$\mathrm{SU}(2)$ structures in $\mathcal{N} = 2$ compactifications of M-theory

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$\mathcal{N} = 2$ compactifications of M-theory on eight-manifolds M

General compactifications of M-theory on eight-manifolds provide a rich class of geometries which are of physical interest due to their relation to F-theory. Such compactifications are of warped product type:

$$\hat{M} = N \times M$$

where (M, g) is an oriented Riemannian eight-manifold while N is an AdS_3 space with cosmological constant $\lambda = -8\kappa^2$, the case $\kappa = 0$ being the Minkowski case. The warp factor is a smooth function $\Delta : M \to \mathbb{R}$. The understanding of such backgrounds was quite limited before our work. The notable exception is the class of compactifications down to 3-dimensional Minkowski space, which were studied intensively starting with the work of the Becker sisters. Supersymmetry generators satisfy:

$$D\eta = 0 \ , \quad \eta = e^{\frac{\Delta}{2}} \sum_{i=1}^{s} \zeta_i \otimes \xi_i \quad , \quad \zeta_i \in \Gamma(N, S_3) \ , \quad \xi_i \in \Gamma(M, S)$$

where:

$$s = \dim \mathcal{K} \ , \quad \mathcal{K} = \{\xi \in \Gamma(M, S) | \ \mathbb{D}\xi = Q\xi = 0\}$$

Given a frame $(\xi_1, \ldots, \xi_s)$ of $\mathcal{K}$, the stabilizer group H_p of $\xi_i(p)$ is:

$$H_p = \mathrm{Stab}_{\mathrm{Spin}(T_pM, g_p)}(\xi_1(p), \ldots, \xi_s(p)) = \{h \in \mathrm{Spin}(T_pM, g_p) | \ h\xi_i(p) = \xi_i(p) \ , \ \forall i = 1 \ldots s\} \ .$$

The *stabilizer stratification* of M is the stratification whose strata are the subsets of M on which the isomorphism type of H_p is fixed.

For the case $s = 2$ ($\mathcal{N} = 2$ compactifications of M-theory down to AdS_3), we described in previous work the stabilizer stratification of M using a certain semi-algebraic body, thus giving a complete description of the "stratified G-structure" of such backgrounds.

Differential forms induced by two Majorana spinors ξ_1 and ξ_2

For $\mathscr{B}$ a symmetric bilinear pairing of type $+1$, we can construct the following differential bilinear forms in ξ:

$$\begin{aligned}
b_i &= \mathscr{B}(\xi_i, \gamma(\nu)\xi_i)\ ,\ b_3 = \mathscr{B}(\xi_1, \gamma(\nu)\xi_2)\ ,\ V_i = \mathscr{B}(\xi_i, \gamma_a\xi_i)e^a\ ,\ V_3 = \mathscr{B}(\xi_1, \gamma_a\xi_2)e^a \\
Y_i &= \frac{1}{4!}\mathscr{B}(\xi_i, \gamma_{a_1\ldots a_4}\xi_i)e^{a_1\cdots a_4}\ ,\ Y_3 = \frac{1}{4!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_4}\xi_2)e^{a_1\cdots a_4} \\
Z_i &= \frac{1}{5!}\mathscr{B}(\xi_i, \gamma_{a_1\ldots a_5}\xi_i)e^{a_1\cdots a_5}\ ,\ Z_3 = \frac{1}{5!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_5}\xi_2)e^{a_1\cdots a_5} \\
K &= \frac{1}{2!}\mathscr{B}(\xi_1, \gamma_{a_1a_2}\xi_2)e^{a_1a_2}\ ,\ \Psi = \frac{1}{3!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_3}\xi_2)e^{a_1\cdots a_3} \\
\Lambda &= \frac{1}{6!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_6}\xi_2)e^{a_1\cdots a_6}\ ,\ \Theta = \frac{1}{7!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_7}\xi_2)e^{a_1\cdots a_7}\ .
\end{aligned}$$

We also define:

$$\phi_i \stackrel{\text{def.}}{=} *Z_i = \frac{1}{3!}\mathscr{B}(\xi_i, \gamma_{a_1\ldots a_3}\gamma(\nu)\xi_i)e^{a_1\cdots a_3}\ ,\ \phi_3 \stackrel{\text{def.}}{=} *Z_3 = \frac{1}{3!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_3}\gamma(\nu)\xi_2)e^{a_1\cdots a_3}$$

$$W \stackrel{\text{def.}}{=} -*\Theta = \mathscr{B}(\xi_1, \gamma_a\gamma(\nu)\xi_2)e^a$$

and:

$$V_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(V_1 \pm V_2)\ ,\ \phi_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(\phi_1 \pm \phi_2)$$

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The distributions $\mathcal{D}$ and $\mathcal{D}_0$

Consider the cosmooth generalized distributions:

$$\mathcal{D} \stackrel{\rm def.}{=} \ker V_+ \cap \ker V_- \cap \ker V_3 \subset TM \quad , \quad \mathcal{D}_0 \stackrel{\rm def.}{=} \mathcal{D} \cap \ker W \subset \mathcal{D} \quad . \tag{1}$$

We showed in a previous paper that the rank stratifications of M induced by $\mathcal{D}$ and $\mathcal{D}_0$ have the same open stratum, the so-called generic locus of M:

$$\mathcal{U} \stackrel{\rm def.}{=} \{p \in M |\ \mathrm{rk}\mathcal{D}(p) = 5\} = \{p \in M |\ \mathrm{rk}\mathcal{D}_0(p) = 4\}$$

while the complement $\mathcal{W} \stackrel{\rm def.}{=} M \setminus \mathcal{U}$ (the *non-generic locus*) decomposes as:

$$\mathcal{W} = \mathcal{W}_2 \sqcup \mathcal{W}_1 \sqcup \mathcal{W}_0 = \mathcal{Z}_2 \sqcup \mathcal{Z}_1 \sqcup \mathcal{Z}_0 \quad ,$$

where:

$$\mathcal{W}_k \stackrel{\rm def.}{=} \{p \in \mathcal{W} |\ \mathrm{rk}\mathcal{D}(p) = 8 - k\}$$

$$\mathcal{Z}_k \stackrel{\rm def.}{=} \{p \in \mathcal{W} |\ \mathrm{rk}\mathcal{D}_0(p) = 8 - k\}$$

and $\mathcal{Z}_3 = \emptyset$.

The rank stratifications of M induced by $\mathcal{D}$ and $\mathcal{D}_0$ give the disjoint union decompositions:

$$M = \mathcal{U} \sqcup \mathcal{W}_2 \sqcup \mathcal{W}_1 \sqcup \mathcal{W}_0 \quad , \quad M = \mathcal{U} \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_1 \sqcup \mathcal{Z}_0 \quad .$$

The G-structure stratification

Theorem. The G-structure stratification coincides with the rank stratification of $\mathcal{D}_0$. For $p \in M$, the stabilizer group H_p is given by:

- $H_p \simeq \mathrm{SU}(2)$ if $p \in \mathcal{U}_0 = \mathcal{U}$
- $H_p \simeq \mathrm{SU}(3)$ if $p \in \mathcal{Z}_2$
- $H_p \simeq \mathrm{G}_2$ if $p \in \mathcal{Z}_1$
- $H_p \simeq \mathrm{SU}(4)$ if $p \in \mathcal{Z}_0$.

The situation is summarized in the following table:

$\mathcal{D}$-stratum	$\mathcal{D}_0$-stratum	$\mathrm{rk}\mathcal{D}$	$\mathrm{rk}\mathcal{D}_0$	H_p
$\mathcal{W}_0$	$\mathcal{Z}_0$	8	8	$\mathrm{SU}(4)$
$\mathcal{W}_1^1$	$\mathcal{Z}_1$	7	7	G_2
$\mathcal{W}_1^0$	$\subset \mathcal{Z}_2$	7	6	$\mathrm{SU}(3)$
$\mathcal{W}_2$	$\subset \mathcal{Z}_2$	6	6	$\mathrm{SU}(3)$
$\mathcal{U}$	$\mathcal{U}_0$	5	4	$\mathrm{SU}(2)$

The ranks of $\mathcal{D}$ and $\mathcal{D}_0$ on various loci and the isomorphism type of H_p.

Conceptual summary of the main result

Hence the internal eight-manifold of an $\mathcal{N}=2$ compactification of eleven-dimensional supergravity down to AdS_3 carries a natural stratification whose open stratum, when non-empty, is endowed with an $\mathrm{SU}(2)$ structure. This is the generic locus $\mathcal{U}$ mentioned above and it is the *largest* stratum when it is non-empty (which is the generic case).

Main result

We show that the generic stratum $\mathcal{U}$ admits a codimension three foliation $\mathcal{F}$ which integrates the distribution $\mathcal{D}|_{\mathcal{U}}$ and whose five-dimensional leaves support this $\mathrm{SU}(2)$ structure. We give explicit formulas for the defining forms of this structure in terms of the two Majorana spinors ξ_1, ξ_2 defined on M which correspond to the supersymmetry generators.

Remark. We never appeal to any auxiliary 9-manifold. As we showed in previous work, using an auxiliary 9-manifold $\hat{M}$ does not simplify the problem but moves the complication somewhere else.

- $\mathrm{SU}(2)$ structures on $\mathcal{D}|_{\mathcal{U}} \subset T\mathcal{U}$ can be parameterized as explained by Conti and Salamon using three 2-forms and one 1-form obeying certain algebraic conditions. Namely, they are characterized by quadruplets $(\boldsymbol{\alpha}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$, where $\boldsymbol{\alpha} \in \Gamma(\mathcal{U}, \mathcal{D}^*)$ is a nowhere-vanishing 1-form along $\mathcal{D}$ and $\boldsymbol{\omega}_i \in \Gamma(\mathcal{U}, \wedge^2\mathcal{D}^*)$ are nowhere-vanishing 2-forms along $\mathcal{D}$ satisfying:

$$\boldsymbol{\omega}_i \wedge \boldsymbol{\omega}_j = \delta_{ij}\mathbf{v}\ , \quad \boldsymbol{\alpha} \wedge \mathbf{v} \neq 0\ . \qquad (2)$$

  for some nowhere-vanishing $\mathbf{v} \in \Gamma(\mathcal{U}, \wedge^4\mathcal{D}^*)$.
- Using the *method of spinor rotations*, we will express $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ through forms constructed as bilinears in the two spinors ξ_1, ξ_2 given on the eight-manifold M.

Parameterising the SU(2) structure

Recall that $V_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(V_1 \pm V_2) \in \Omega^1(M)$. Let ν be the normalized volume form of (M, g). One can show that $V_+ \wedge V_- \wedge V_3$ does not vanish anywhere on the generic locus $\mathcal{U}$. Let:

$$\nu_\perp \stackrel{\text{def.}}{=} -\frac{1}{||V_+ \wedge V_- \wedge V_3||} \iota_{V_+ \wedge V_- \wedge V_3} \nu \in \Gamma(\mathcal{U}, \wedge^5 \mathcal{D}^*)$$

be the normalized volume form of $(\mathcal{D}|_\mathcal{U}, g|_\mathcal{D})$ with respect to the orientation of $\mathcal{D}$ induced from that of TM. We introduce the following $\mathcal{D}$-longitudinal 2-forms:

$$U_+ \stackrel{\text{def.}}{=} V_- \wedge V_3 \quad , \quad U_- \stackrel{\text{def.}}{=} V_+ \wedge V_3 \quad , \quad U_3 = V_- \wedge V_+$$

and the following sign factors:

$$\epsilon_+ \stackrel{\text{def.}}{=} +1 \quad , \quad \epsilon_- = \epsilon_3 \stackrel{\text{def.}}{=} -1 \quad .$$

With these notations, we have:

$$\langle U_r, U_s \rangle = (-1)^{r+s} \epsilon_r \epsilon_s \det G^{r|s} \quad , \quad \forall r \in \{+, -, 3\} \quad ,$$

where $G^{r|s}$ is the 2x2 matrix obtained by deleting row r and column s of the Gram matrix G of V_+, V_-, V_3.

Parameterising the SU(2) structure

Theorem 1. The 3-forms $\phi_{\pm,3}|_\mathcal{U}$ (recall $\phi_\pm \stackrel{\mathrm{def.}}{=} \frac{1}{2}(\phi_1 \pm \phi_2)$) admit unique decompositions:

$$\boxed{\begin{aligned} \phi_+|_\mathcal{U} &= \phi_+^\perp + X_- \wedge \omega_3 - X_3 \wedge \omega_- - X_- \wedge X_3 \wedge \boldsymbol{\alpha} \\ \phi_-|_\mathcal{U} &= \phi_-^\perp + X_+ \wedge \omega_3 - X_3 \wedge \omega_+ + X_+ \wedge X_3 \wedge \boldsymbol{\alpha} \\ \phi_3|_\mathcal{U} &= \phi_+^\perp + X_- \wedge \omega_+ + X_+ \wedge \omega_- + X_- \wedge X_+ \wedge \boldsymbol{\alpha} \quad . \end{aligned}}$$

with the following properties:

1. (X_+, X_-, X_3) is the frame of the distribution $\mathcal{D}^\perp|_\mathcal{U} \subset T\mathcal{U}$ which is contragradient to the frame (V_+, V_-, V_3):
$$\langle X_r, V_s \rangle = \delta_{rs} \ , \quad \forall r, s \in \{+, -, 3\} \quad .$$
2. $\phi^\perp_{\pm,3} \in \Gamma(\mathcal{U}, \wedge^3 \mathcal{D}^*)$ are $\mathcal{D}$-longitudinal 3-forms defined on $\mathcal{U}$.
3. $\boldsymbol{\alpha} \in \Gamma(\mathcal{U}, \mathcal{D}^*)$ is a $\mathcal{D}$-longitudinal 1-form defined on $\mathcal{U}$ which satisfies:
$$||\boldsymbol{\alpha}|| = ||V_+ \wedge V_- \wedge V_3|| \quad .$$
4. $\omega_{\pm,3} \in \Gamma(\mathcal{U}, \wedge^2 \mathcal{D}_0)$ satisfy the relations:
$$\langle \omega_r, \omega_s \rangle = 2\langle U_r, U_s \rangle \quad , \quad \omega_r \wedge \omega_s = \langle U_r, U_s \rangle \mathbf{v} \quad , \quad \text{where} \quad \mathbf{v} \stackrel{\mathrm{def.}}{=} 2\nu_0 \tag{3}$$
   Moreover, we have:
$$\mathcal{D}_0 = \mathcal{D}|_\mathcal{U} \cap \ker \boldsymbol{\alpha} \subset T\mathcal{U} \quad .$$

Remark. The decomposition of $\phi_{\pm,3}$ given in **Theorem 1** determine $\omega_{\pm,3}$ and $\boldsymbol{\alpha}$ in terms of $\phi_{\pm,3}$, which are constructed as spinor bilinears. One can give explicit formulas for $\omega_{\pm,3}$ and $\boldsymbol{\alpha}$ in terms of repeated contractions of $\phi_{\pm,3}$ with $X_{\pm,3}$ (see **Theorem 2** below).

Parameterising the SU(2) structure

Let:

$$\nu_0 \stackrel{\text{def.}}{=} \iota_{\hat{\boldsymbol{\alpha}}}\nu_\perp \in \Gamma(\mathcal{U}, \wedge^4\mathcal{D}_0^*)\ , \quad \text{where} \quad \hat{\boldsymbol{\alpha}} \stackrel{\text{def.}}{=} \frac{\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||} \in \Gamma(\mathcal{U}, \mathcal{D}^*)$$

be the normalized volume form induced on $\mathcal{D}_0$ from $\mathcal{D}$.

Theorem 2. Let $\boldsymbol{\alpha}$ and ω_+, ω_-, ω_3 be defined by the expansions of the previous theorem. Then:

(a) The following relations hold:

$$\boxed{\boldsymbol{\alpha} = -\iota_{V_-\wedge V_+}\phi_3 = -\iota_{V_+\wedge V_3}\phi_- = \iota_{V_-\wedge V_3}\phi_+} \quad ,$$

i.e.:

$$\boldsymbol{\alpha} = \iota_{V_+}J_+ = -\iota_{V_-}J_- = -\iota_{V_3}J_3 \quad ,$$

where the 2-forms $J_{\pm,3}$ are defined through:

$$\boxed{\begin{aligned} J_+ &\stackrel{\text{def.}}{=} \iota_{V_-}\phi_3 = -\iota_{V_3}\phi_- \\ J_- &\stackrel{\text{def.}}{=} \iota_{V_+}\phi_3 = -\iota_{V_3}\phi_+ \\ J_3 &\stackrel{\text{def.}}{=} \iota_{V_-}\phi_+ = -\iota_{V_+}\phi_- \quad . \end{aligned}}$$

(b) The 2-forms $\omega_{\pm,3}$ coincide with the components of $J_{\pm,3}$ which are orthogonal to the one-form $\boldsymbol{\alpha}$:

$$\boxed{\omega_r = J_r - \hat{\boldsymbol{\alpha}} \wedge (\iota_{\hat{\boldsymbol{\alpha}}} J_r)\ , \quad \forall r \in \{+,-,3\}} \quad .$$

Parameterising the $\mathrm{SU}(2)$ structure

Theorem 3. Let:

$$\boxed{\boldsymbol{\omega}_r \stackrel{\text{def.}}{=} \sum_{s=+,-,3} a_{rs}\omega_s \in \Gamma(\mathcal{U}, \wedge^2\mathcal{D}^*)} \quad ,$$

where a_{rs} are any real numbers such that the 3x3 matrix $A \stackrel{\text{def.}}{=} (a_{rs})_{r,s=+,-,3}$ satisfies:

$$ATA^t = I_3 \quad ,$$

with T the positive symmetric 3x3 real matrix with entries:

$$T_{rs} \stackrel{\text{def.}}{=} \langle U_r, U_s\rangle = \epsilon_r\epsilon_s(\det G)(G^{-1})_{rs} \quad ,$$

where $\det G = ||V_+ \wedge V_- \wedge V_3||^2$ and $(G^{-1})_{rs}$ is given by:

$$(G^{-1})_{rs} = \frac{(-1)^{r+s}\det G^{r|s}}{\det G} \quad , \quad \forall r \in \{+,-,3\} \quad .$$

Then $\boldsymbol{\omega}_r$ satisfy the Conti-Salamon relations for an $\mathrm{SU}(2)$ structure defined on the rank five distribution $\mathcal{D}|_{\mathcal{U}}$:

$$\boxed{\boldsymbol{\omega}_r \wedge \boldsymbol{\omega}_s = \delta_{rs}\mathbf{v} \quad , \quad \langle \boldsymbol{\omega}_r, \boldsymbol{\omega}_s\rangle = 2\delta_{rs}} \tag{4}$$

Parameterising the $\mathrm{SU}(2)$ structure

Proposition. One can choose A such that $T = A^{-1}(A^{-1})^t$ is the Cholesky decomposition of T (in which case A is uniquely determined by T and hence by V_+, V_-, V_3). In this case, we have:

$$\begin{aligned}
a_{11} &= \frac{1}{||\boldsymbol{\alpha}||}\sqrt{\frac{\det G}{\det G_{[23|23]}}} = \frac{1}{\sqrt{\det G_{[23|23]}}} = \frac{1}{||V_- \wedge V_3||}\\
a_{21} &= \frac{1}{||\boldsymbol{\alpha}||}\frac{\det G_{[13|23]}}{\sqrt{G_{33}\det G_{[23|23]}}} = \frac{\langle V_+ \wedge V_3, V_- \wedge V_3\rangle}{||\boldsymbol{\alpha}||\ ||V_3||\ ||V_- \wedge V_3||}\\
a_{22} &= \frac{1}{||\boldsymbol{\alpha}||}\sqrt{\frac{\det G_{[23|23]}}{G_{33}}} = \frac{||V_- \wedge V_3||}{||\boldsymbol{\alpha}||\ ||V_3||}\\
a_{31} &= \frac{1}{||\boldsymbol{\alpha}||}\frac{G_{13}}{\sqrt{G_{33}}} = \frac{\langle V_+, V_3\rangle}{||\boldsymbol{\alpha}||\ ||V_3||}\\
a_{32} &= \frac{1}{||\boldsymbol{\alpha}||}\frac{G_{23}}{\sqrt{G_{33}}} = \frac{\langle V_-, V_3\rangle}{||\boldsymbol{\alpha}||\ ||V_3||}\\
a_{33} &= \frac{1}{||\boldsymbol{\alpha}||}\frac{G_{33}}{\sqrt{G_{33}}} = \frac{||V_3||}{||\boldsymbol{\alpha}||}\quad .
\end{aligned}$$

Directly studying the Fierz identities is hopeless

Let ξ_1, ξ_2 be two everywhere-orthonormal global sections of S. Define:

$$\check{E}_{ij} \stackrel{\text{def.}}{=} \check{E}_{\xi_i,\xi_j} = \tau(\check{E}_{ji}) \quad , \quad \forall i,j \in \{1,2\} \quad .$$

where:

$$\begin{aligned}
\check{E}_{11} &= \frac{1}{16}(1 + V_1 + Y_1 + Z_1 + b_1\nu) \\
\check{E}_{22} &= \frac{1}{16}(1 + V_2 + Y_2 + Z_2 + b_2\nu) \\
\check{E}_{12} &= \frac{1}{16}(V_3 + K + \Psi + Y_3 + Z_3 + \Lambda + \Theta + b_3\nu) \quad ,
\end{aligned}$$

The Fierz identities are equivalent with:

$$\check{E}_{ij}\check{E}_{kl} = \delta_{kj}\check{E}_{il} \ , \quad i,j,k,l \in \{1,2\}$$

- **Problem**. Expanding these leads to extremely complicated relations. Hence studying the Fierz identities is hopeless in 8 dimensions with two Majorana spinors (*not* Majorana-Weyl !).
- Crucial idea: the method of parameterized spinors: we can reduce these Fierz identities to the Fierz identities for a *single* spinor, which we already studied.
- To extract the essential information, we only need the method of spinor rotations, which is a particular case of the method of parameterized spinors. I will briefly sketch this.

The method of spinor rotations

We can expand an arbitrary norm one element ξ of $\mathcal{K}$ as:

$$\boxed{\xi(u) = \cos\left(\frac{u}{2}\right)\xi_1 + \sin\left(\frac{u}{2}\right)\xi_2 \quad (u \in \mathbb{R})} \tag{5}$$

Using $\xi(u)$, we construct the inhomogeneous differential form :

$$\check{E}(u) \stackrel{\text{def.}}{=} \check{E}_{\xi(u),\xi(u)} \qquad (\ \check{E}(u) \textit{ is periodic in } u \textit{ with period } 2\pi)$$

and find:

$$\check{E}(u) = \frac{1}{16}\left[1 + V(u) + Y(u) + Z(u) + b(u)\nu\right] \ ,$$

where:

$$b(u) \stackrel{\text{def.}}{=} \mathscr{B}(\xi(u), \xi(u)) = b_+ + b_- \cos u + b_3 \sin u \ ,$$

$$V(u) \stackrel{\text{def.}}{=} \mathscr{B}(\xi(u), \gamma_a \xi(u))e^a = V_+ + V_- \cos u + V_3 \sin u$$

$$Y(u) \stackrel{\text{def.}}{=} \frac{1}{4!}\mathscr{B}(\xi(u), \gamma_{a_1\ldots a_4}\xi(u))e^{a_1\cdots a_4} = Y_+ + Y_- \cos u + Y_3 \sin u$$

$$Z(u) \stackrel{\text{def.}}{=} \frac{1}{5!}\mathscr{B}(\xi(u), \gamma_{a_1\ldots a_5}\xi(u))e^{a_1\cdots a_5} = Z_+ + Z_- \cos u + Z_3 \sin u$$

$$\phi(u) \stackrel{\text{def.}}{=} *Z(u) = \phi_+ + \phi_- \cos u + \phi_3 \sin u \ ,$$

and:

$$b_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(b_1 \pm b_2)\ , \quad V_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(V_1 \pm V_2)\ , \quad Y_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(Y_1 \pm Y_2)\ , \quad Z_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(Z_1 \pm Z_2) \ ,$$

$$\phi_\pm \stackrel{\text{def.}}{=} *Z_\pm = \frac{1}{2}(\phi_1 \pm \phi_2) \ , \quad \phi_3 \stackrel{\text{def.}}{=} *Z_3 = \frac{1}{3!}\mathscr{B}(\xi_1, \gamma_{a_1\ldots a_3}\gamma(\nu)\xi_2)e^{a_1\cdots a_3} \ .$$

The Fierz identities for $\xi(u)$ imply the main result

The Fierz identities for a single Majorana spinor on M were studied (in particular, by us) and can be characterized using a G_2 structure defined on a codimension one distribution. We can apply these conclusions to the single spinor $\xi(u)$ for *any* $u \in \mathbb{R}$. This implies that the following relations hold globally on $\mathcal{U}$ for any $u \in \mathbb{R}$:

$$\boxed{\begin{array}{l} ||V(u)||^2 = 1 - b(u)^2 \geq 0 \ \ , \ \ ||Y_{\pm}(u)||^2 = \frac{7}{2}(1 \pm b(u))^2 \\ \iota_{V(u)}\phi(u) = 0 \ \ , \ \ \iota_{V(u)}Z(u) = Y(u) - b(u) * Y(u) \\ (\iota_\lambda \phi(u)) \wedge (\iota_\rho \phi(u)) \wedge \phi(u) = 6\langle \lambda \wedge V(u), \rho \wedge V(u)\rangle \iota_{V(u)}\nu \ \ , \ \ \forall \lambda, \rho \in \Omega^1(\mathcal{U}) \end{array}} \tag{6}$$

Using Fourier expansion in $u \in \mathbb{R}$, relations (6) imply all of the Fierz identities which are relevant to the problem, thus allowing us to avoid working directly with the Fierz identities for ξ_1 and ξ_2. Performing this Fourier expansion leads to a system of relations which can be analyzed (using a few tricks !), thus leading to the theorems stated above.