Three point function in N=4 SYM from integrability

D. Serban

IPhT SACLAY

w/ O. Foda, Y. Jiang, S. Komatsu, I. Kostov, F. Loebbert, A. Petrovskii

GATIS
Gauge Theory as an Integrable System

Bucharest, GAP seminar
August 27, 2015

Overview

- AdS/CFT dictionary
- Integrability at strong and weak coupling
- All-loop solution for the spectrum from integrability
- Correlation functions: weak coupling vs. strong coupling, the spin vertex
- The bootstrap approach
- Conclusions

Why study the N=4 planar gauge theory?

“more is less”: it is simpler than QCD and presumably exactly solvable due to the high amount of symmetry

first example of precise duality with a string theory (type IIB on $AdS_5 \times S^5$)

[Maldacena 97; Witten 98; Gubser, Klebanov, Polyakov 98]

although not realized in nature, it can help our understanding of strongly coupled gauge theories

Symmetries

- the beta function is zero, presumably at all orders $\longrightarrow$ the Poincaré group gets promoted to the **conformal group** SO(4,2) $\cong$ SU(2,2)

- there are four copies of **supersymmetry** generators, which are rotated into one another by the **R-symmetry** SO(6) $\cong$ SU(4)

- the total symmetry super-group is PSU(2,2|4)

$$\begin{pmatrix} L,\ \bar{L},\ P,\ K,\ D & Q,\ \bar{S} \\ \bar{Q},\ S & R \end{pmatrix}$$

Dictionary

$\mathcal{N} = 4$ gauge theory

- fundamental fields: A_μ, Φ_I $(I = 1, ..., 6)$, $\Psi_\alpha (\alpha = 1, ..., 4)$ and derivatives

- symmetries: – conformal group $SO(2,4) \simeq SU(2,2)$
  – R-symmetry $SO(6) \simeq SU(4)$
  – supersymmetry $\Rightarrow$ global symmetry: $PSL(2,2|4)$

Correspondence

Local operators in the gauge theory

e.g. $\mathrm{Tr}\,(\Phi_{I_1}\Phi_{I_2}...\Phi_{I_L})$	$\leftrightarrow$	One-string states
Anomalous dimensions	$\leftrightarrow$	Energy spectrum
R-charges	$\leftrightarrow$	Angular momenta J_i

[Lipatov, 98]

Integrability

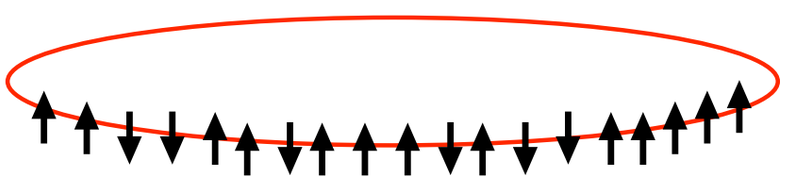
One loop dilatation operator
=
integrable spin chain

[Minahan, Zarembo, 02]

$$Z = \Phi_1 + i\Phi_2$$

$$W = \Phi_3 + i\Phi_4$$

$$\mathrm{tr}\, ZZZWWZZZWWWZWZZZZ\ldots$$



$$\hat{D}_1 = 2\sum_{l=1}^{L}(1 - P_{l,l+1})$$

solution in terms of Bethe Ansatz equations

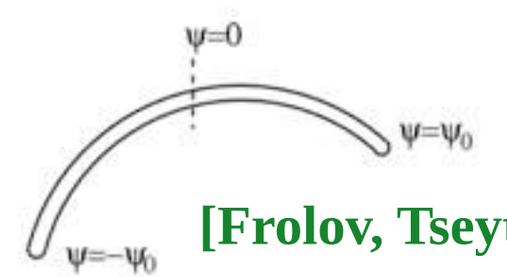
String sigma model
is
classically integrable

[Bena, Polchinski, Roiban, 02]

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \left[G^{(AdS_5)}_{mn}\,\partial_a X^m\,\partial^a X^n + G^{(S^5)}_{mn}\,\partial_a Y^m\,\partial^a Y^n\right]$$

$$+\ \text{fermions}$$

string solution, *e.g.*



[Frolov, Tseytlin, 02]

[Kazakov, Marshakov, Minahan, Zarembo, 04]

solution of the classical sigma model in terms of an algebraic curve

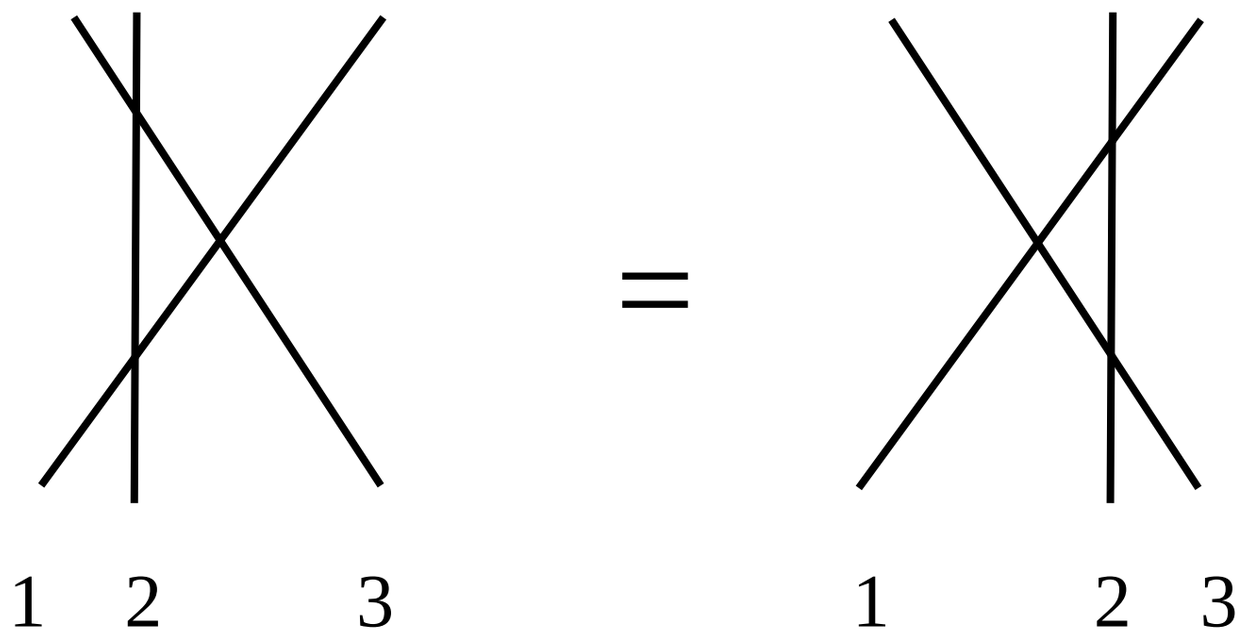
Integrability

existence of an infinite number of integrals of motion $[I_m, I_n]=0$

factorized scattering (no particle production)

Yang-Baxter equation

$$S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$$



exact solution for some 2d field theories

[Zamolodchikov, Zamolodchikov, 70ties]

extra difficulty for the AdS/CFT integrable system: lack of relativistic invariance for the excitations of the string in light cone gauge

Integrability: solution of the spectrum

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\rangle = \frac{\delta_{AB}}{|x-y|^{2\Delta_A(\lambda)}}$$

- the scattering matrix of excitations over the BMN vacuum fixed solely by symmetry **[Beisert, 05-06]**

the BMN vacuum: $\mathrm{Tr}\, Z^L$ **[Berenstein, Maldacena, Nastase, 02]**

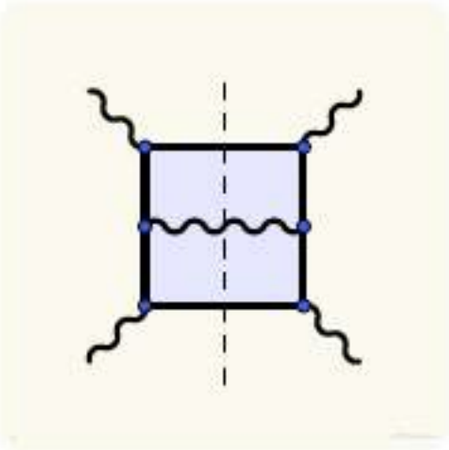
breaks the symmetry to the centrally extended $PSU(2|2)^2$

- crossing equation written down by **[Janik, 06]** and solved by **[Beisert, Eden, Staudacher, 06]**

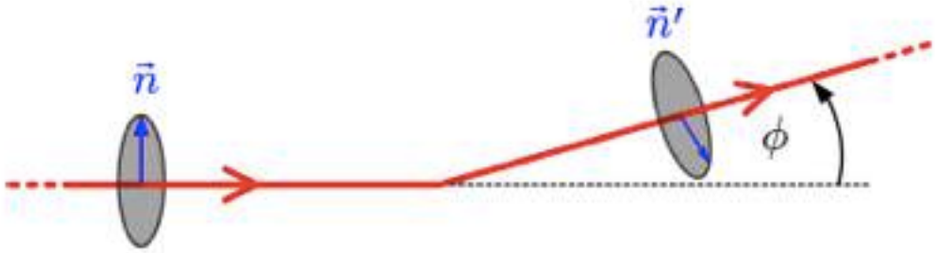
- full solution of the spectral problem via Thermodynamic Bethe Ansatz **[Gromov, Kazakov, Vieira, 09]** and Riemann-Hilbert problem a.k.a Quantum Spectral Curve (QSC) **[Gromov, Kazakov, Leurent, Volin, 13-15]**:

Other objects computed using integrability

(Gluon) amplitudes



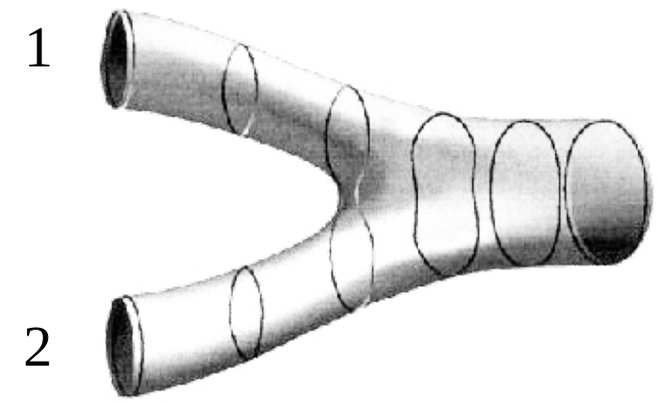
Wilson loops (lines)



Correlators of local fields

The three point function in N=4 SYM

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}$$



initial data: three states with definite conformal dimensions and psu(2,2|4) charges

$$\mathcal{O}_\alpha(x)\ , \quad \alpha = 1,2,3$$

- each characterized by a set of rapidities $\{u_{i,\alpha} = \frac{1}{2}\cot\frac{p_{i,\alpha}}{2}\sqrt{1+16g^2\sin^2\frac{p_{i,\alpha}}{2}}\}$

$$\Delta_\alpha = L_\alpha - M_\alpha + \sum_{i=1}^{M_\alpha}\sqrt{1+16g^2\sin^2\frac{p_{i,\alpha}}{2}}$$

efficiently encoded in the zeros of the Baxter functions $\{Q_a^\alpha(u;u_{i,\alpha})\ , \quad a=1,\ldots 8\}$

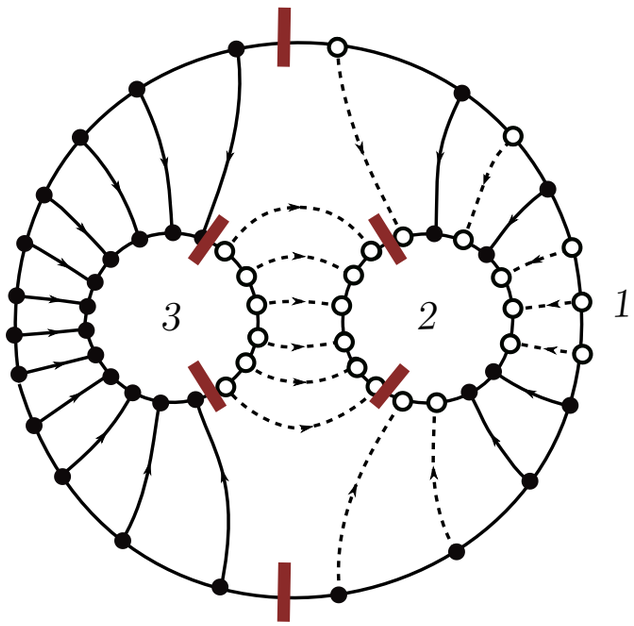
- and polarizations (or global rotations with respect to some reference BPS state, e.g Tr Z^L)

$$g_\alpha = e^{\zeta_\alpha^A J^A}$$

The three point function at weak coupling

At tree level the three point function can be computed using gaussian contraction

$\longrightarrow$ pure combinatorics



$$\langle \bar{Z}(x) Z(y) \rangle \sim \frac{1}{|x-y|^2}$$

$$\langle \bar{X}(x) X(y) \rangle \sim \frac{1}{|x-y|^2}$$

Spin chain language: the combinatorics can be expressed in terms of scalar products of states of (pieces of) spin chain **[Roiban, Volovich, 04]**

- use Algebraic Bethe Ansatz (ABA) to build and cut the chains into pieces

$\longrightarrow$ "tailoring" of spin chains **[Escobedo, Gromov, Sever, Vieira, 10]**

The three point function at weak coupling

Cutting the chains into pieces generates sums over partitions of magnons

Resumming the contribution of magnons and taking the limit of large number of magnons are among the open problems.

In some special cases these sums can explicity taken, and obtain determinant representations **[Foda, 11]** whose semiclassical limit is rather straightforward **[Escobedo, Sever, Vieira, 11; Kostov, 12; Kostov, Bettelheim, 14]**

e.g. in some of the su(2) sector at tree level and one loop **[Jiang, Kostov, Loebbert, DS, 14]** the semiclassical limit is

$$\log C_{123}(g) \simeq \oint_{\mathcal{C}^{(12|3)}} \frac{du}{2\pi}\, \mathrm{Li}_2\big(e^{ip^{(1)}(u)+ip^{(2)}(u)-ip^{(3)}(u)}\big)$$
$$+ \oint_{\mathcal{C}^{(13|2)}} \frac{du}{2\pi}\, \mathrm{Li}_2\big(e^{ip^{(3)}(u)+ip^{(1)}(u)-ip^{(2)}(u)}\big) - \tfrac{1}{2}\sum_{a=1}^{3} \int_{\mathcal{C}^{(a)}} \frac{dz}{2\pi}\, \mathrm{Li}_2\big(e^{2ip^{(a)}(z)}\big)$$

in agreement with strong coupling computations **[Kazama, Komatsu, 13 & unpublished]**

Similar expressions for some special cases of higher rank, su(3) cases **[Foda, Jiang, Kostov, DS, 13]**

The three point function at weak coupling

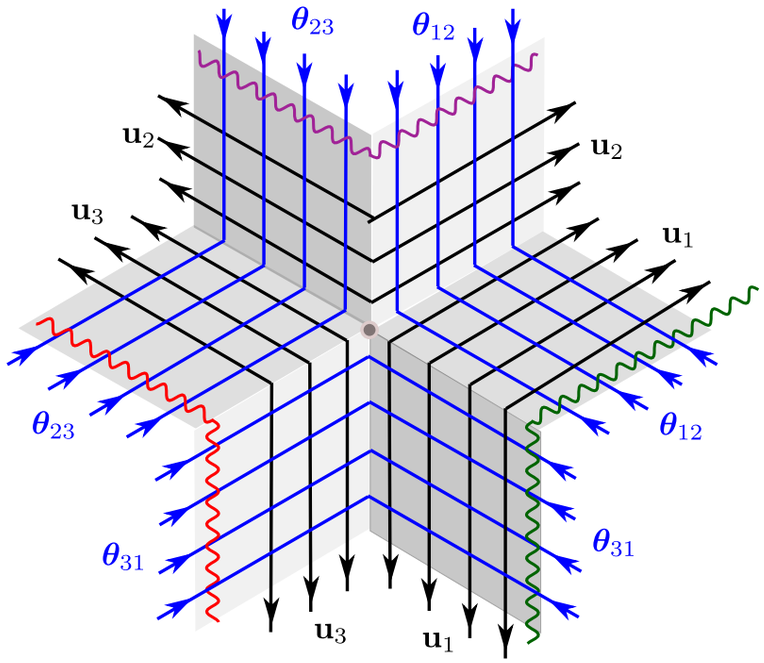
We expect to get similar expressions for general three point functions at higher rank and higher loop order

[Kazama, Komatsu, 13]

fermionic representations (~ partition function for relativistic fermions)?

methods based on Sklyanin's Separation of Variables
-> integral representattion **[Jiang, Komatsu, Kostov, DS, 15]**

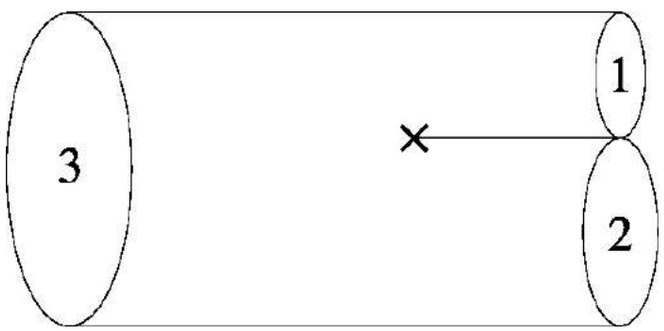
generic su(2) three point function:
six vertex partition function on a lattice
with a conical defect



The string vertex (strings in the pp-wave limit)

Simple configuration at strong coupling: near-extremal configuration with one string of length J_3 spliting into two strings of length J_1 and J_2 with $J_3 = J_1 + J_2$

with transverse excitations with polarizations $j = 1,\ldots,8$



[Spradlin, Volovich, 02-03; Dobashi, Yoneya, Shimada, 04;...]

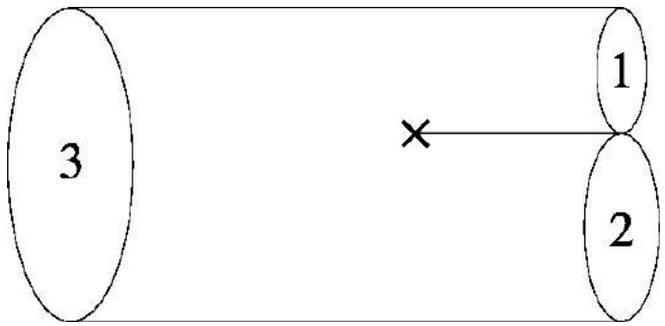
BMN excitations: (dilute gas of magnons with momentum ~1/L)

modes (massive bosons/fermions): $$E = J + \sum_k \sqrt{1 + \frac{\pi \lambda n_k^2}{L^2}}$$

the S matrix in the BMN limit is trivial: $$S(p_1, p_2) = \pm 1$$

$$a_n^{(s)j\dagger} \text{ creates excitation with momentum } p = \frac{2\pi n}{J_s} \text{ in the } s\text{-th string}$$

The string vertex (strings in the pp-wave limit)



$$C_{123} = f(\Delta_1, \Delta_2\Delta_3)\langle 1|\langle 2|\langle 3|V_3\rangle$$

string vertex state:

$$|V_3\rangle = \mathcal{P}|E_a\rangle,$$

where

$$|E_a\rangle = \exp\left(-\frac{1}{2}\sum_{i=1}^{8}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a_m^{(r)i\dagger}\widetilde{N}_{mn}^{rs}a_n^{(s)i\dagger}\right)|0\rangle,$$

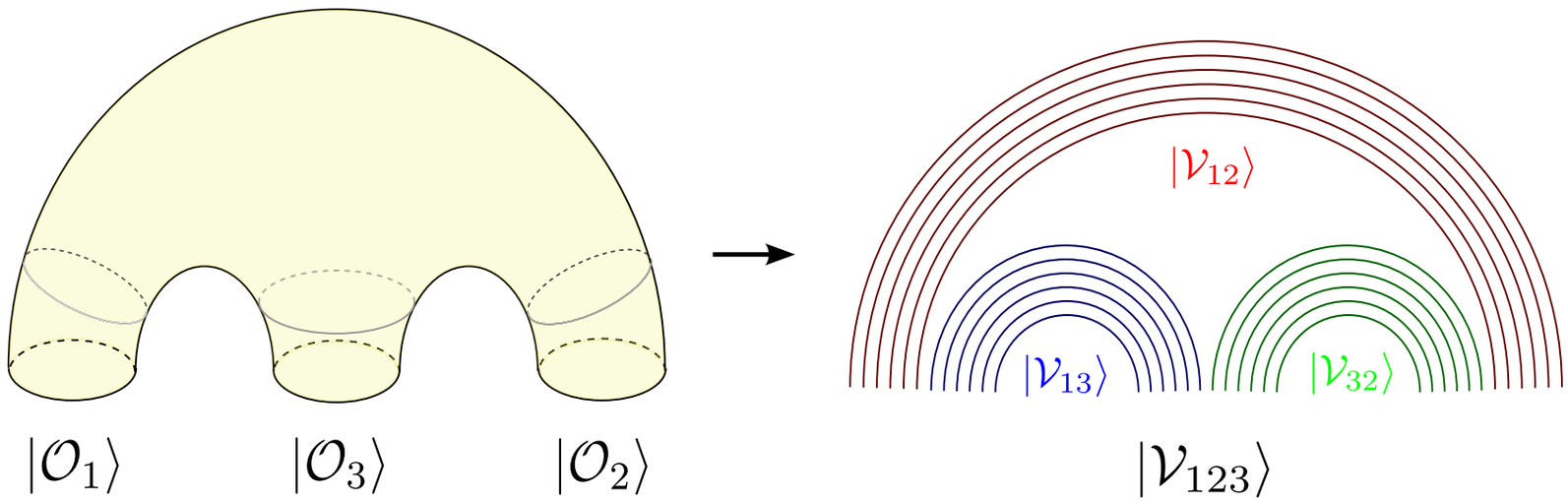
and $\mathcal{P}$ is a polynomial in the creation/annihilation operators.

comparison with the computation at weak coupling in the BMN limit: agreement at the leading order; disagreement at one loop **[Schulgin, Zayakin, 13]**

The spin vertex

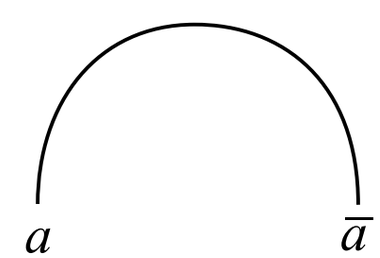
A structure similar to the string vertex can be built at weak coupling, too

[Alday, David, Gava, Narain, 05; Y. Jiang, I. Kostov, A. Petrovskii, D.S., 14
Y. Kazama, S. Komatsu, T. Nishimura 14-15]



At tree level the spin vertex mimics the planar Wick contractions

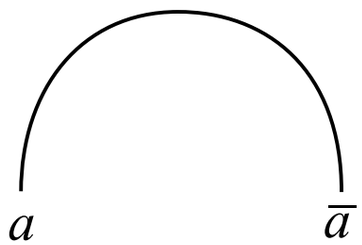
$\longrightarrow$ combining incoming states into singlets



all the three states are treated equally

$$\sum_a |a\rangle \otimes |\bar{a}\rangle$$

Constructing the singlet states



$$= \sum_a |a\rangle \otimes |\bar{a}\rangle$$

a is a state in a particular lowest weight module V_+ of psu(2,2|4)

$$a = Z, X, Y, \bar{Z}, \bar{X}, \bar{Y} + \text{fermions, derivatives, etc}$$

since psu(2,2|4) is non-compact $\bar{a}$ should be in the highest weight module $\mathbf{V}_-$ of dual to $\mathbf{V}_+$

- Build $\mathbf{V}_+$ and $\mathbf{V}_-$ via the oscillator representation (spin chain language) **[Bars, Gunaydin, 83,...]**

The oscillator representation

emphasizing the maximally compact subalgebra su(2) x su(2) x u(1) x su(4)

$$[a_i, a_j^\dagger] = \delta_{ij} \ , \quad [b_i, b_j^\dagger] = \delta_{ij} \ , \quad \{c_k, c_l^\dagger\} = \delta_{kl} \ , \quad i,j = 1,2 \ , \quad k,l = 1,\ldots,4 \ .$$

optional particle-hole transformation $\quad d_i = c_{i+2}^\dagger \ , \qquad d_i^\dagger = c_{i+2} \quad i = 1,2$

u(2,2|4) generators (spin chain) $\quad E^{AB} = \bar{\psi}^A \psi^B$

$$\psi = (\ a_i \quad -b_i^\dagger \quad c_i \quad d_i^\dagger\ ), \qquad \bar{\psi} = (\ a_i^\dagger \quad b_i \quad c_i^\dagger \quad d_i\ )$$

psu(2,2|4): vanishing central central charge condition:

$$\sum_A E^{AA} = \sum_{i=1,2} (N_{a_i} - N_{b_i} + N_{c_i} - N_{d_i}) = 0$$

The spin vertex

there exists a non-unitary (Wick-like) rotation U which transforms a direction of positive signature (5) into one of negative (0) signature and viceversa

$$\eta^{\mathrm{D}}_{PQ} = \mathrm{diag}(- + + + | + -)$$

$$\updownarrow \qquad \updownarrow \qquad U = \exp -\frac{\pi}{2} M_{05} = \exp -\frac{\pi}{4}(P_0 - K_0)$$

$$\eta^{\mathrm{E}}_{PQ} = \mathrm{diag}(+ + + + | - -)$$

[Alday, David, Gava, Narain, 05; Govil, Günaydin, 13]

at tree level:

$$U = \exp -\frac{\pi}{4} \sum_{i=1,2} (a_i^\dagger b_i^\dagger + a_i b_i)$$

transformed oscillators
(Bogoliubov-like transformation):

$$\lambda_\alpha \equiv U\, a_\alpha\, U^{-1} = a_\alpha - b_\alpha^\dagger, \quad \widetilde{\lambda}_{\dot\alpha} \equiv U\, b_\alpha\, U^{-1} = b_\alpha - a_\alpha^\dagger,$$

$$\mu_\alpha \equiv U\, a_\alpha^\dagger\, U^{-1} = a_\alpha^\dagger + b_\alpha, \quad \widetilde{\mu}_{\dot\alpha} \equiv U\, b_\alpha^\dagger\, U^{-1} = b_\alpha^\dagger + a_\alpha$$

- "D-scheme" **[Kazama, Komatsu, Nishimura, 15]:** the action of the conformal group is manifest

The U^2 transformation

the operator U^2 realizes a PT transformation (changes the sign of x_0 and x_5)

transforms positive energy state into negative energy states

$$U^{-2} D U^2 = -D$$

all loop property

- positive energy (lowest weight) module $\mathbf{V}_+$: built on the oscillator vacuum

$$|0\rangle = |0\rangle_B \otimes |0\rangle_F \qquad (a_i, b_i, c_i, d_i)|0\rangle = 0$$

- negative energy (highest weight) module $\mathbf{V}_-$: built on the dual vacuum

$$|\bar{0}\rangle = |\bar{0}\rangle_B \otimes |\bar{0}\rangle_F \qquad (a_i^\dagger, b_i^\dagger, c_i^\dagger, d_i^\dagger)|\bar{0}\rangle = 0$$

bosonic particle-hole transformation implemented by U^2

$$|\bar{0}\rangle_B = U^2 |0\rangle_B$$

necessary to construct the psu(2,2|4) singlets

Two-point function and the “vertex”

Operator-state correspondence (E-scheme):

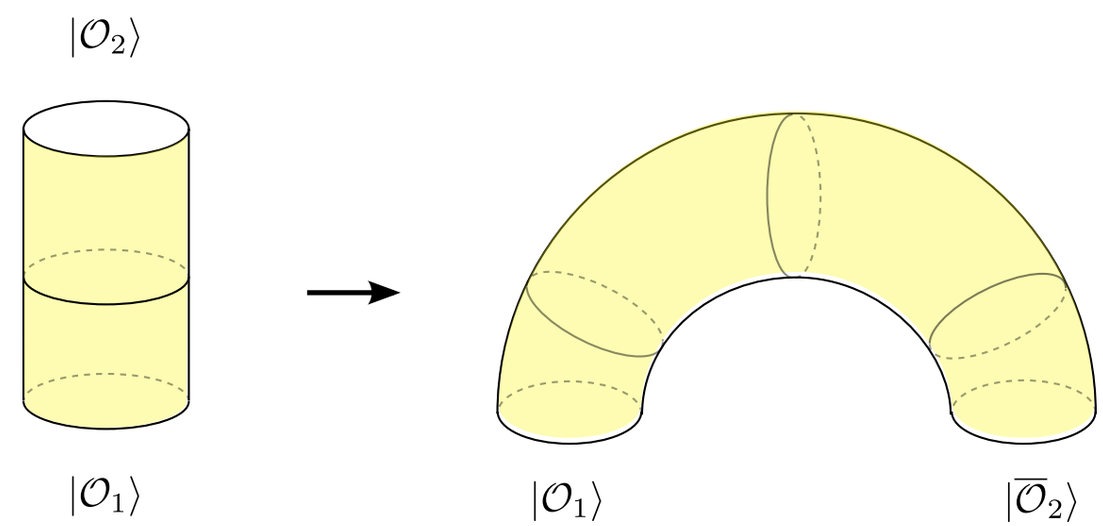
$$\mathcal{O}(x) = e^{iPx}\mathcal{O}(0)e^{-iPx} \qquad \longrightarrow \qquad e^{iPx}U|\mathcal{O}\rangle$$

$$\langle \mathcal{O}_2^\dagger(y)\mathcal{O}_1(x)\rangle = \langle \mathcal{O}_2|U^\dagger e^{iP(x-y)}U|\mathcal{O}_1\rangle$$

Flip the outgoing state into an incoming state

and pair the two states into the singlet $\langle \mathcal{V}_{12}|$:

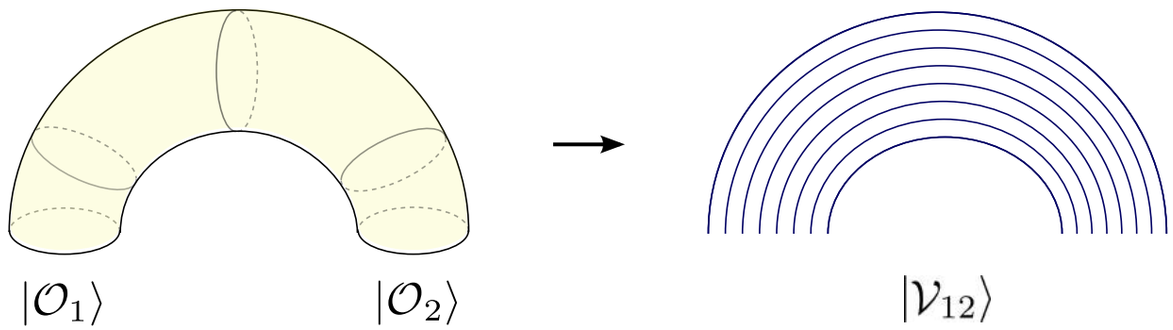
à la **[EGSV, 10]**



$$\langle \mathcal{O}_2^\dagger(y)\mathcal{O}_1(x)\rangle = \langle \mathcal{V}_{12}|\, e^{i[L^+_{(1)}x + L^+_{(2)}y]}\, |\bar{\mathcal{O}}_2\rangle^{(2)} \otimes |\mathcal{O}_1\rangle^{(1)} \qquad U^{-1}P_\mu U = L^+_\mu$$

Two-point function and the "vertex"

Tree level: Wick contractions:



$$|\mathcal{V}_{12}\rangle = \exp - \sum_{s=1}^{L} \sum_{i=1,2} \left(a_{i,s}^{(1)} a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)} b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)} d_{i,s}^{(2)\dagger} - c_{i,s}^{(1)} c_{i,s}^{(2)\dagger} \right) |0\rangle^{(2)} \otimes |\bar{0}\rangle^{(1)}$$

$$|0\rangle^{(2)} \otimes |0\rangle^{(1)} = \left(|0\rangle_L^{(2)} \otimes \cdots \otimes |0\rangle_1^{(2)} \right) \otimes \left(|0\rangle_1^{(1)} \otimes \ldots \otimes |0\rangle_L^{(1)} \right)$$

[Alday, David, Gava, Narain, 05]

delta function-like expression

$$|\mathcal{V}_{12}\rangle = \sum_{N_a, N_b, N_c, N_d} |N_a, N_b, N_c, N_d\rangle^{(2)} \otimes |\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle^{(1)}$$

$$|N_a, N_b, N_c, N_d\rangle = \frac{1}{\sqrt{N_a!\, N_b!}} \prod_{k=1,2} (d_k^\dagger)^{N_{d_k}} (c_k^\dagger)^{N_{c_k}} (b_k^\dagger)^{N_{b_k}} (a_k^\dagger)^{N_{a_k}} |0\rangle \ ,$$

$$|\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle = \frac{(-1)^{N_a + N_c}}{\sqrt{N_a!\, N_b!}} \prod_{k=1,2} a_k^{N_{a_k}} b_k^{N_{b_k}} c_k^{N_{c_k}} d_k^{N_{d_k}} |\bar{0}\rangle \ ,$$

Two-point function and the "vertex"

$$|\mathcal{V}_{12}\rangle = \exp - \sum_{s=1}^{L} \sum_{i=1,2} \left(a_{i,s}^{(1)} a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)} b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)} d_{i,s}^{(2)\dagger} - c_{i,s}^{(1)} c_{i,s}^{(2)\dagger} \right) |0\rangle^{(2)} \otimes |\bar{0}\rangle^{(1)}$$

$$|\mathcal{V}_{12}\rangle = \sum_{N_a,N_b,N_c,N_d} |N_a, N_b, N_c, N_d\rangle^{(2)} \otimes |\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle^{(1)}$$

the exponential form includes states with arbitrary (integer) central charge C_s at each site s

the simple form is introduces at the expenses of enlarging the Hilbert space

one can easily project on the $C_s = 0$ modules; these are automatically selected when projected on the incoming states respecting this condition

main property: local psu(2,2|4) symmetry

$$\left(E_s^{AB(1)} + E_s^{AB(2)} + (-1)^{|B|} \delta^{AB} \right) |\mathcal{V}_{12}\rangle = 0 \, , \qquad s = 1, \ldots L \, .$$

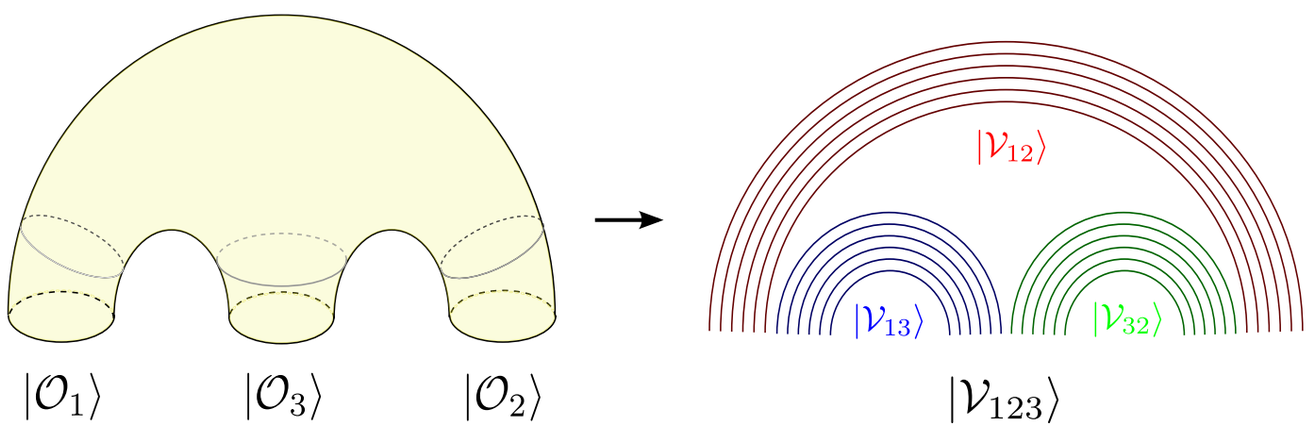
proven using the action of the oscillators on the vertex

Three-point function and the vertex

straightforward generalization to the three point function at tree level
(one singlet for every “bridge” ij)

$$|\mathcal{V}_{123}\rangle = |\mathcal{V}_{12}\rangle \otimes |\mathcal{V}_{13}\rangle \otimes |\mathcal{V}_{32}\rangle$$

$$\begin{aligned}
|\mathcal{O}_1\rangle &\simeq |\mathcal{O}_{13}\rangle \otimes |\mathcal{O}_{12}\rangle \ , \\
|\mathcal{O}_2\rangle &\simeq |\mathcal{O}_{21}\rangle \otimes |\mathcal{O}_{23}\rangle \ , \\
|\mathcal{O}_3\rangle &\simeq |\mathcal{O}_{32}\rangle \otimes |\mathcal{O}_{31}\rangle \ .
\end{aligned}$$



$$\langle \mathcal{O}_2(y)\mathcal{O}_3(z)\mathcal{O}_1(x)\rangle = \langle \mathcal{V}_{123}|\, e^{i[L^+_{(1)}x + L^+_{(2)}y + L^+_{(3)}z]} |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \otimes |\mathcal{O}_1\rangle$$

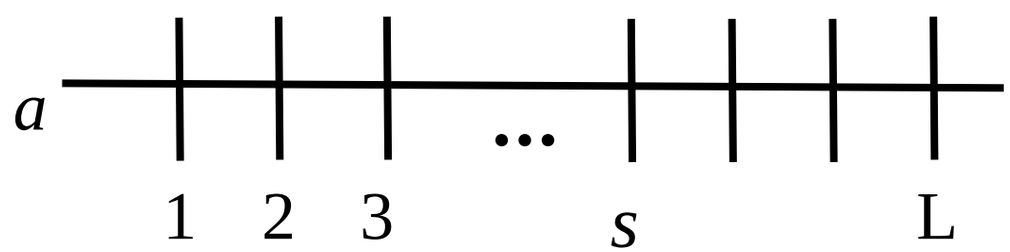
Local symmetry vs. Yangian symmetry

promote the local symmetry to Yangian symmetry

$$\left(E_s^{AB(1)} + E_s^{AB(2)} + (-1)^{|B|}\delta^{AB}\right)|\mathcal{V}_{12}\rangle = 0\ , \qquad s = 1, \ldots L\ . \qquad \textbf{(*)}$$

start by defining the monodromy matrix (which generates the Yangian)

$$T(u) = L_1(u)\ldots L_L(u)$$



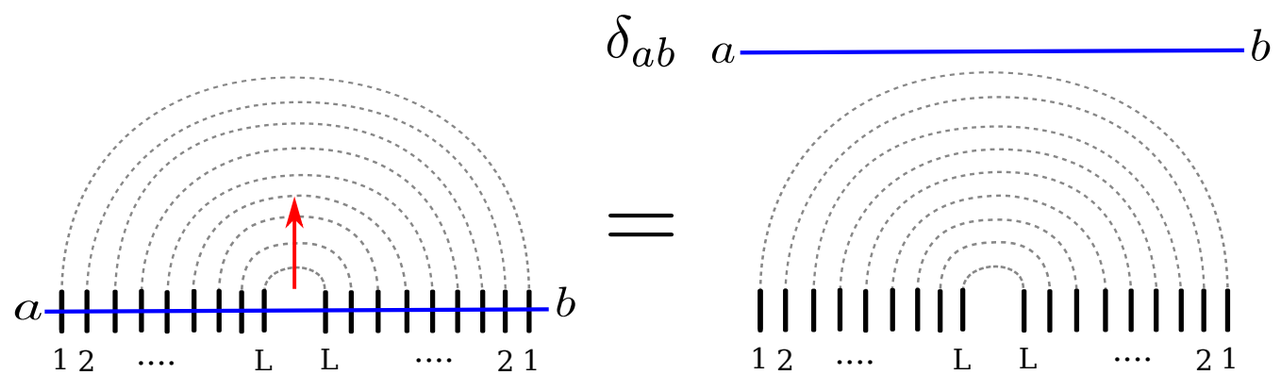
Lax matrix at site s: $\quad L_s(u) = u - i/2 - i(-1)^{|A|}E_0^{AB}E_s^{BA}$

auxiliary space in defining, (4|4) representation $\quad (E_0^{AB})_{CD} = \delta_C^A\delta_D^B$

quantum space in the oscillator representation $\quad E_s^{AB} = \bar{\psi}_s^A\psi_s^B$

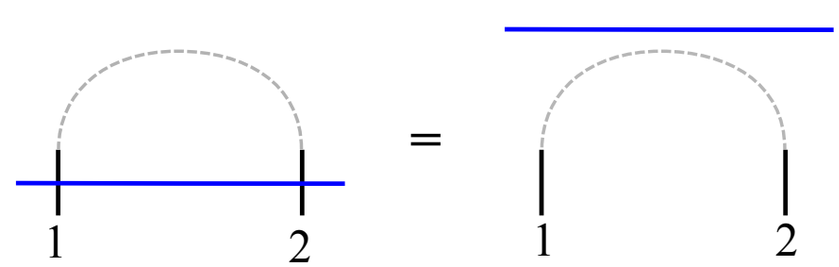
Monodromy condition for the vertex

the vertex (singlet) is also an Yangian invariant for the monodromy matrix $T_{12}(u)$



$$T_{12}(u)|\mathcal{V}_{12}\rangle = f(u)|\mathcal{V}_{12}\rangle$$

it is sufficient to prove the above relation for two chains of one site



Monodromy condition for the vertex

proof of the monodromy relation for two sites 1 and 2
(with the auxiliary space 0 in the defining representation)

R matrix $$R_{01}(u) = u - i\Pi_{01} \equiv L_1(u + i/2)$$

with the graded "permutation" $\Pi_{01} = (-1)^{|A|} E_0^{AB} E_1^{BA}$

$$\Pi_{01}^2 = c + (c-1)\Pi_{01}$$

$$c = E_1^{BB}$$

c =0 for psu(2,2|4)

c =1 for su(2)

1) unitarity condition $$R_{01}(u) R_{01}(i(c-1) - u) \sim 1$$

2) symmetry of the vertex $$R_{02}(u)|\mathcal{V}_{12}\rangle = -R_{01}(-i-u)|\mathcal{V}_{12}\rangle$$

$$R_{01}(u) R_{02}(u - ic)|\mathcal{V}_{12}\rangle \sim |\mathcal{V}_{12}\rangle$$

Monodromy condition for the vertex

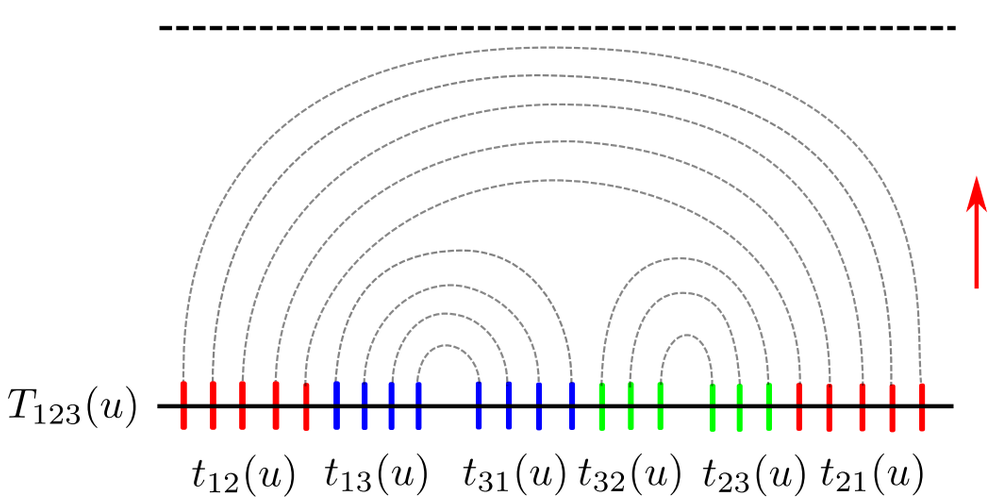
$$R_{01}(u)R_{02}(u-ic)|\mathcal{V}_{12}\rangle \sim |\mathcal{V}_{12}\rangle$$

the monodromy condition depends on the sector through the value of *c*

in the full psu(2,2|4) there is no relative shift in the rapidity $c = 0$

the significance of this fact not fully understood (relation to crossing?)

monodromy relation for three chains:



strong coupling semiclassical equivalent:

$$\Omega_1(u)\Omega_2(u)\Omega_3(u) = 1$$

[Kazama, Komatsu, 13]

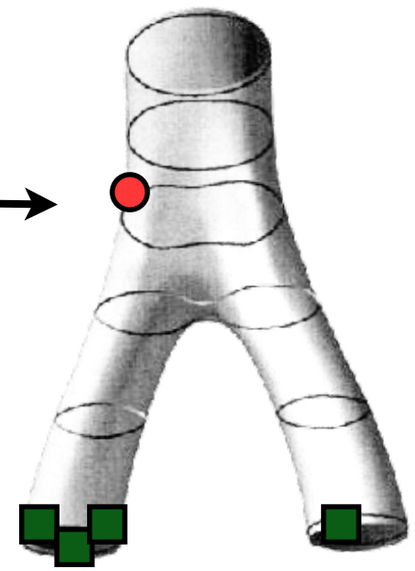
monodromy in physical space **[Kazama, Komatsu, Nishimura, 15]** $\longrightarrow$ conserved charges

Three point function at all loop from bootstrap

[Basso, Komatsu, Vieira, 15]

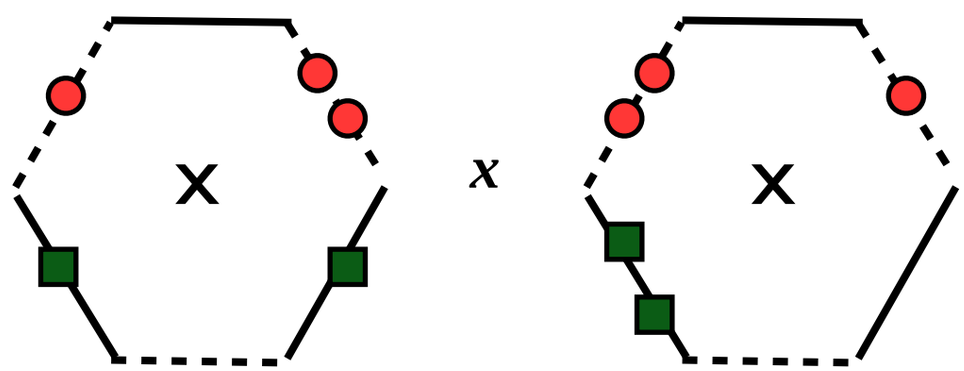
virtual excitations
propagationg in the mirror channel

physical excitations
(magnons)



pants decomposition into two hexagons
(worldsheets with conical defect)

$$C_{123} = \sum_{\beta_i \cup \bar{\beta}_i = \alpha_i} \sum_{\bullet}$$



$$e^{-E_{\bullet\, ij}\ell_{ij}}$$

hexagon contribution should satisfy form-factor like axioms and symmetry requirements

minimal solution conjectured which passes comparison with known cases

Conclusion and outlook

- various integrability-based methods used to determine the three point function in different limits
- we have built a weak coupling version of the string vertex based on the oscillator representation of psu(2,2|4) **[Y. Jiang, I. Kostov, A. Petrovskii, D.S., 14; Y. Kazama, S. Komatsu, T. Nishimura 14-15]**
- adapted for the spin chain language and perturbative computations; complementary to the hexagon bootstrap method **[Basso, Komatsu, Vieira, 15]**
- implementation of the symmetries at tree level/monodromy condition

- performing the sums in the hexagon approach and take the semiclassical limit (small parameter: 1/L); connection with fermion partition functions, Mayer expansion, etc.
- implement a general Quantum Spectral Curve-like approach for the three point function