 0$ for $k \neq 0$ and there exists a natural isomorphism of $\mathrm{O}(X)$-modules:

$$\mathrm{HPV}^0(X) \simeq_{\mathrm{O}(X)} H^0(Jac_W) = \mathrm{Jac}(X, W) \quad .$$

Definition

- $\mathcal{J}_W \stackrel{\text{def.}}{=} \mathrm{im}(\boldsymbol{\iota}_W : TX \to O_X)$ (the critical sheaf of W)
- $Jac_W \stackrel{\text{def.}}{=} O_X/\mathcal{J}_W$ (the Jacobi sheaf of W)
- $\mathrm{Jac}(X, W) \stackrel{\text{def.}}{=} \Gamma(X, Jac_W)$ (the Jacobi algebra of (X, W))

Definition

The holomorphic dG category of holomorphic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category $\mathrm{F}(X, W)$ defined as follows:

- The objects are the holomorphic factorizations of W.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$, $a_2 = (E_2, D_2)$ of W:

$$\mathrm{Hom}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom(E_1, E_2))$$

  endowed with the $\mathbb{Z}_2$-grading with homogeneous components:

$$\mathrm{Hom}^{\kappa}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom^{\kappa}(E_1, E_2)), \forall \kappa \in \mathbb{Z}_2$$

  and with the differentials d_{a_1,a_2} determined uniquely by the condition:

$$\mathrm{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^{\kappa} f \circ D_1, \forall f \in \Gamma(X, Hom^{\kappa}(E_1, E_2)), \forall \kappa \in \mathbb{Z}_2$$

- The composition of morphisms (induced by that of $\mathrm{VB}(X)$, which is the *full* subcategory of $\mathrm{Coh}(X)$ whose objects are the locally-free sheaves of finite rank.)

Theorem

Suppose that X is Stein. Then $\mathrm{HDF}(X, W)$ and the cohomological category of holomorphic factorizations $\mathrm{HF}(X, W) \stackrel{\mathrm{def.}}{=} H(\mathrm{F}(X, W))$ are equivalent.

Definition

An $\mathrm{O}(X)$-*supermodule* is a $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-module M endowed with a direct sum decomposition $M = M^{\hat{0}} \oplus M^{\hat{1}}$ into submodules.

$\mathrm{O}(X)$-supermodules form an $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded category $\mathrm{Mod}^s_{\mathrm{O}(X)}$ if we define the Hom space $\mathrm{Hom}(M_1, M_2)$ from a supermodule M_1 to a supermodule M_2 to be the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-module with homogeneous components:

$$\mathrm{Hom}^{\hat{0}}(M_1, M_2) \stackrel{\text{def.}}{=} \mathrm{Hom}(M_1^{\hat{0}}, M_2^{\hat{0}}) \oplus \mathrm{Hom}(M_1^{\hat{1}}, M_2^{\hat{1}})$$
$$\mathrm{Hom}^{\hat{1}}(M_1, M_2) \stackrel{\text{def.}}{=} \mathrm{Hom}(M_1^{\hat{0}}, M_2^{\hat{1}}) \oplus \mathrm{Hom}(M_1^{\hat{1}}, M_2^{\hat{0}}) \quad .$$

The composition is defined in the obvious manner. Given an $\mathrm{O}(X)$-supermodule M:

$$\mathrm{End}(M) \stackrel{\text{def.}}{=} \mathrm{Hom}(M, M) \ .$$

Definition

An $\mathrm{O}(X)$-supermodule $M = M^{\hat{0}} \oplus M^{\hat{1}}$ is called *finitely-generated* if both of its $\mathbb{Z}_2$-homogeneous components $M^{\hat{0}}$ and $M^{\hat{1}}$ are finitely-generated over $\mathrm{O}(X)$. It is called *projective* if both $M^{\hat{0}}$ and $M^{\hat{1}}$ are projective $\mathrm{O}(X)$-modules.

Let $\mathrm{Mod}^s_{\mathrm{O}(X)}$ denote the category of $\mathrm{O}(X)$-supermodules and $\mathrm{mod}^s_{\mathrm{O}(X)}$ denote the full sub-category of finitely-generated $\mathrm{O}(X)$-supermodules.

Definition

A projective analytic factorization of W is a pair (P, D), where P is a finitely-generated projective $\mathrm{O}(X)$-supermodule and $D \in \mathrm{End}^1_{\mathrm{O}(X)}(P)$ is an odd endomorphism of P such that $D^2 = W\mathrm{id}_P$.

Definition

The dG category $\mathrm{PF}(X, W)$ of projective analytic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category defined as follows:

- The objects are the projective analytic factorizations of W.
- Given two projective analytic factorizations (P_1, D_1) and (P_2, D_2) of W:

$$\mathrm{Hom}_{\mathrm{PF}(X,W)}((P_1, D_1), (P_2, D_2)) = \Gamma(X, \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)) \quad ,$$

  endowed with the $\mathbb{Z}_2$-grading and with the $\mathrm{O}(X)$-linear odd differential $\mathrm{d} := \mathrm{d}_{(P_1,D_1),(P_2,D_2)}$ determined uniquely by the condition:

$$\mathrm{d}(f) = D_2 \circ f - (-1)^{\mathrm{deg} f} f \circ D_1$$

  for all elements $f \in \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$ which have pure $\mathbb{Z}_2$-degree.
- The corresponding composition of morphisms (inherited from $\mathrm{mod}^s_{\mathrm{O}(X)}$).

Definition

The cohomological category $\mathrm{HPF}(X, W)$ of analytic projective factorizations of W is the total cohomology category $\mathrm{HPF}(X, W) \stackrel{\mathrm{def.}}{=} H(\mathrm{PF}(X, W))$, which is a $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear category.

Theorem

The categories $\mathrm{HDF}(X, W)$ and $\mathrm{HPF}(X, W)$ are equivalent when X is Stein. When X is Stein and Z_W is compact, the category of topological D-branes of the B-type Landau-Ginzburg theory can be identified with $\mathrm{HPF}(X, W)$.

The worldsheet Lagrangian

The bulk action is:

$$\tilde{S}_{bulk} = S_B + S_W + s \quad ,$$

where:

$$\begin{aligned} S_B &= \int_\Sigma d^2\sigma\sqrt{g}\Big[G_{i\bar{j}}\Big(g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \mathrm{i}\varepsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \frac{1}{2}g^{\alpha\beta}\rho^i_\alpha D_\beta\eta^{\bar{j}} \\ &\quad -\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha D_\beta\theta^{\bar{j}} - \tilde{F}^i\tilde{F}^{\bar{j}}\Big) + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}R_{i\bar{l}k\bar{j}}\rho^i_\alpha\bar{\chi}^{\bar{l}}\rho^k_\beta\chi^{\bar{j}}\Big] \end{aligned}$$

is the action of the B-twisted sigma model and $S_W = S_0 + S_1$ is the potential-dependent term, with:

$$\begin{aligned} S_0 &= -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\bar{\chi}^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}}\Big] \\ S_1 &= -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[(\partial_i W)\tilde{F}^i + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}D_i\partial_j W\rho^i_\alpha\rho^j_\beta\Big] \quad . \end{aligned}$$

Here:

$$s := \mathrm{i}\int_\Sigma d^2\sigma\sqrt{g}\varepsilon^{\alpha\beta}\partial_\alpha(G_{\bar{i}j}\chi^{\bar{i}}\rho^j_\beta) = \mathrm{i}\int_\Sigma d(G_{\bar{i}j}\chi^{\bar{i}}\rho^j) \quad .$$

is a correction needed to solve the so-called "Warner problem".

The fields involved are:

- the Grassmann even fields:
  - the scalar field $\phi : \Sigma \to X$
  - the Riemannian metric g on Σ,
  - the auxiliary fields $\tilde{F} \in \Gamma_\infty(\phi^*(\mathcal{T}_\mathbb{C} X))$
- the Grassmann odd fields:
  - $\eta, \chi, \bar{\chi} \in \Gamma_\infty(\phi^*(\bar{T}X))$, $\theta \in \Gamma_\infty(\phi^*(T^*X))$, $\rho \in \Gamma_\infty(\phi^*(TX) \otimes \mathcal{T}^*\Sigma)$

Here $\mathcal{T}X$ is the real tangent bundle of X and $\mathcal{T}_\mathbb{C} X = \mathcal{T}X \otimes \mathbb{C} = TX \oplus \bar{T}X$ is its complexification, while TX and $\bar{T}X$ are the holomorphic and antiholomorphic tangent bundles of X. $\mathcal{T}\Sigma$ is the real tangent bundle of Σ.

Consider a complex superbundle $E = E^{\hat{0}} \oplus E^{\hat{1}}$ on X and a superconnection $\mathcal{B}$ on E. The bundle $End(E)$ is $\mathbb{Z}_2$-graded:

$$
\begin{aligned}
End^{\hat{0}}(E) &:= End(E^{\hat{0}}) \oplus End(E^{\hat{1}}) \\
End^{\hat{1}}(E) &:= Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \quad .
\end{aligned}
$$

In a local frame of E compatible with the grading, $\mathcal{B}$ corresponds to a matrix:

$$
\mathcal{B} = \begin{bmatrix} A^{(+)} & v \\ u & A^{(-)} \end{bmatrix}
$$

where v and u are smooth sections of $Hom(E^{\hat{1}}, E^{\hat{0}})$ and $Hom(E^{\hat{0}}, E^{\hat{1}})$, while the diagonal entries $A^{(+)}$ and $A^{(-)}$ are connection one-forms on $E^{\hat{0}}$ and $E^{\hat{1}}$, such that $A^{(+)} \in \Omega^{(0,1)}(End(E^{\hat{0}}))$ and $A^{(-)} \in \Omega^{(0,1)}(End(E^{\hat{1}}))$.

We define the partition function on an oriented Riemann surface Σ with corners by:

$$
Z := \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta] e^{-\tilde{S}_{bulk}} \mathcal{U}_1 \dots \mathcal{U}_h \quad ,
$$

where h is the number of holes and the factors $\mathcal{U}_h$ have complicated expressions depending on the superconnection $\mathcal{B}$ and the fields as well as on "boundary condition changing operators" inserted at the corners of each hole. ($\mathcal{U}_1 \dots \mathcal{U}_h = e^{-\tilde{S}_{boundary}}$)