Holomorphic vector bundles on non-kähler manifolds

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Construction methods (1)

Serre's method

Theorem (J.-P. Serre) X=complex manifold, $L_1, L_2 \in Pic(X)$ line bundles, $Z \subset X$ with $codim_X(Z) = 2.$ Under some cohomological conditions, the sheaf E sitting in

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Definition. A coherent sheaf $\mathcal{F}$ is called *filtrable* if there exists

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{F}$$

with each $\mathcal{F}_i$ =coherent of rank i.

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Existence results

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- (*Bănică - Le Potier*) If X is a non-projective surface then there exists a non-negative function m such that if E is a filtrable rank-2 vector bundle, then $\Delta(E) = 4c_2(E) - c_1^2(E) \geq m(c_1, c_2) \geq 0$.

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Non-filtrable vector bundles

- Let X be a K3 surface with $Pic(X) = 0$. Then the tangent bundle $\mathcal{T}_X$ is non-filtrable.
- (*Bănică - Le Potier*) If X is a non-projective surface and N is big enough, there exists rank-2 vector bundles on X with $c_2(E) = N$ which are non-filtrable.

Construction methods (1)

Restrictions on non-projective surfaces

Theorem (Bănică - Le Potier) If X is non-projective surface and E is a rank-n holomorphic vector bundle then

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Theorem(-) If X is non-projective surface and E is a holomorphic vector bundle then

$$\chi(E) = h^0(E) - h^1(E) + h^2(E) \leq 0$$

Construction methods (2)

Double cover method

Prop. If $f : Y \to X$ is a $2:1$ map and $L \in Pic(Y)$ then $f_*(L)$ is a rank-2 vector bundle on X.

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- (*Schwarzenberger*) If X is a projective surface and E is any topological rank-2 vector bundle with $det(E) \in Pic(X)$, then E has a holomorphic structure constructed in this way.
- Pick again a K3 surface X with $Pic(X) = 0$; then X has no double cover at all!
- (*Brînzănescu - Moraru*) If X is a non-Kähler elliptic surface, all rank-2 vector bundles can be constructed in this way.

Construction methods (3)

Elementary transformations

Theorem. Let X be a complex manifold, $i : D \hookrightarrow X$ closed with $dim(D) = dim(X) - 1$, E=vector bundle on X, V =vector bundle on D. Assume there exist a surjection

$$E \to i_*(V) \to 0$$

Then $\mathcal{E}$ defined by

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- (*Ballico*) If X is a projective surface and $\mathcal{E}$ is any topological rank-2 vector bundle with $det(\mathcal{E}) \in Pic(X)$, then $\mathcal{E}$ has a holomorphic structure by applying elementary transformations starting from the trivial bundle.

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- (*Ballico*) If X is a projective surface and $\mathcal{E}$ is any topological rank-2 vector bundle with $det(\mathcal{E}) \in Pic(X)$, then $\mathcal{E}$ has a holomorphic structure by applying elementary transformations starting from the trivial bundle.
- The method fails on non-projective surfaces due to lack of curves!

Stability

Slope stability

Definition. X=compact complex manifold of complex dimension n , ω=fixed Gauduchon metric (i.e. $\partial\overline{\partial}\omega^{n-1} = 0$) , E=holomorphic vector bundle.

- The *degree* of E is

$$deg_\omega(E) = \int_X c_1(E, h) \wedge \omega^{n-1}$$

- The *slope* of E is

$$\mu_\omega(E) = \frac{deg_\omega(E)}{rank(E)}$$

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Definition. E is called *ω-(semi)stable* if

$$\mu_{\omega}(\mathcal{F}) < \mu_{\omega}(E)$$

(resp. $\leq$) for all coherent $\mathcal{F} \subset E$ with $rank(\mathcal{F}) < rank(E)$.

Stability

The Kobayashi-Hitchin correspondence

Definition. (X, ω)= compact complex manifold, ω =Gauduchon metric. E is said *Hermite-Einstein* if it has a Hermitian metric h whose curvature F_h satisfies

$$\Lambda_\omega F_h = \gamma_E id_E$$

for some $\gamma_E \in \mathbb{R}$.

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Main Theorem. E is (poly)stable iff E has a Hermite-Einstein metric.

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Main Theorem. E is (poly)stable iff E has a Hermite-Einstein metric.

Corollary. (Bogomolov-Lübke) If X is a surface and E is a stable rank-r vector bundle, then

$$2rc_2(E) - (r-1)c_1^2(E) \geq 0$$

Stability

Examples of stable vector bundles.

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- Line bundles are stable.
- Non-filtrable vector bundles are stable.
- On non-Kähler surfaces X one can easily prove:

Theorem. For any $n > 0$ there exists a stable rank-2 vector bundle E with trivial determinant, $(det(E) = \mathcal{O}_X)$ and $c_2(E) = n$.

More about non-filtrable vector bundles

Can one produce more examples?

Theorem. If X is a compact complex *non-projective* surface and $\hat{X}$ is its blow-up at some point, then every holomorphic vector bundle on $\hat{X}$ can be turned into a pull-back form X (twisted by a line bundle) after finitely many elementary transformations along the exceptional divisor.

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Theorem. (Verbitsky) If $\pi : X \to B$ is an elliptic principal bundle (with $dim_{\mathbb{C}}(X) \geq 3$) over a projective manifold B which is *positive* (i.e. $\pi^*(\omega)$ =exact, for some Kähler class ω on B) then every stable vector bundle E on X is a pull-back form B (up to a twist by a line bundle),

$$E = \pi^*(\mathcal{E}) \otimes L.$$

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Theorem. (Brînzănescu-/) The same is true if B is a surface with no curves (with a topological condition on the fibration instead of positivity).

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Question. If X is an LCK manifold, is it true that moduli spaces of stable vector bundles have (natural) LCK metrics?

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Theorem.(Aprodu-Moraru-Toma) If X is a Kodaira surface, and if $\mathcal{M}$ is a (well-chosen) moduli space of *non-filtrable* rank-2 vector bundles on X, then $\mathcal{M}$ is itself a primary Kodaira surface.