Topological obstructions for bundles of Clifford modules

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Dirac operators and pinor bundles

Question

Given a smooth pseudo-Riemannian manifold (M, g) and a smooth **real** vector bundle S over M, when can one define a Dirac operator $D : \Gamma(M, S) \to \Gamma(M, S)$?

The Dirac operator can be defined on S iff. S admits a globally well-defined Clifford multiplication $TM \otimes S \to S$, which amounts to the condition that S is a bundle of modules over the Clifford bundle $\mathrm{Cl}(TM)$. Thus, we are asking:

Equivalent question

When does S admit a smooth morphism of vector bundles $\gamma : \mathrm{Cl}(TM) \to \mathrm{End}(S)$ such that γ is a unital morphism of bundles of algebras ?

Definition. A pair (S, γ) with this property is called a *real pinor bundle*. When γ is fiberwise-irreducible, it is called an *elementary* real pinor bundle.

Related questions

- What are the topological obstructions to existence of real pinor bundles ?
- Classify (elementary) real pinor bundles on (M, g) up to isomorphism.

Quadratic spaces over $\mathbb{R}$ and their morphisms form a category whose unit groupoid $\mathrm{Quad}^\times$ we call the *groupoid of real quadratic spaces*; its objects coincide with those of Quad while its morphisms are the invertible isometries.

The Clifford algebra construction gives a functor $\mathrm{Cl} : \mathrm{Quad} \to \mathrm{Alg}$, where Alg denotes the category of unital associative $\mathbb{R}$-algebras and unital algebra morphisms. For each object (V, h) of Quad, $\mathrm{Cl}(V, h)$ is the Clifford algebra of the quadratic space (V, h) while for each isometry $\varphi : (V, h) \to (V', h')$, $\mathrm{Cl}(\varphi) : \mathrm{Cl}(V, h) \to \mathrm{Cl}(V', h')$ denotes the unique unital morphism of algebras which satisfies the condition $\mathrm{Cl}(\varphi)|_V = \varphi$. The image of the functor Cl is a *non-full* sub-category of Alg which we denote by Cl and whose unit groupoid we denote by $Cl^\times$.

Definition A *morphism of Clifford algebras* is a morphism $\alpha : \mathrm{Cl}(V, h) \to \mathrm{Cl}(V', h')$ in the category Cl, i.e. a morphism of unital algebras which satisfies $\alpha(V) \subset V'$ and hence is necessarily of the form $\alpha = \mathrm{Cl}(\varphi)$ for a (uniquely-determined) isometry $\varphi : (V, h) \to (V', h')$, given by $\varphi \stackrel{\mathrm{def.}}{=} \alpha|_V$.

The corestriction of the functor Cl to its image gives an isomorphism of categories $\mathrm{Quad} \simeq Cl$, which in turn restricts to an isomorphism $\mathrm{Quad}^\times \simeq Cl^\times$.

The category of Clifford representations over $\mathbb{R}$

Definition. A *Clifford representation* is a morphism of unital algebras $\gamma : \mathrm{Cl}(T,g) \to \mathrm{End}_{\mathbb{R}}(S)$, where S is a finite-dimensional $\mathbb{R}$-vector space.

Definition. Let $\gamma : \mathrm{Cl}(T,g) \to \mathrm{End}_{\mathbb{R}}(S)$ and $\gamma' : \mathrm{Cl}(T',g') \to \mathrm{End}_{\mathbb{R}}(S')$ be two Clifford representations. A *morphism* from γ to γ' is a pair (f_0, f) such that:

1. $f_0 : T \to T'$ is an isometry from (T,g) to (T',g')
2. $f : S \to S'$ is an $\mathbb{R}$-linear map
3. $\gamma'(\mathrm{Cl}(f_0)(x)) \circ f = f \circ \gamma(x)$ for all $x \in \mathrm{Cl}(T,g)$.

A *based morphism* is a morphism (f_0, f) such that $f_0 = \mathrm{id}_T$.

With this definition, Clifford representations form a category denoted ClRep. The forgetful functor $\pi : \mathrm{ClRep} \to Cl$ which takes γ into $\mathrm{Cl}(T,g)$ and (f_0, f) into $\mathrm{Cl}(f_0)$ is a fibration whose fiber above $\mathrm{Cl}(T,g)$ is the usual category $\mathrm{Rep}(Cl(T,g))$ of representations of $\mathrm{Cl}(T,g)$ (whose morphisms are the based morphisms of representations). Hence equivalences of representations of Clifford algebras coincide with the based isomorphisms of ClRep; in particular, any isomorphism class of Clifford representations decomposes as a disjoint union of equivalence classes.

Definition. A Clifford representation $\gamma : \mathrm{Cl}(T,h) \to \mathrm{End}_{\mathbb{R}}(S)$ is called *weakly faithful* if the restriction $\gamma_0 \stackrel{\mathrm{def.}}{=} \gamma|_T : T \to \mathrm{End}_{\mathbb{R}}(S)$ is an injective map. It is called *rigid* if $\gamma_T = \mathrm{id}_T$.

Real Lipschitz groups

Let $\gamma : \mathrm{Cl}(V,h) \rightarrow \mathrm{End}_{\mathbb{R}}(S)$ be a weakly faithful Clifford representation. Since γ is weakly-faithful, the restriction of γ gives a linear isomorphism:

$$\gamma|_V : V \xrightarrow{\sim} W(\gamma)$$

which transports h to a symmetric non-degenerate pairing $g_\gamma : W(\gamma) \times W(\gamma) \rightarrow \mathbb{R}$:

$$g_\gamma(w_1, w_2) \stackrel{\mathrm{def.}}{=} h((\gamma|_V)^{-1}(w_1), (\gamma|_V)^{-1}(w_2)) \ \ \forall w_1, w_2 \in W(\gamma) \ \ .$$

Thus $(W(\gamma), g_\gamma)$ is a quadratic space and $\gamma|_V : (V,h) \xrightarrow{\sim} (W(\gamma), g_\gamma)$ is an invertible isometry.

Definition. The group $\mathrm{L}(\gamma) \stackrel{\mathrm{def.}}{=} \mathrm{Aut}_{\mathrm{ClRep}}(\gamma) = \{a \in \mathrm{Aut}_{\mathbb{R}}(S) | \mathrm{Ad}(a)(W) \subset W\}$ is called the *Lipschitz group* of γ.

For any $a \in \mathrm{L}, w \in W$, we have $\mathrm{Ad}(a)(w) \in W$ and $\mathrm{Ad}(a)(w)^2 = \mathrm{Ad}(a)(w^2)$. Since $w^2 = g(w,w)\mathrm{id}_S$ and $\mathrm{Ad}(a)(w)^2 = g(\mathrm{Ad}(a)(w), \mathrm{Ad}(a)(w))\mathrm{id}_S$, this implies $\mathrm{Ad}(a) \in \mathrm{O}(W, g)$.

Definition. The group morphism $\mathrm{Ad}_0^\gamma : \mathrm{L}(\gamma) \rightarrow \mathrm{O}(W(\gamma), g_\gamma)$ given by:

$$\mathrm{Ad}_0^\gamma(a) \stackrel{\mathrm{def.}}{=} \mathrm{Ad}(a)|_W$$

is called the *vector representation* of $\mathrm{L}(\gamma)$.

Irreducible Clifford representations

A Clifford irrep $\gamma : \mathrm{Cl}(V,h) \to \mathrm{End}_{\mathbb{R}}(S)$ is faithful iff $\mathrm{Cl}(V,h)$ is simple as an associative $\mathbb{R}$-algebra, which happens when $p-q \not\equiv_8 1,5$ (the *simple case*). The pinor volume form $\omega = \gamma(\nu)$ is proportional to id_S iff we are in the *non-simple case* $p-q \equiv_8 1,5$. In the simple case, all real irreps of $\mathrm{Cl}(V,h)$ are equivalent. In the non-simple case, $\mathrm{Cl}(V,h)$ admits two inequivalent irreps, which can be realized in the same space S. In each of these, the Clifford volume form $\nu \in \mathrm{Cl}(V,h)$ defined by a given orientation of V satisfies:

$$\omega = \gamma(\nu) = \epsilon_\gamma \mathrm{id}_S \quad ,$$

where $\epsilon_\gamma \in \{-1,1\}$ is the *signature* of γ. We have:

$$\gamma_+ = \gamma_- \circ \pi \quad , \tag{1}$$

where $\pi : \mathrm{Cl}(V,h) \to \mathrm{Cl}(V,h)$ is the main automorphism, which satisfies $\pi(\nu) = -\nu$ since $d = \dim V = p+q$ is odd in the non-simple case.

Proposition. Let (V,h) be a quadratic space. Then the real irreps of $\mathrm{Cl}(V,h)$ are weakly faithful. Moreover, there exits a single isomorphism class of such representations, which is uniquely determined by the isomorphism class of (V,h) and hence by the signature of h. In the simple cases, this isomorphism class is also an equivalence class of representations. In the non-simple cases, this isomorphism class decomposes into two equivalence classes of representations, each of which is determined by the signature of h.

Reduced Lipschitz groups of irreducible Clifford representations

Let L be the Lipschitz group of a Clifford irrep $\gamma : \mathrm{Cl}(V, h) \to \mathrm{End}(S)$.

Definition. The *reduced Lipschitz group* $\mathcal{L}$ is the the kernel of a certain group morphism $|\mathcal{N}_e| : \mathrm{L} \to \mathbb{R}_{>0}$ which is called the *canonical Lipschitz norm*.

Proposition. $\mathcal{L}$ is homotopy equivalent with L.

Let $\alpha \in \{-1, 1\}$ and $\mathrm{Pin}_2(\alpha) \stackrel{\mathrm{def.}}{=} \begin{cases} \mathrm{Pin}_{2,0} & \text{if } \alpha = +1 \\ \mathrm{Pin}_{0,2} & \text{if } \alpha = -1 \end{cases}$.

Definition.

$$\mathrm{Spin}^o_\alpha(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Spin}(V, h) \cdot \mathrm{Pin}_2(\alpha) \stackrel{\mathrm{def.}}{=} [\mathrm{Spin}(V, h) \times \mathrm{Pin}_2(\alpha)]/\{-1, 1\} \ .$$

Theorem. The reduced Lipschitz group $\mathcal{L}$ is isomorphic with the *canonical spinor group* $\Lambda(V, h)$ of (V, h), which is defined as follows:

1. In the normal simple case, we set $\Lambda(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Pin}(V, h)$.
2. In the complex case, we set $\Lambda(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Spin}^o_{\alpha_{p,q}}(V, h)$, where $\alpha_{p,q} = (-1)^{\frac{p-q+1}{4}}$.
3. In the quaternionic simple case, we set $\Lambda(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Pin}^q(V, h) = \mathrm{Pin}(V, h) \cdot \mathrm{Sp}(1)$
4. In the normal non-simple case, we set $\Lambda(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Spin}(V, h)$
5. In the quaternionic non-simple case, we set $\Lambda(V, h) \stackrel{\mathrm{def.}}{=} \mathrm{Spin}^q(V, h) = \mathrm{Spin}(V, h) \cdot \mathrm{Sp}(1)$.

Canonical spinor groups

$p-q$ mod 8	type	$\Lambda(V,h)$
$0,2$	normal simple	$\mathrm{Pin}(V,h)$
$3,7$	complex simple	$\mathrm{Spin}^o_-(V,h), \mathrm{Spin}^o_+(V,h)$
$4,6$	quaternionic simple	$\mathrm{Pin}^q(V,h)$
1	normal non-simple	$\mathrm{Spin}(V,h)$
5	quaternionic non-simple	$\mathrm{Spin}^q(V,h)$

Canonical spinor groups

Remark. The isomorphism $\mathcal{L} \simeq \Lambda$ is generally not natural.

Real pinor bundles and real Lipschitz structures

Definition. A real pinor bundle (S, γ) is called *weakly faithful* if $\gamma_0 \stackrel{\text{def.}}{=} \gamma|_{T^*M}$ is a monomorphism of vector bundles from T^*M to $\mathrm{End}(S)$.

Let $\mathrm{ClB}(M, g)$ denote the category of real pinor bundles over (M, g) and based pinor bundle morphisms and $\mathrm{ClB}_w(M, g)$ denote the full sub-category whose objects are the weakly faithful real pinor bundles.

Definition. The *type* of a real pinor bundle (S, γ) is the isomorphism class $\lambda(S, \gamma)$ of its fiberwise Clifford representation $\gamma_p : (T^*_p M, g^*_p) \to \mathrm{Aut}_{\mathbb{R}}(S_p)$.

Let $\mathrm{ClB}^\eta_w(M, g)$ denote the full sub-category of $\mathrm{ClB}_w(M, g)$ consisting of all real pinor bundles of type equal $[\eta]$ and $\mathrm{Cl}^\eta_w(M, g)^\times$ the corresponding unit groupoid. Let $\mathrm{L} \stackrel{\text{def.}}{=} \mathrm{Aut}(\eta)$ and $\mathrm{Ad}_0 : \mathrm{L} \to \mathrm{O}(V, h)$ be the vector representation of L.

Definition. Let P be a principal $\mathrm{O}(V, h)$-bundle over M. A *Lipschitz structure on P relative to η* is an Ad_0-reduction of P to L. A *Lipschitz structure on (M, g) relative to η* is an Ad_0-reduction of $P_{\mathrm{O}(V,h)}$.

Definition. Let $\tau : P \to P_{\mathrm{O}(V,h)}(M, g)$ and $\tau' : P' \to P_{\mathrm{O}(V,h)}(M, g)$ be two Lipschitz structures relative to η. An *isomorphism of Lipschitz structures* from τ to τ' is a based isomorphism of principal L-bundles $f : P \to P'$ such that $\tau' \circ f = \tau$.

Let $L_\eta(M, g)$ be the groupoid of Lipschitz structures of (M, g) relative to η.

Classification of real pinor bundles

Theorem. The groupoids $\mathrm{ClB}_w^\eta(M,g)^\times$ and $\mathrm{L}_\eta(M,g)$ are equivalent.

Remark. We construct an explicit equivalence.

Definition. An *elementary real pinor bundle* over (M,g) is a bundle (S,γ) of real pinors which is fiberwise irreducible. A *(reduced) elementary real Lipschitz structure* of (M,g) is a (reduced) Lipschitz structure relative to an irreducible Clifford representation.

Theorem. The following groupoids are equivalent for any pseudo-Riemannian manifold (M,g):

(a) The groupoid of elementary real pinor bundles of (M,g).
(b) The groupoid of elementary Lipschitz structures.
(c) The groupoid of elementary reduced Lipschitz structures.
(d) The groupoid of canonical spinor structures.

Depending on the dimension and signature, this groupoid equals:

1. When $p-q\equiv_8 0,2$ (normal simple case): the groupoid of untwisted Pin structures.
2. When $p-q\equiv_8 3,7$ (complex case): the groupoid of Spin^o structures
3. When $p-q\equiv 4,6$ (quaternionic simple case): the groupoid of untwisted Pin^q structures.
4. When $p-q\equiv_8 1$ (normal non-simple case): the groupoid of Spin structures.
5. When $p-q\equiv_8 5$ (quaternionic non-simple case): the groupoid of Spin^q structures.

Topological obstructions

Let $\sigma := \sigma_{p,q} \stackrel{\text{def.}}{=} (-1)^{q+\left[\frac{d}{2}\right]}$ and $\alpha := \alpha_{p,q} = (-1)^{\frac{p-q+1}{4}}$.

Theorem.

- In the normal simple case ($p - q \equiv_8 0, 2$), the following statements are equivalent:
  - (a) (M, g) admits a Lipschitz structure
  - (b) (M, g) admits an untwisted $\mathrm{Pin}(V, h)$ structure
  - (c) (M, g) admits a twisted $\mathrm{Pin}(V, -\sigma h)$ structure
  - (d) We have $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^\sigma(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = 0$.
- In the complex case, the following statements are equivalent:
  - (a) (M, g) admits a Lipschitz structure
  - (b) (M, g) admits a Spin^o structure
  - (c) We have $\mathrm{w}_1(M) = \mathrm{w}_1(E)$ and $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = \mathrm{w}_2(E) + \mathrm{w}_1(E)(p\mathrm{w}_1^+(M) + q\mathrm{w}_1^-(M)) + \frac{1}{2}\left[p(p+\alpha) + q(q+\alpha)\right]\mathrm{w}_1(E)^2$.
- In the quaternionic simple case ($p - q \equiv_8 4, 6$), the following statements are equivalent:
  - (a) (M, g) admits a Lipschitz structure
  - (b) (M, g) admits an untwisted Pin^q structure
  - (c) $(M, -\sigma g)$ admits a twisted Pin^q structure
  - (d) The exists a principal $\mathrm{SO}(3)$-bundle E over M such that $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^\sigma(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = \mathrm{w}_2(E)$.

- In the normal non-simple case ($p - q \equiv_8 1$), the following statements are equivalent:
  (a) (M, g) admits a Lipschitz structure
  (b) (M, g) admits a Spin structure
  (c) We have $\mathrm{w}_1(M) = 0$ and $\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = 0$.
- In the quaternionic non-simple case ($p - q \equiv_8 5$), the following statements are equivalent:
  (a) (M, g) admits a Lipschitz structure
  (b) (M, g) admits a Spin^q structure
  (c) The exists a principal $\mathrm{SO}(3)$-bundle E over M such that
  $$w_1(M) = 0 \quad \text{and} \quad \mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = \mathrm{w}_2(E).$$

Other results and open problems

Other results:

- We studied the so-called basic representations of Lipschitz groups (vector, Schur and anticommutant representations)
- Study of Lipschitz norms
- Conditions when a pinor bundle has an admissible bilinear form
- Various other characterizations of topological obstructions
- Description of the set of all $\mathrm{Spin}^o_\pm$ structures
- Applications to structure of Dirac operators and to the Kähler-Atiyah formalism, to reconstruction theorems for spinors etc.
- Equivalent descriptions of $\mathrm{Spin}^o_\pm$ groups
- Connected components of Lipschitz groups.
- Interpretation of Lipschitz groups through quantization of spin systems.
- Various examples

Open problems:

- Applications to supergravity. Non-standard compactifications etc.
- Index theory.
- Applications to various generalizations of monopole equations.