U-folds, section sigma models and fibered supergravity

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Outline

Local versus global formulations of supergravity

Virtually all computations found in the supergravity literature are formulated only locally, in the sense that it is tacitly assumed that all manifolds involved are contractible. Any fiber bundle E defined on a contractible manifold U is topologically trivial and hence admits a global trivialization. As a consequence, any section of such a bundle can be identified with the graph of a map from U into the fiber F of E.

The fundamental limitation of the current formulation

With very few exceptions, the computations found in the supergravity literature only suffice to define supergravity theories on *contractible space-times* and under the assumption that all other manifolds involved (such as the scalar manifolds of various sigma models) are contractible.

A contractible manifold has vanishing homology and cohomology groups (except in dimension zero and top dimension), hence it can never have non-trivial cycles.

Local doesn't suffice

General relativity, string theory, M-theory etc often have solutions whose space-times are *not* contractible but have non-trivial topology. Most supergravity theories have simply never been properly defined for such space-times. Moreover, moduli spaces arising "in nature" are often not contractible, so conjectures attempting to identify these with the scalar manifolds of sigma models of various supergravity theories cannot be true since the latter are always tacitly assumed to be contractible when formulating supergravity theories merely through local computations.

Toward globally-valid formulations

How can we formulate supergravity theories if we relax the assumption that all manifolds involved are contractible ? How can we find geometric formulations which work for reasonably general manifold topologies ?

Aspects to be addressed:

- Find the correct global nature of the bosonic and fermionic fields arising in the local formulas, i.e. interpret them as sections of appropriate globally-defined fiber bundles, which need not be topologically trivial.
- Find the correct formulation of the action and equations of motion when fields are interpreted as such global sections, in such a way that the correct amount of supersymmetry is preserved.
- Find the most general topological and geometric conditions under which such global formulations are possible.

Caution: Supergravity theories have "local supersymmetry" in the sense that they admit a supergroup of *gauge transformations*. The notion of "local" (as opposed to "global") symmetry of a theory should not be confused with the notion of "global formulation" of the theory. The latter means a formulation which makes sense for non-contractible manifolds. For example, ordinary non-Abelian gauge theories have an elegant mathematical formulation on non-contractible manifolds given in terms of fiber bundles, even though such theories have "local" symmetries under gauge transformations. We are interested in finding similar formulations for supergravity theories, which apply to non-contractible manifolds.

Some aspects of this program

- Spinor fields are complicated:
  - We need real spinor fields. The treatment of such fields in the supergravity literature is exceedingly poor, due largely to many physicists' ignorance of the mathematical theory of real spinor fields (as developed by Atiyah, Bott, Shapiro, Karoubi etc).
  - Some naive attempts at globalizing spinor fields make unwarranted mathematical assumptions. For example, supergravity theories coupled to matter can often be formulated globally *without* requiring that the space-time admit a spin structure. To clarify such aspects, one has to develop real spin geometry from the perspective of real Clifford bundles and real Lipschitz structures, something which wasn't done previously in the mathematics literature.
  - In supergravity theories, spinor fields must be understood as sections of bundles which depend on the bosonic field configuration. Hence a thorough understanding of correct global formulation of the bosonic sector is needed first.
- Tensor fields aren't that simple:
  - One must take into account duality transformations for Abelian gauge fields, some of which are locally described by higher rank forms. Globalization requires manifestly duality-invariant formulations of the equations of motion which make no recourse to the crutch of "duality frames".
  - The scalar manifold need not be contractible (in particular, it need not be simply connected) – as has been assumed until now in most of the supergravity literature.
  - In general, scalar fields are *not* described globally by a map from space-time into a scalar manifold, but by sections of a fiber bundle (or even of a submersion) defined over space-time.

Do we get anything new ?

Already in four dimensions, we get a global geometric description of U-folds in supergravity. This takes away much of the "mystery" of U-folds.

Removing assumptions

Theory of maps $\varphi : M \to \mathcal{M}$, where:

- The space-times (M, g) is a four-dimensional Lorentzian manifold (we use the "mostly plus" convention)
- $(\mathcal{M}, \mathcal{G})$ is a Riemannian manifold, which need not be contractible and may be non-compact.

coupled to n Abelian gauge fields whose coupling constants and "theta angles" (encoded by the gauge-kinetic functions) are defined on $\mathcal{M}$. Want global formulation which:

- Does not assume that the space-time M is contractible.
- Does not assume that the scalar manifold $\mathcal{M}$ of the sigma model is contractible.
- Is manifestly covariant under electro-magnetic duality transformations of the Abelian gauge theory.

Key ideas:

- Duality transformations act non-locally on the Abelian gauge-connections. We can achieve a duality-invariant formulation for the Maxwell equations only (*first order formalism*).
- When $\mathcal{M}$ is not simply-connected, the "duality structure" of the Abelian gauge theory is encoded by a flat symplectic vector bundle $(\mathcal{S}, D, \omega)$ defined over the scalar manifold $\mathcal{M}$
- The gauge-kinetic functions are encoded in a duality-invariant manner by a taming J of $(\mathcal{S}, \omega)$, which need not be D-flat.

$(\mathcal{S}, D, J, \omega)$ pulls back to a tamed flat symplectic vector bundle $(\mathcal{S}^\varphi, D^\varphi, J^\varphi, \omega^\varphi)$ defined over space-time. Both the electric and magnetic field strengths are encoded in a duality-invariant manner by a two-form $\mathcal{V} \in \Omega^2(M, \mathcal{S}^\varphi)$ valued in the vector bundle $\mathcal{S}^\varphi$. This form is subject to two conditions:

(a) The negative polarization condition with respect to J^φ:

$$*_g \mathcal{V} = -J^\varphi \mathcal{V}$$

(b) The duality-invariant form of the Maxwell equations:

$$\mathrm{d}_{D^\varphi} \mathcal{V} = 0 \quad .$$

The action for φ is:

$$\mathcal{S}_{\mathrm{bos}}[\varphi] = \int_M \nu[g] e(\varphi)$$

where $e(\varphi) = \frac{1}{2} Tr_g \varphi^*(\mathcal{G})$, which gives the usual equations of motion:

$$\tau(\varphi) = 0 \quad , \tag{1}$$

where $\tau(\varphi) \stackrel{\mathrm{def.}}{=} \mathrm{Tr}_g \Sigma(\varphi) \in \Gamma(M, (T\mathcal{M})^\varphi)$ is the tension of φ. Here, $\Sigma(\varphi) \stackrel{\mathrm{def.}}{=} \nabla \mathrm{d}\varphi \in \Gamma(M, \mathrm{Sym}^2(T^*M) \otimes \varphi^*(T\mathcal{M}))$ is the second fundamental form of φ. Solutions of (1) are called *pseudoharmonic maps* from (M, g) to $(\mathcal{M}, \mathcal{G})$.

Recovering the local formulation

Remark

In the Abelian gauge theory sector, there is a strong similarity between the data of our formulation and the formalism of geometric quantization, which suggests that a manifestly duality-covariant quantization of Abelian gauge theories in a background field φ is in fact possible.

Definition. A *duality frame* is a local flat symplectic frame of $(\mathcal{S}, D, \omega)$.

Note that $\mathcal{S}$ admits a global duality frame iff $\mathcal{S}$ is topologically trivial.

Local character of duality frames

In general, duality frames exist only locally.

Remark. Flat symplectic vector bundles $(\mathcal{S}, D, \omega)$ of rank $2n$ over $\mathcal{M}$ are classified up to isomorphism by equivalence classes of symplectic representations $\rho : \pi_1(\mathcal{M}) \to \mathrm{Sp}(2n, \mathbb{R})$. When $\mathcal{M}$ is simply-connected, any such bundle is trivial and global duality frames exist; this is the only case that was considered previously in the supergravity literature.

Recovering the local formulation

Given a local duality frame $\mathcal{E} \stackrel{\mathrm{def.}}{=} (e_1 \ldots e_n, f_1 \ldots f_n)$ defined above $U \subset \mathcal{M}$, we have:

$$\mathcal{V}|_{\varphi^{-1}(U)} = [F, G] \left[\begin{array}{c} e \\ f \end{array} \right]$$

where $F, G \in \mathrm{Mat}(n, 1, \Omega^2(\varphi^{-1}(U))$ and the matrix of J in the frame $\mathcal{E}$ can always be written as:

$$\hat{J} = \left[\begin{array}{cc} \theta\gamma^{-1} & -\gamma - \theta\gamma^{-1}\theta \\ \gamma^{-1} & -\gamma^{-1}\theta \end{array} \right] \quad ,$$

where $\tau = \tau^{\mathcal{E}}(J) \stackrel{\mathrm{def.}}{=} \theta + i\gamma \in \mathcal{C}^\infty(U, \mathbb{SH}^n)$ is a smooth map from U to the n-th Siegel upper half space, called the *modulus of J in the frame $\mathcal{E}$*. Thus $\theta(p)$ and $\gamma(p)$ are symmetric real-valued square matrices of size n and $\gamma(p)$ is strictly positive-definite for any $p \in U$. They encode the theta angles and inverse coupling constants in the duality frame $\mathcal{E}$. The negative polarization condition for $\mathcal{V}$ amounts to the usual condition which expresses G as the Lagrangian conjugate of F:

$$G = \theta^\varphi \ \hat{F} - \gamma^\varphi *_g F$$

with respect to the local Lagrangian density of the Abelian gauge fields written in the frame $\mathcal{E}$:

$$\mathcal{L}_{\mathrm{gauge}} = \frac{1}{4}\left(\gamma^\varphi_{ij} F^i \wedge *_g F^j - \theta^\varphi_{ij} F^i \wedge F^j\right) \quad .$$

The Maxwell equations reduce to the usual local form:

$$\mathrm{d}F = \mathrm{d}G = 0 \quad .$$

Recovering the local formulation

Remark. One can use the relation between tamings of (S, ω) and positive polarizations of the complexified symplectic vector bundle $(\mathcal{S}^{\mathbb{C}}, \omega^{\mathbb{C}})$ to also recover the complex version of the local formalism.

Under a change of local symplectic frame:

$$\mathcal{E}' = M^{-\boldsymbol{T}}\mathcal{E}$$

with $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n, R)$, one finds that τ changes by fractional transformations (the modular action of $\mathrm{Sp}(2n, \mathbb{R})$ on the Siegel upper half space):

$$\tau^{M^{-\boldsymbol{T}}\mathcal{E}}(J) = (A\tau^{\mathcal{E}}(J) + B)(C\tau^{\mathcal{E}}(J) + D)^{-1} \quad .$$

The scalar field as a section

Until now, we merely gave a manifestly-duality invariant formulation of the usual sigma model coupled to Abelian gauge fields, while removing some unnecessary assumptions. We still treated φ as a map from space-time into the scalar manifold $(\mathcal{M}, \mathcal{G})$. In the usual local formulation, this is warranted because sections of a trivial fiber bundle are graphs of maps into the fiber, but we can globalize the local formulas by taking φ to be a section of a fiber bundle over $\mathcal{M}$. *Locally, we would never know the difference.* Thus:

It is natural to promote φ to a section $s \in \Gamma(\pi)$ of a fiber bundle $\pi : E \to M$ with typical fiber $(\mathcal{M}, \mathcal{G})$, i.e. a fiber bundle associated to a principal G-bundle Π defined over M, where $G = \mathrm{Iso}(\mathcal{M}, \mathcal{G})$ and we associate E to Π using the tautological action ρ of G on $\mathcal{M}$:

$$E = \Pi \times_\rho (\mathcal{M}, \mathcal{G}) \quad .$$

This crucial observation will lead us to the conclusion that:

The classical geometric nature of supergravity U-folds

Classical (i.e., non-quantum) U-folds are not "exotic". Rather, they are global solutions of classical supegravity theories, when the later are *properly formulated* mathematically in a way that takes into account the topology of scalar manifolds and of spacetime, while also being manifestly invariant under duality transformations.

Supergravity U-folds are "exotic" only from the perspective of someone who insists on *computations in local coordinates*.

Since E is a fiber bundle, its projection $\pi : E \to M$ is a submersion. Let $V = \ker \mathrm{d}\pi$ be the vertical distribution of π. We can endow the total space E with a metric h of Lorentzian signature such that π becomes a pseudo-Riemannian submersion in the sense of O'Neill. In fact:

Proposition. There exists a bijection between the set $\mathrm{Met}(E, \pi, g)$ of Lorentzian signature metrics h on E which are *adapted to* π *and* g (i.e., they have the property that π is a pseudo-Riemannian submersion from (E, h) to (M, g)) and the set $\mathcal{H}(E, \pi, M)$ of horizontal distributions (a.k.a. Ehresmann connections) of the fiber bundle (E, π, M). This correspondence takes a π-horizontal distribution $H \subset TE$ into the corresponding *Kaluza-Klein metric* $h_{g,H}$ determined by H and g on the total space E:

$$h_{g,H}(X, Y) = g(\pi_*(X_V), \pi_*(Y_V)) + \mathcal{G}(X_H, Y_H) \quad ,$$

where X_V, X_H are the vertical and horizontal projections of X determined by the splitting $TE = V \oplus H$.

Remark. The associated bundle construction induces a bijection between connections on the principal G-bundle Π and Ehresmann connections H on (E, π, M), so Kaluza-Klein metrics on E with respect to g are in bijection with connections on Π. The latter can be interpreted as an auxiliary *background gauge field* parameterizing our model (in general, these background field is non-Abelian and is *not* required to satisfy the Yang-Mills equations, since we do not require that the metric h be an Einstein metric). Kaluza-Klein metrics in this generality were studied in the literature on the *generalized Kaluza-Klein Ansatz* by Kobayashi, O'Neill, Hermann, Trautman, Bourguignon and others.

The section sigma model

Let h be a metric on E which is adapted to π and g and let $H \in \mathcal{H}(E, \pi, M)$ be the corresponding Ehresmann connection.

Definition. For any smooth section $s \in \Gamma(\pi)$, define the *vertical differential* of s through:

$$\mathrm{d}s = P_V \circ \mathrm{d}s : TM \to V \quad ,$$

where $\mathrm{d}s : TM \to TE$ is the ordinary differential and $P_V : TE \to V$ is the projection on V parallel to H. Define the vertical connection induced by (E, π, h, g) on V through:

$$\nabla^v_X(Y) \stackrel{\mathrm{def.}}{=} P_V \nabla_X Y \quad \forall X \in \Gamma(E, TE) \;\; \forall Y \in \Gamma(E, V) \quad ,$$

where ∇ is the Levi-Civita connection of (E, h). Define the *vertical Lagrange density of s* through:

$$e^v(s) \stackrel{\mathrm{def.}}{=} \mathrm{Tr}_g s^*(h) \in \mathcal{C}^\infty(M, \mathbb{R}) \quad ,$$

the *vertical second fundamental form of s* through:

$$\Sigma^v(s) \stackrel{\mathrm{def.}}{=} \nabla^v \mathrm{d}^v s \in \Gamma(M, T^*M \otimes T^*M \otimes s^*(V))$$

(where ∇^v is the connection induced on $s^*(V)$ by ∇^v and by the Levi-Civita connection of (M, g)) and the *vertical tension* of s through:

$$\tau^v(s) \stackrel{\mathrm{def.}}{=} \mathrm{Tr}_g \Sigma^v \in \Gamma(M, s^*(V)) \quad .$$

The section sigma model

The *section sigma model* is defined by the action functional $S_{\rm sc}:\Gamma_c(\pi)\to\mathbb{R}$ given by:

$$S_{\rm sc}[s]\stackrel{\rm def.}{=}\int_M \nu[g]e^v(s)$$

Proposition.[C. M. Wood] The Euler-Lagrange equations of the action $S_{\rm sc}$ are:

$$\tau^v(s)=0\quad . \tag{2}$$

Moreover, we have $\tau^v(s)=P_V\tau(s)$.

Solutions of (2) are called *pseudo-harmonic sections* of (E,h,π,M,g).

Remark. A section of π which is a pseudo-harmonic map from (M,g) to (E,h) is automatically a pseudo-harmonic section, but the converse is true only if the horizontal distribution H is integrable.

Local equivalence with the ordinary sigma model

Let $U \subset M$ be a contractible subset of space-time. Then the restriction $(E|_U, \pi_U, U)$ of (E, π, M) to U is isomorphic with the trivial fiber bundle $(U, U \times \mathcal{M}, \pi_1)$, where $\pi_1 : U \times \mathcal{M} \to U$ is the projection on the first factor. Any smooth section s of (E_U, π_U, U) is the graph of the smooth map $\varphi = \pi_2 \circ s : U \to \mathcal{M}$:

$$s(x) = (x, \varphi(x)) \quad \forall x \in U$$

where $\pi_2 : U \times \mathcal{M} \to \mathcal{M}$ is the projection on the second factor. Conversely, the graph of any smooth map into the fiber is a sooth section of (E_U, π_U, U). We have:

$$\mathrm{d}^v s = \mathrm{d}\varphi \quad , \quad e^v(s) = e(\varphi) \quad , \quad \tau^v(s) = \tau(\varphi)$$

so s is a harmonic section of $(E_U, h_U, \pi_U, U, g_U)$ iff φ is a harmonic map from (U, g_U) to $(\mathcal{M}, \mathcal{G})$. Thus:

Local equivalence with the ordinary sigma model.

The section sigma model is locally equivalent with the ordinary sigma model on any contractible open subset U of space-time. In particular, the local pseudo-harmonic sections of (E, h, π, g, M) defined on U are the graphs of the pseudo-harmonic functions from (U, g_U) into the fiber $(\mathcal{M}, \mathcal{G})$.

The section sigma model is locally indistinguishable from the ordinary sigma model and hence provides an allowed extension of the latter to non-contractible space-times.

Scalar locally geometric U-folds

Definition. A four-dimensional *locally geometric scalar U-fold* of vertical type $(\mathcal{M},\mathcal{G})$ is a global solution of the equations of motion of the section sigma model defined by (E,h,π,M,g) (i.e., a pseudo-harmonic section s of (E,h,π,M,g)), where (M,g) is any four-dimensional space-time and (E,π,M) is any fiber bundle with base M, endowed with a Lorentzian signature metric h which is compatible with π and g.

The U-fold interpretation arises by picking an open cover $(U_\alpha)_{\alpha\in I}$ of M which is a trivializing cover for (E,π,M). Representing $s_\alpha \stackrel{\mathrm{def.}}{=} s|_{U_\alpha}$ as the graph of a map $\varphi_\alpha : U_\alpha \to \mathcal{M}$, the pseudoharmonic section equation for s amounts to the usual harmonic map equations:

$$\tau(\varphi_\alpha) = 0 \quad \forall \alpha \in I \quad .$$

If $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \stackrel{\mathrm{def.}}{=} \mathrm{Iso}(\mathcal{M},\mathcal{G})$ are the transition functions of (E,π,M) defined by the trivializing cover, then $g_{\alpha\beta}$ satisfy the *cocycle conditions*:

$$g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma} \quad \forall \alpha,\beta,\gamma \in I$$

while φ_α are subject to the *gluing conditions*:

$$g_{\alpha\beta}(x)\varphi_\beta(x) = \varphi_\alpha(x) \quad \forall \alpha,\beta \in I \quad \forall x \in U_\alpha \cap U_\beta \quad .$$

This realizes Chris Hull's ideology of "gluing local solutions through symmetries of the equations of motion".

Since fiber bundles over (M, g) endowed with adapted metrics are parameterized by principal bundles with connection, it follows that locally geometric scalar U-folds with space-time (M, g) and type $(\mathcal{M}, \mathcal{G})$ are parameterized by principal $\mathrm{Iso}(\mathcal{M}, \mathcal{G})$-bundles with connection defined over M. The topology of such U-folds can be studied with standard methods of fiber bundle theory, such as the Ehresmann parallel transport, bundle monodromy, classifying space theory etc. The analytic aspects can be approached using the theory of pseudo-harmonic sections of fiber bundles.

Numerous new constructions, as well as systematic mathematical study, become possible using the global description of this class of U-folds given above.

Let (E, h, π, M, g) be as above. The duality structure of the Abelian field theory is described by a flat symplectic vector bundle $(\mathcal{S}, D, \omega)$ defined over the total space E of the fiber bundle (E, π, M).

Definition. A taming J of $(\mathcal{S}, \omega)$ is called *vertical* if:

$$D_X \circ J = J \circ D_X \quad \forall X \in \Gamma(E, H) \ \ ,$$

i.e. if J is preserved by the parallel transport of D along H-horizontal curves.

The coupling constants and theta angles of the Abelian gauge theory are described by a vertical taming J of $(\mathcal{S}, D, \omega)$. Notice that J need not be D-flat. Let $s \in \Gamma(\pi)$ be a smooth section of the fiber bundle (E, π, M). The electric and magnetic field strengths in the scalar field background described by s are encoded by a 2-form $\mathcal{V} \in \Omega^2(M, S^s)$ defined on space-time and taking values in the pulled-back bundle S^s, subject to the negative polarization condition:

$$*_g \mathcal{V} = -J^s \mathcal{V}$$

and to the Maxwell equations:

$$\mathrm{d}_{D^s} \mathcal{V} = 0 \ \ .$$

Bosonic U-folds with scalar and Abelian gauge fields

If U is a contractible open subset of the space-time M, then one can show that the model reduces on U to the ordinary sigma model coupled to Abelian gauge fields (with the extension to non-simply connected scalar manifolds $\mathcal{M}$ explained before, which already describes certain U-folds since it encodes twisting by non-trivial duality transformations of the Abelian gauge theory). Thus:

Local equivalence with the ordinary sigma model coupled to Abelian gauge fields

The section sigma model coupled to Abelian gauge fields is locally indistinguishable from the ordinary sigma model and hence it provides an extension of the latter to non-contractible space-times.

Using an open cover of M which is a trivializing cover for (E, π, M), one finds that global solutions of the section sigma model coupled to Abelian gauge fields can be obtained by gluing local solutions of the ordinary sigma model coupled to Abelian gauge fields using symmetries of the latter, thus realizing the ideology of U-folds. For any $x \in M$, let $(S_x, D_x, J_x, \omega_x)$ denote the pull-back of (S, D, J, ω) to the fiber $E_x \stackrel{\mathrm{def.}}{=} \pi^{-1}(x)$ of E above x.

Locally geometric U-folds with scalar and Abelian gauge fields

Definition. The *horizontal transport* of a flat vector bundle $(\mathcal{S}, D)$ along a curve $\gamma : [0,1] \to M$ is the isomorphism of flat vector bundles $\mathbb{U}_\gamma : (S_{\gamma(0)}, D_{\gamma(0)}) \to (S_{\gamma(1)}, D_{\gamma(1)})$ defined by:

$$\mathbb{U}_\gamma(e) = U_{\gamma_e} : \mathcal{S}_e \to \mathcal{S}_{\gamma_e(1)} \quad \forall e \in E_{\gamma(0)} \quad ,$$

where U is the parallel transport of D and γ_e is the horizontal lift of γ starting at the point e (thus $\gamma_e(0) = e$ and $\gamma_e(1) \in E_{\gamma(1)}$). The horizontal transport covers the Ehresmann transport of (E, π, M) defined by the horizontal distribution H.

Proposition. Let J be a vertical taming of $(\mathcal{S}, D, J, \omega)$. Then $\mathbb{U}_\gamma$ is an isomorphism of tamed flat symplectic vector bundles from $(S_{\gamma(0)}, D_{\gamma(0)}, J_{\gamma(0)}, \omega_{\gamma(0)})$ to $(S_{\gamma(1)}, D_{\gamma(1)}, J_{\gamma(1)}, \omega_{\gamma(1)})$. In particular, the isomorphism type of $(S_x, D_x, J_x, \omega_x)$ is independent on $x \in M$ (since M is connected) and is called the *vertical type* of $(\mathcal{S}, D, J, \omega)$.

Definition. Let $(\mathcal{M}, \mathcal{G})$ be a Riemannian manifold and $(\mathcal{S}_0, D_0, J_0, \omega_0)$ be a tamed symplectic vector bundle defined on $(\mathcal{M}, \mathcal{G})$. A four-dimensional locally geometric U-fold coupled to Abelian gauge fields of type $(\mathcal{M}, \mathcal{G}, \mathcal{S}_0, D_0, J_0)$ is a global solution $(s, \mathcal{V})$ of the equations of motion of the section sigma model coupled to Abelian gauge fields defined by some fiber bundle (E, π, M) defined over a four-dimensional space-time (M, g) and endowed with a (π, h) compatible Lorentzian signature metric h and with a vertically-tamed flat symplectic vector bundle $(\mathcal{S}, D, J, \omega)$ defined over E and having vertical type $(\mathcal{S}_0, D_0, J_0, \omega_0)$.

Final remarks

We gave a mathematical description of bosonic locally geometric U-folds, which can be studied with standard tools of global differential geometry. We conjecture that this is the most general description possible.

To prove the conjecture, one would first have to give a proper mathematical definition of locally geometric supergravity U-folds, which no one never did. One way to view our work is as providing such a definition. Some further aspects:

- One can extend our work to include fermions fields and to extract the conditions for a locally-geometric bosonic U-fold as defined above to preserve a certain amount of supersymmetry.
- Our models can be extended to the case when $\pi : E \to M$ is a submersion which is not locally trivial (i.e., which is not a fiber bundle). It is tempting to conjecture that this corresponds to the classical limit of "locally non-geometric U-folds".
- One can study the global geometry and topology of U-folds using our results. For example, one can formulate a global topological classification problem for U-fold solutions, which appears to be rather hard.
- It would be interesting to construct new interesting solutions using our approach, for example "U-fold black holes".
- One can follow the same approach to construct global formulations of supergravity theories coupled to matter in other dimensions and or in Euclidean signature and to give a global description of the corresponding U-folds.
- Quantum aspects, extension to string theory.