Two-field cosmological models and the uniformization theorem

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- Inflation in the early universe can be described reasonably well by so-called α-attractor models. In the two-field version, they arise from cosmological solutions of four-dimensional gravity coupled to a nonlinear sigma model whose scalar manifold Σ is the open unit disk endowed with its unique complete metric $\mathcal{G}$ of Gaussian curvature $K(\mathcal{G}) = -\frac{1}{3\alpha}$ (Poincaré disk).
- The "universal" behavior of such models in the radial one-field truncation close to the conformal boundary of the unit disk is a consequence of the hyperbolic character of $\mathcal{G}$.
- We consider a generalization of such models obtained by replacing the scalar manifold with a general non-compact hyperbolic surface $(\Sigma, \mathcal{G})$. For technical reasons, we concentrate on *geometrically finite* surfaces.
- This generalization is extremely rich.

Consider:

- $(\Sigma, \mathcal{G})$ = oriented, connected, non-compact, borderless and complete two-dimensional Riemannian manifold (called the scalar manifold)
- $\Phi : \Sigma \to \mathbb{R}$ a smooth function (called the scalar potential)

and the Einstein-Scalar theory defined by $(\Sigma, \mathcal{G}, \Phi)$ on an oriented four-manifold X:

$$S[g, \varphi] = \int_X \mathrm{vol}_g \Big[-\frac{M^2}{2}\mathrm{R}(g) - \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) - \Phi \circ \varphi \Big] \qquad (1)$$

where:

- g = Lorentzian metric on X
- $\varphi : X \longrightarrow \Sigma$ is a smooth map which locally describes two real scalar fields
- $\varphi^*(\mathcal{G})$= the pull-back through φ of the metric $\mathcal{G}$ on Σ
- vol_g = the volume form of X w.r.t. g
- M = the reduced Planck mass
- $\mathrm{R}(g)$ = the scalar curvature of g

Let:

- $X = \mathbb{R}^4$ with global coordinates (t, x), $x = (x^1, x^2, x^3)$
- $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \sum_{i=1}^{3} (\mathrm{d}x^i)^2$ (FLRW metric with flat spatial section)
- $\varphi = \varphi(t)$ (independent of x).

The equations of motion reduce to:

$$\nabla_t \dot{\varphi} + 3H\dot{\varphi} + (\mathrm{grad}\Phi) \circ \varphi = 0 \tag{2}$$

$$\frac{1}{3}\dot{H} + H^2 - \frac{\Phi \circ \varphi}{3M^2} = 0 \tag{3}$$

$$\dot{H} + \frac{||\dot{\varphi}||^2}{2M^2} = 0\,, \tag{4}$$

where:

$$H \stackrel{\mathrm{def.}}{=} \frac{\dot{a}}{a} \quad , \quad \nabla_t \stackrel{\mathrm{def.}}{=} \nabla_{\dot{\varphi}(t)} \quad , \quad \dot{} \stackrel{\mathrm{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}t} \quad .$$

$H(t)$ is the Hubble parameter.

Assuming that $H(t) > 0$, we have:

$$H(t) = \frac{1}{M\sqrt{6}} \left[||\dot{\varphi}(t)||^2 + 2\Phi(\varphi(t)) \right]^{1/2}$$

and the e.o.m. reduce to the single equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M}\sqrt{\frac{3}{2}} \left[||\dot{\varphi}(t)||^2 + 2\Phi(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}\Phi)(\varphi(t)) = 0$$

with initial conditions $\varphi(t_0) = \varphi_0, \ \dot{\varphi}(t_0) = v_0$.

The conditions for inflation $\dot{a} > 0$ and $\ddot{a} > 0$ are equivalent with:

$$\{H > 0 \ \text{and} \ \Phi > 2M^2H^2\} \Longleftrightarrow \left\{ \Phi(\varphi(t)) > 0 \ \text{and} \ 0 < H(t) < \frac{1}{M}\sqrt{\frac{\Phi(\varphi(t))}{2}} \right\}$$

These conditions on φ and $\dot{\varphi}$ define the inflationary region of the phase space.

In the gradient flow approximation the cosmological trajectories of the model are replaced by gradient flow lines of the scalar potential Φ. This allows one to derive features of the model using the well-known properties of gradient flows on Riemann surfaces, which is especially useful when Φ is a Morse function (smooth function with non-degenerate Hessian for all critical points).

Assuming $H(t) > 0$, define the gradient flow vector parameter through:

$$\eta(t) \stackrel{\text{def.}}{=} -\frac{1}{H}\frac{\nabla_t \dot{\varphi}}{||\dot{\varphi}||} = M\sqrt{6}\left[||\dot{\varphi}(t)||^2 + 2\Phi(\varphi(t))\right]^{-1/2}\frac{\nabla_t \dot{\varphi}}{||\dot{\varphi}||} \quad .$$

The gradient flow condition on the potential is:

$$||\eta|| \ll 1 \iff \frac{1}{9H^4}\frac{||\nabla_{\mathrm{grad}\Phi}\mathrm{grad}\Phi||^2}{||\mathrm{d}\Phi||^2} - \frac{1}{54M^2H^6}\partial_{\mathrm{grad}\Phi}||\mathrm{d}\Phi||^2 + \frac{||\mathrm{d}\Phi||^4}{324M^4H^8} \ll 1$$

When this condition is satisfied, solutions of the e.o.m are well approximated by gradient flow lines of Φ.

This approximation is much less restrictive than SRST.

Let (τ, n) be a Frenet frame and $\eta = \eta_{\parallel}\tau + \eta_{\perp} n$, $\xi \stackrel{\mathrm{def.}}{=} \xi_{\parallel}\tau + \xi_{\perp} n$ where:

$$\eta_{\parallel} \stackrel{\mathrm{def.}}{=} \frac{\ddot{\sigma}}{H\dot{\sigma}} \ , \ \eta_{\perp} \stackrel{\mathrm{def.}}{=} \frac{\dot{\sigma}}{H}\kappa \ , \ \xi_{\parallel} \stackrel{\mathrm{def.}}{=} -\frac{\dddot{\sigma}}{H\ddot{\sigma}} \ , \ \xi_{\perp} \stackrel{\mathrm{def.}}{=} -\frac{\dot{\eta}_{\perp}}{H\eta_{\perp}}$$

$\sigma(t)$ is the proper length parameter on the curve $\varphi : \mathbb{R} \to \Sigma$

$\kappa(t)$ is the extrinsic curvature of φ.

The kinematic slow-roll/slow-turn (SRST) conditions are:

$$0 < \epsilon \stackrel{\mathrm{def.}}{=} -\frac{\dot{H}}{H^2} \ll 1 \ , \ ||\eta|| \ll 1 \ , \ ||\xi|| \ll 1$$

1^{st}, 2^{nd}, 3^{rd} slow-roll parameters, respectively 1^{st}, 2^{nd} slow-turn parameters are:

$$\{\epsilon \ , \ \eta_{\parallel} \ , \ \xi_{\parallel}\} \ , \ \{\eta_{\perp} \ , \ \xi_{\perp}\}$$

These imply the SRST conditions on the potential:

$$M||\mathrm{d}\log\Phi|| \ll 1 \ , \ M^2 \left|\frac{\mathrm{Hess}(\Phi)(\tau,\tau)}{\Phi}\right| \ll 1 \ , \ M^2 \left|\frac{\mathrm{Hess}(\Phi)(n,\tau)}{\Phi}\right| \ll 1 \ .$$

It turns out that the SRST approximation is not very well suited for the study of our models, being too restrictive.

A hyperbolic surface Σ is an orientable, connected, possibly non-compact 2-dim. manifold endowed with a hyperbolic metric G (i.e. a **complete** Riemannian metric with constant negative Gaussian curvature $K(G) = -1$).

Definition

Let (Σ, G) be a complete hyperbolic surface and $\Phi : \Sigma \to \mathbb{R}$ be a smooth potential function. The *generalized two-field α-attractor model* is the two-field cosmological model defined by the triplet $(\Sigma, \mathcal{G}, \Phi)$, where $\mathcal{G} \stackrel{\text{def.}}{=} 3\alpha G$ with $\alpha > 0$.

The only hyperbolic surface considered before in the study of two-field α-attractors was the Poincaré disk.

For simplicity we restrict to geometrically finite hyperbolic surfaces (Σ, G), i.e. those for which the fundamental group $\pi_1(\Sigma)$ is finitely-generated.

- The hyperbolic disk (Poincaré disk) $(\mathbb{D}, \mathrm{d}s^2_{\mathbb{D}})$ is the open unit disk:

$$\mathbb{D} \stackrel{\text{def.}}{=} \{u \in \mathbb{C} | |u| < 1\} \quad ,$$

endowed with its unique hyperbolic metric (Poincaré disk metric):

$$\mathrm{d}s^2_{\mathbb{D}} = \lambda^2_{\mathbb{D}}(u, \bar{u})|\mathrm{d}u|^2 \quad , \quad \lambda_{\mathbb{D}}(u, \bar{u}) = \frac{2}{1 - |u|^2} \quad .$$

For $\Sigma = \mathbb{D}$ we get the two-field version of the α-attractor model of Linde & Kallosh.

- The hyperbolic plane (Poincaré half-plane) $(\mathbb{H}, \mathrm{d}s^2_{\mathbb{H}})$ is the upper half-plane:

$$\mathbb{H} \stackrel{\text{def.}}{=} \{\tau \in \mathbb{C} | \mathrm{Im}\tau > 0\}$$

endowed with its unique hyperbolic metric (Poincaré plane metric):

$$\mathrm{d}s^2_{\mathbb{H}} = \lambda_{\mathbb{H}}(\tau, \bar{\tau})^2 \mathrm{d}\tau^2 \quad \text{with} \quad \lambda_{\mathbb{H}}(\tau, \bar{\tau}) = \frac{1}{\mathrm{Im}\tau} \quad .$$

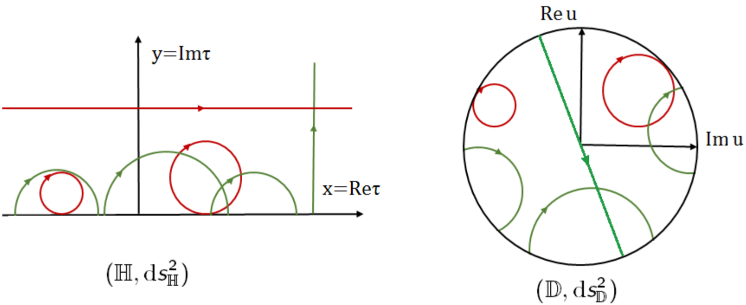


Figure: The Poincaré plane and Poincaré disk and some geodesics and horocycles.

Their conformal boundaries:

$$\partial_\infty \mathbb{H} = \mathbb{R} \cup \{\infty\} \simeq S^1 \quad , \quad \partial_\infty \mathbb{D} \simeq S^1 \quad \Longrightarrow \quad \partial_\infty \mathbb{H} \simeq \partial_\infty \mathbb{D}$$

$(\mathbb{H}, \mathrm{d}s^2_{\mathbb{H}})$ is isometric with $(\mathbb{D}, \mathrm{d}s^2_{\mathbb{D}})$ through the Cayley transform:

$$f : \mathbb{D} \longrightarrow \mathbb{H} \quad , \quad \tau = f(u) \stackrel{\text{def.}}{=} \frac{u + \mathbf{i}}{\mathbf{i}u + 1} \Rightarrow u = \frac{\mathbf{i} - \tau}{\mathbf{i}\tau - 1} \quad . \tag{5}$$

The group of orientation-preserving isometries of $\mathbb{H}$ is $\mathrm{Iso}^+(\mathbb{H}) \simeq \mathrm{PSL}(2, \mathbb{R})$, acting on $\mathbb{H}$ through the Möbius transformation:

$$\tau \longrightarrow A\tau = \frac{a\tau + b}{c\tau + d} \quad , \qquad \forall \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$$
$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$$

The group of orientation-preserving isometries of $\mathbb{D}$ is $\mathrm{Iso}^+(\mathbb{D}) = \mathrm{PSU}(1, 1)$, which is isomorphic with $\mathrm{PSL}(2, \mathbb{R})$.

An element $A \in \mathrm{PSL}(2, \mathbb{R})$ is called:

- elliptic if $|\mathrm{tr}(A)| < 2$
- parabolic, if $|\mathrm{tr}(A)| = 2$
- hyperbolic, if $|\mathrm{tr}(A)| > 2$.

Definition

A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. A surface group is a Fuchsian group without elliptic elements.

Uniformization theorem (Poincaré-Hopf)

For any hyperbolic surface (Σ, G) there is a surface group Γ and a holomorphic covering map (uniformization map) $\pi_{\mathbb{H}} : \mathbb{H} \longrightarrow \Sigma$ such that $\Sigma \simeq \mathbb{H}/\Gamma$.

The projection

$$\pi_{\mathbb{D}} : \mathbb{D} \longrightarrow \Sigma$$

is also called uniformization map, where

$$\pi_{\mathbb{D}} = \pi_{\mathbb{H}} \circ f \quad , \quad f : \mathbb{D} \longrightarrow \mathbb{H}$$

To study the cosmological trajectories $\varphi(t)$ on the hyperbolic surface Σ it is convenient to first study lifted trajectories $\tilde{\varphi}(t)$ to $\mathbb{H}$ (or to $\mathbb{D}$) and then to project them back to Σ.

Projecting back to Σ

The projection from $\mathbb{H}$ to Σ can be computed if we know the tiling of $\mathbb{H}$ determined by a fundamental polygon of Γ. This tiling can be very non-trivial.

A fundamental polygon $\mathcal{D}_{\mathbb{H}} \subset \mathbb{H}$ is a convex polygon whose sides can be either geodesic segments or free sides (i.e. intervals of the conformal boundary $\partial_\infty \mathbb{H}$).

Computing fundamental polygons

There is no fully general stopping algorithm known for computing fundamental polygons of surface groups. But a general algorithm is known for the case when Γ is an arithmetic Fuchsian group such that $\mathbb{H}/\Gamma$ has finite hyperbolic area.

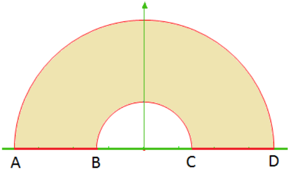


Figure: Example of a fundamental polygon on $\mathbb{H}$ (for the annulus)

For aplications to cosmology it is important to allow (Σ, G) to be non-compact and of possibly infinite area.
For example, $(\mathbb{D}, \mathrm{d}s^2_{\mathbb{D}})$ (which gives the original two-field alpha-attractor models of Kallosh and Linde) is non-compact and of infinite area.

Cosmological applications require sophisticated results from uniformization theory, closely connected to number theory.

Let $\Lambda_{\mathbb{H}} \subset \partial_\infty \mathbb{H}$ denote the set of limit points of the orbit of Γ on $\mathbb{H}$.

Theorem (Poincare, Fricke, Klein)

One has the trichotomy:

- $\Lambda_{\mathbb{H}}$ is finite (contains 0, 1 or 2 points), in which case Γ (and (Σ, G)) is called elementary.
- $\Lambda_{\mathbb{H}} = \partial_\infty \mathbb{H}$, in which case Γ (and (Σ, G)) is called of the first kind.
- $\Lambda_{\mathbb{H}}$ is a perfect and nowhere-dense[a] subset of $\partial_\infty \mathbb{H}$, in which case Γ (and (Σ, G)) is called of the second kind.

[a] Namely a closed set with empty interior and without isolated points.

Elementary hyperbolic surfaces

There are only of three types:

- $\Sigma = \mathbb{D}$ (hyperbolic disk)
- $\Sigma = \mathbb{D}^*$ (hyperbolic punctured disk)
- $\Sigma = \mathbb{A}(R)$ for $R > 1$ (hyperbolic annuli)

All elementary hyperbolic surfaces have infinite hyperbolic area.

Hyperbolic surfaces of the first kind

The following statements are equivalent:

(a) (Σ, G) is of the first kind

(b) $\mathrm{area}_G(\Sigma)$ is finite

(c) the fundamental polygon has no free sides.

Let Σ be a topologically finite surface, i.e. Σ is homeomorphic with $\hat{\Sigma} \setminus \{p_1, \ldots, p_n\}$, where $\hat{\Sigma}$ is a borderless compact oriented surface and $p_1, \ldots, p_n$ are a finite number of distinct points. $\hat{\Sigma}$ can be identified with the end (a.k.a. Kerekjarto-Stoilow) compactification of Σ, where $p_1, \ldots, p_n$ correspond to the Kerekjarto-Stoilow *ideal points*, a.k.a. *Freudenthal ends* of Σ.

Consider a partition $\{1, \ldots, n\} = \{i_1, \ldots, i_{n_c}\} \sqcup \{j_1, \ldots, j_{n_f}\}$ where $n_c \geq 0$ and $n_f \geq 0$ are natural numbers. Since any annulus is diffeomorphic with the punctured unit disk, Σ is diffeomorphic with the borderless surface $\hat{\Sigma} \setminus (\{p_{i_1}, \ldots, p_{i_{n_c}}\} \cup \bar{D}_{j_1} \cup \ldots \cup \bar{D}_{j_{n_f}})$, where $\bar{D}_j$ are closed disks embedded in $\hat{\Sigma}$ and centered at the points $p_{j_1}, \ldots, p_{j_{n_f}}$, such that no two closed disks meet each other and no closed disk meets any of the points $p_{i_1}, \ldots, p_{i_{n_c}}$.

Let J be an orientation-compatible complex structure on Σ. Then it was shown by Maskit that there exists a unique complex structure $\hat{J}$ on $\hat{\Sigma}$ such that (Σ, J) is biholomorphic with the surface Σ_J obtained from $(\hat{\Sigma}, \hat{J})$ by removing a finite set of points and a disjoint union of so-called "closed circular disk domains".

The conformal compactification $\bar{\Sigma}_J$ of Σ with respect to J is the surface obtained by taking the closure of Σ_J inside $\hat{\Sigma}$. The topological boundary $\partial^J_\infty \Sigma = \bar{\Sigma}_J \setminus \Sigma$ of Σ_J consists of n_c isolated points and n_f disjoint simple closed curves; this is called the *conformal boundary* of (Σ, J).

We shall concentrate on geometrically finite hyperbolic surfaces.

Proposition

Let (Σ, G) be a hyperbolic surface uniformized by the surface group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$. One says that Γ and (Σ, G) are geometrically finite iff the following statements are equivalent:

- $\Sigma \simeq \mathbb{H}/\Gamma$ is topologically finite (i.e. Σ has finite Euler characteristic $\chi(\Sigma) = 2 - 2g - 2n$, where $g =$ genus, $n =$ number of ends).
- Γ (which is isomorphic with $\pi_1(\Sigma)$) is finitely-generated.
- Γ admits a fundamental polygon with a finite number of sides (some of which may be free).

All elementary surfaces, all surfaces of the first kind and part of those of the second kind are geometrically finite. (Siegel)

The explicit forms of the hyperbolic metric in semi-geodesic coordinates (r, θ) (geodesic polar coordinates) on the canonical neighborhood of each type of end are as follows:

- **cusp end**: $\mathrm{d}s^2 = \mathrm{d}r^2 + e^{-2r}\frac{\mathrm{d}\theta^2}{(2\pi)^2}$
- **funnel end**: $\mathrm{d}s^2 = \mathrm{d}r^2 + \ell_p^2 \cosh(r)^2 \frac{\mathrm{d}\theta^2}{(2\pi)^2}$
- **plane end**: $\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh(r)^2 \mathrm{d}\theta^2$
- **horn end**: $\mathrm{d}s^2 = \mathrm{d}r^2 + e^{2r}\frac{\mathrm{d}\theta^2}{(2\pi)^2}$

The plane and horn end arise only for the Poincaré disk and for the hyperbolic punctured disk, while the hyperbolic annulus has two funnel ends:

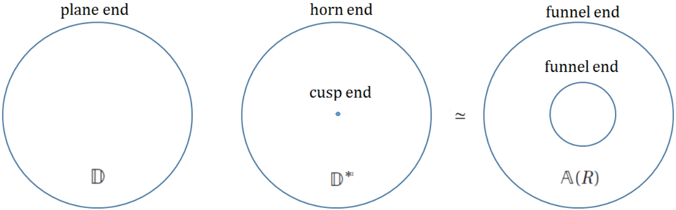


Figure: Elementary hyperbolic surfaces and the hyperbolic type of their corresponding ends. Their end compactification is S^2.

Theorem (Borthwick)

A **non-elementary** geometrically-finite hyperbolic surface (Σ, G) can have only cusp and/or funnel ends. The surface Σ can be decomposed as:

$$\Sigma = K \sqcup C_1 \sqcup \ldots \sqcup C_{n_c} \sqcup F_1 \sqcup \ldots \sqcup F_{n_f}$$

where K is a compact surface, C_i are hyperbolic cusps and F_j are hyperbolic funnels. Moreover:

- (Σ, G) is of the first kind (has finite area) iff it has no funnels ($n_f = 0$).
- (Σ, G) is compact iff it has no ends (no cusps and no funnels).

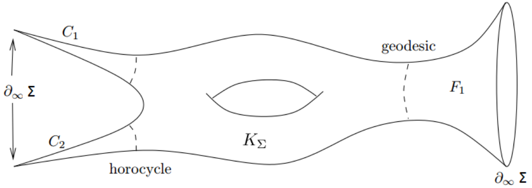


Figure: Example of a non-elementary geometrically-finite hyperbolic surface with 2 cusp ends and one funnel ens. Its end compactification is T^2

Let $\hat{\Sigma}$ be the end compactification of Σ. A scalar potential $\Phi : \Sigma \to \mathbb{R}$ is called well-behaved at an end $p \in \hat{\Sigma} \setminus \Sigma$ if there exists a smooth function $\hat{\Phi}_p : \Sigma \sqcup \{p\} \to \mathbb{R}$ such that $\Phi = \hat{\Phi}_p|_\Sigma$.

The potential Φ is called globally well-behaved if there exists a globally-defined smooth function $\hat{\Phi} : \hat{\Sigma} \to \mathbb{R}$ such that $\Phi = \hat{\Phi}|_\Sigma$. Thus Φ is globally well-behaved if it is well-behaved at each end of Σ.

We consider for example the following well-behaved scalar potentials on $\hat{\Sigma} = \mathrm{S}^2$ written in spherical coordinates:

$$\hat{\Phi}_0(\psi, \theta) = M\sqrt{\frac{3}{2}}(1 + \sin\psi\cos\theta) \tag{6}$$

$$\hat{\Phi}_+(\psi) = M\sqrt{\frac{3}{2}}\cos^2\left(\frac{\psi}{2}\right) \tag{7}$$

$$\hat{\Phi}_-(\psi) = M\sqrt{\frac{3}{2}}\sin^2\left(\frac{\psi}{2}\right) \tag{8}$$

Suppose that Φ is independent of θ in semigeodesic coordinates (r, θ) near some end and that it has an asymptotic expansion:

$$\Phi(x) =_{x \ll 1} V_0 \left(1 - cx + O(x^2)\right) \tag{9}$$

where $x = e^{-r}$ and $V_0 > 0,\ c > 0$.

Then the generalized α-attractor model admits a local **naive** truncation to a one-field model, obtained by setting $\theta =$ constant. In the slow-roll approximation for this truncated model, the spectral index $\mathbf{n_s}$ and the tensor to scalar ratio $\mathbf{r}$ are given by:

$$\mathbf{n_s} \approx 1 - \frac{2}{N}, \quad \mathbf{r} \approx \frac{12\alpha}{N^2} \quad . \tag{10}$$

where $N \stackrel{\mathrm{def.}}{=} \log \frac{a(t_1)}{a(t_0)}$ is the number of e-folds.

Universal behavior near the ends

For fixed α, all generalized two-filed α-attractor models lead to the same values of $\mathbf{n_s}$ and $\mathbf{r}$ in the leading order of the slow-roll approximation near any end, in the **naive** one-field truncation near those ends.

A In semi-geodesic coordinates near an end $p \in \hat{\Sigma} \setminus \Sigma$, we have:

$$\mathrm{d}s^2_{\mathcal{G}} \simeq_{r\gg 1} 3\alpha \left[\mathrm{d}r^2 + \left(\frac{C_p}{4\pi}\right)^2 e^{2\epsilon_p r}\mathrm{d}\theta^2\right] \quad ,$$

where C_p and ϵ_p are known constants which depend on the type of end. We find that the e.o.m. in a vicinity of an end reduces to:

$$\ddot{r} - 3\epsilon\alpha\left(\frac{C_p}{4\pi}\right)^2 e^{2\epsilon_p r}\dot{\theta}^2 + 3H\dot{r} + \frac{1}{3\alpha}\partial_r\Phi = 0 \quad , \tag{11}$$

$$\ddot{\theta} + 2\epsilon_p\dot{r}\dot{\theta} + 3H\dot{\theta} + \frac{1}{3\alpha}\left(\frac{4\pi}{C_p}\right)^2 e^{-2\epsilon_p r}\partial_\theta\Phi = 0 \quad . \tag{12}$$

The generic solution of this system has $\dot{r} \neq 0$ and $\dot{\theta} \neq 0$, thus being a portion of a spiral which “winds” around the ideal point. This gives a form a spiral inflation in our class of models.

Spiral trajectories near the ends

Since θ is periodic, a generic trajectory will spiral around the ends. Our models can admit spiral trajectories when Φ is well-behaved at the ends already for $\Phi = 0$.

The hyperbolic punctured disk is the punctured unit disk endowed with the unique complete hyperbolic metric:

$$\mathrm{d}s^2 = \lambda^2_{\mathbb{D}^*}(u,\bar{u})|\mathrm{d}u|^2 \quad , \text{ where } \lambda_{\mathbb{D}^*}(u,\bar{u}) = \frac{1}{|u|\log(1/|u|)} \quad (0<|u|<1) \quad .$$

$\Gamma \simeq \mathbb{Z}$ is the parabolic cyclic group generated by the translation $\tau \to \tau + 1$.
A holomorphic covering map $\pi_{\mathbb{H}} : \mathbb{H} \to \mathbb{D}^*$ is given by $\pi_{\mathbb{H}}(\tau) = e^{2\pi \mathrm{i}\tau}$.
A fundamental polygon: $\mathcal{D}_{\mathbb{H}} = \{\tau \in \mathbb{H} \mid 0 \leq \mathrm{Re}(\tau) < 1\}$.

Let's choose the globally well-behaved potential $\hat{\Phi}_0$ given in (6) which takes the following form in polar coordinates on $\mathbb{D}^*$:

$$\Phi_0 = M\sqrt{\frac{3}{2}}\left[1 + \frac{2|\log\rho|}{1+(\log\rho)^2}\cos\theta\right] \qquad (u = \rho e^{\mathrm{i}\theta})$$

and which lifts to $\mathbb{H}$ as:

$$\tilde{\Phi}_0 = \Phi_0 \circ \pi_{\mathbb{H}} = M\sqrt{\frac{3}{2}}\left[1 + \frac{4\pi y\cos(2\pi x)}{1+4\pi^2 y^2}\right]$$

where $x = \mathrm{Re}\tau$ and $y = \mathrm{Im}\tau$.

For any $u_0 \in \Sigma = \mathbb{D}^*$ take $\tau_0 = x_0 + \mathbf{i}y_0 \in \mathbb{H}$ such that
$\pi_{\mathbb{H}}(\tau_0) = u_0$, $\tilde{v}_0 = v_0 \circ \pi_{\mathbb{H}} = \dot{\tau}_0$

trajectory	τ_0	$\tilde{v}_0$
orange	$0.3 + \mathbf{i}/(2\pi)$	0
red	$(1 + 2\mathbf{i})/10$	$2 + 3\mathbf{i}$
blue	$\mathbf{i}$	$1 + \mathbf{i}$
magenta	$\mathbf{i}/10$	$1.3 + 7\mathbf{i}$

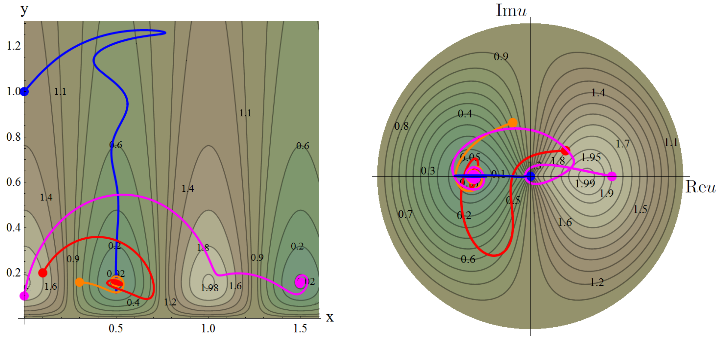
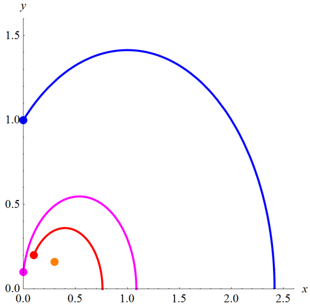


Figure: Examples of trajectories for the potential Φ_0 on $\mathbb{H}$ and $\mathbb{D}^*$

For the same initial conditions, but with zero potential:

trajectory	τ_0	$\tilde{v}_0$
orange	$0.3 + \mathbf{i}/(2\pi)$	0
red	$(1 + 2\mathbf{i})/10$	$2 + 3\mathbf{i}$
blue	$\mathbf{i}$	$1 + \mathbf{i}$
magenta	$\mathbf{i}/10$	$1.3 + 7\mathbf{i}$



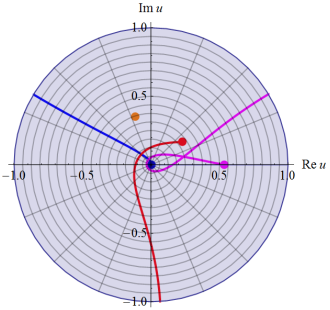


Figure: Examples of trajectories on $\mathbb{H}$ and $\mathbb{D}^*$ for $\Phi = 0$

The annulus:

$$\mathbb{A}(R) = \{u \in \mathbb{C} \mid \frac{1}{R} \leq |u| \leq R\} \quad (R > 1)$$

of modulus $\mu = 2\log R > 0$ has the unique complete hyperbolic metric:

$$\mathrm{d}s^2 = |\lambda_R(u)|^2 |\mathrm{d}u|^2 \;, \text{ where } \; \lambda_R(u) = \frac{\pi}{2\log R} \frac{1}{|u| \cos\left(\frac{\pi \log|u|}{2\log R}\right)} \;.$$

It is uniformized to $\mathbb{H}$ by the hyperbolic cyclic group Γ generated by the dilation $\tau \to e^{\ell}\tau$, where $\ell = \frac{\pi^2}{\log R} = \frac{2\pi^2}{\mu}$.

The same potential (6) takes the following form on $\mathbb{A}(R)$:

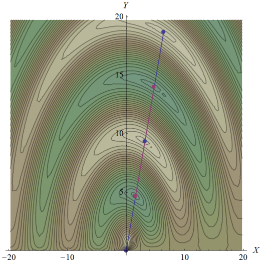
$$\Phi_0 = M\sqrt{\frac{2}{3}}\left[1 + \frac{2\log\frac{R-\frac{1}{R}}{\rho-\frac{1}{R}}}{1+\left(\log\frac{R-\frac{1}{R}}{\rho-\frac{1}{R}}\right)^2}\cos\theta\right]$$

and lifts to $\mathbb{H}$ as:

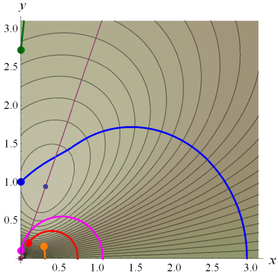
$$\tilde{\Phi}_0(\tau) = M\sqrt{\frac{2}{3}}\left[1 + \frac{2\log\frac{R-\frac{1}{R}}{\rho(\tau)-\frac{1}{R}}}{1+\left(\log\frac{R-\frac{1}{R}}{\rho(\tau)-\frac{1}{R}}\right)^2}\cos\left(\frac{2\pi}{\ell}\log|\tau|\right)\right]$$

Choices of trajectories on $\mathbb{H}$ and $\mathbb{A}(R)$

For any $u_0 \in \Sigma = \mathbb{A}(R)$ take $\tau_0 = x_0 + \mathbf{i}y_0 \in \mathbb{H}$ s.t $\pi_{\mathbb{H}}(\tau_0) = u_0$, $\tilde{v}_0 = \dot{\tau}_0$



trajectory	τ_0	$\tilde{v}_0$
orange	$0.3 + \mathbf{i}/(2\pi)$	0
red	$(1 + 2\mathbf{i})/10$	$2 + 3\mathbf{i}$
blue	$\mathbf{i}$	$1 + \mathbf{i}$
magenta	$\mathbf{i}/10$	$1.3 + 7\mathbf{i}$



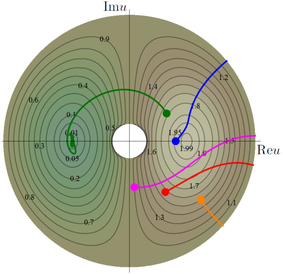


Figure: Examples of trajectories for the potential Φ_0 on $\mathbb{H}$ and $\mathbb{A}(R)$

The triply punctured sphere $\Sigma = Y(2) \stackrel{\text{def.}}{=} \mathbb{CP}^1 \setminus \{p_1, p_2, p_3\}$ endowed with the hyperbolic metric:

$$\mathrm{d}s^2 = \rho(\zeta,\bar{\zeta})^2 |\mathrm{d}\zeta^2 \ ,$$

where:

$$\rho(\zeta,\bar{\zeta}) = \frac{\pi}{8|\zeta(1-\zeta)|} \frac{1}{\mathrm{Re}[\mathcal{K}(\zeta)\mathcal{K}(1-\bar{\zeta})]} \ , \quad \mathcal{K}(\zeta) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\zeta t^2)}}$$

Each of the three punctures corresponds to a cusp end.
Its end compactification $\hat{\Sigma} = \mathbb{CP}^1 \simeq \mathrm{S}^2$.
It is conformal to $\mathbb{C} \setminus \{0, 1\}$.

$Y(2)$ is uniformized by the principal congruence subgroup of level 2:

$$\Gamma(2) \stackrel{\text{def.}}{=} \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \mid a, d = \mathrm{odd}\ ,\ b, c = \mathrm{even} \right\}$$

with uniformization map $\lambda : \mathbb{H} \to Y(2)$ given by the elliptic modular lambda function:

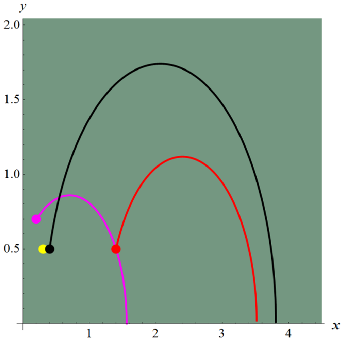
$$\lambda(\tau) = \frac{\wp_\tau\left(\frac{1+\tau}{2}\right) - \wp_\tau\left(\frac{\tau}{2}\right)}{\wp_\tau\left(\frac{1}{2}\right) - \wp_\tau\left(\frac{\tau}{2}\right)}$$

where $\wp$ is the Weierstrass elliptic function of modulus τ.

For scalar potentials which are invariant under the natural action on $Y(2)$ of the anharmonic group $\mathrm{PSL}(2,\mathbb{Z}_2)$, the generalized α-attractor model defined by $Y(2)$ is related to the $\mathrm{PSL}(2,\mathbb{Z})$ modular inflation models of Schimmrigk through the $\infty : 1$ field redefinition given by λ.

Let's consider $\Phi = 0$ and the following initial conditions:

trajectory	τ_0	$\tilde{v}_0$
black	$0.4 + 0.5\mathbf{i}$	$0.3 + \mathbf{i}$
red	$1.4 + 0.5\mathbf{i}$	$0.1 + 0.2\mathbf{i}$
magenta	$0.2 + 0.7\mathbf{i}$	$0.7 + 0.5\mathbf{i}$
yellow	$0.3 + 0.5\mathbf{i}$	0



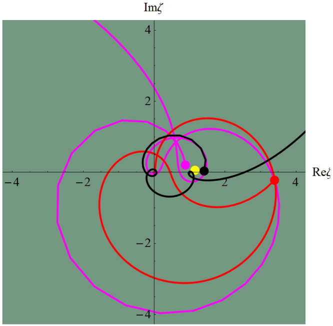


Figure: Corresponding trajectories for $\Phi = 0$ on $\mathbb{H}$ and $Y(2)$.

Let's choose the same initial conditions and the scalar potential $\hat{\Phi}_0$ given in (6)

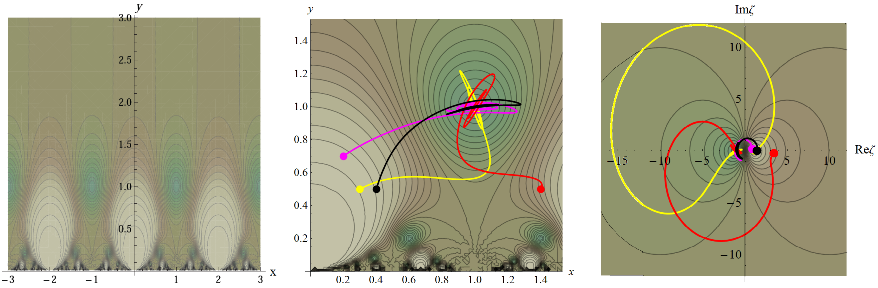


Figure: Level plof of $\tilde{\Phi}_0$ on $\mathbb{H}$. Corresponding trajectories on $\mathbb{H}$ and $Y(2)$

Let's choose the same initial conditions and the scalar potential $\hat{\Phi}_{-}$ given in (8)

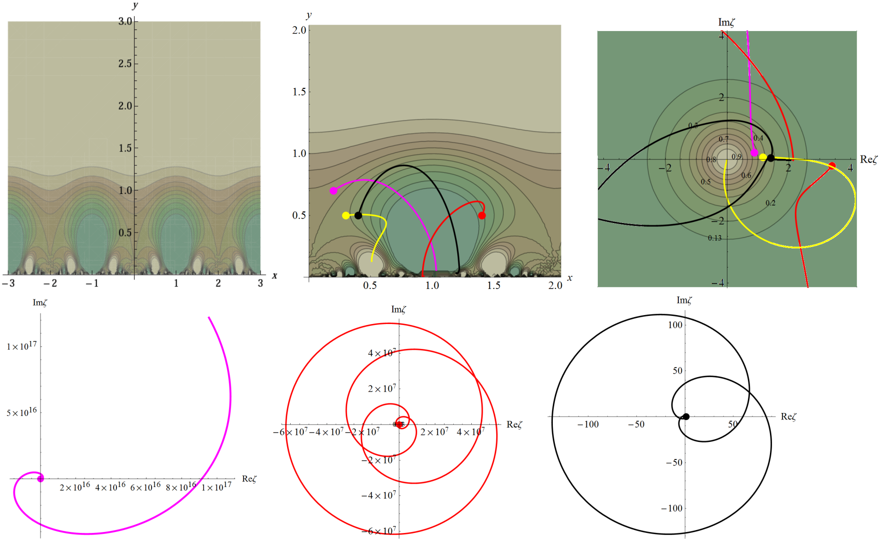


Figure: Level plof of $\tilde{\Phi}_{-}$ on $\mathbb{H}$. Corresponding trajectories on $\mathbb{H}$ and $Y(2)$

Conclusions:

- We proposed a wide generalization of two-field α-attractor models obtained by promoting the scalar manifold from the Poincaré disk to a more general geometrically finite non-compact hyperbolic surface.
- We proposed a general procedure for studying such models through uniformization techniques and without using one-field truncations.
- We showed that such models have the same universal behavior as ordinary α-attractors in a naive one-field truncation near each end, provided that the scalar potential is well-behaved near that end.
- The SRST approximation is not very well suited for the study of our models since it is too restrictive and can in particular fail near cusp ends.

Further directions:

- Investigating cosmological perturbation (radiative corrections for CMB), adapting the numerical approach developed by Mulryne et all.
- Study of embeddings into $N = 1$ supergravity with a single chiral multiplet.
- Extension of our models to the cases when Σ is not orientable, or when Σ is not topologically finite.

This talk was based on the following papers:

- C. I. Lazaroiu, C. S. Shahbazi *Generalized α-attractor models from geometrically finite hyperbolic surfaces*, arXiv:1702.06484.
- Mirela Babalic, Calin Iuliu Lazaroiu, *Generalized α-attractor models from elementary hyperbolic surfaces*, arXiv:1703.01650.
- Mirela Babalic, Calin Iuliu Lazaroiu, *Generalized α-attractors from the hyperbolic triply-punctured sphere*, arXiv:1703.06033.