# The landscape of G-structures in eight-manifold compactifications of M-theory

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# CGK equations

Supersymmetry-preserving compactifications of supergravity theories on a Riemannian spin $d$-manifold $(M, g)$ are characterizes by the condition that the space $\mathcal{K}(D, Q) \subset \Gamma(M, S)$ of solutions to the CGK equations:

$$D\xi = Q\xi = 0 \quad , \tag{1}$$

is non-trivial. Here:

- $S$ is a vector bundle over $M$ associated to a given spin structure $P$ of $M$ through a given (not necessarily irreducible) real representation of $\mathrm{Spin}(d)$.
- $D$ is a connection on $M$ parameterized by $g$ and by an inhomogeneous form $\omega \in \Omega^{\gamma}(M, \mathbb{S})$, where $\mathbb{S}$ is the Schur bundle of $S$.

We will be interested in the case when $\mathbb{S}$ is the trivial real line bundle, with application to compactifications of eleven-dimensional supergravity on an eight-manifold $M$.

Let $\mathscr{B}$ be an admissible pairing on $S$.

**Definition.** A subspace $\mathcal{K} \subset \Gamma(M, S)$ is *locally non-degenerate* (lnd) if the restriction $\mathrm{ev}_p|_{\mathcal{K}} : \mathcal{K} \to S_p$ of the evaluation map is injective for all $p \in M$. A locally nondegenerate subspace $\mathcal{K} \subset \Gamma(M, S)$ is *$\mathscr{B}$-compatible* if:

$$\mathscr{B}(\xi, \xi') = \text{constant on } M \quad , \quad \forall \xi, \xi' \in \mathcal{K} \; .$$

**Proposition.** $\mathcal{K}(D, Q)$ is an lnd subspace of $\Gamma(M, S)$. If $D$ is $\mathscr{B}$-compatible, then $\mathcal{K}(D, Q)$ is $\mathscr{B}$-compatible.

Let:

- $\mathrm{Grn}_s(M, S)$ = the set of $s$-dimensional lnd subspaces of $\Gamma(M, S)$.
- $\mathrm{Trivf}_s(M, S)$ = the set of pairs $(K, \mathbf{D})$, where $K$ is a globally trivializable smooth rank $s$ sub-bundle of $S$ and $\mathbf{D}$ is a trivial flat connection on $K$.
- $\mathrm{Grn}_s(M, S, \mathscr{B})$ = the subset of $\mathrm{Grn}_s(M, S)$ consisting of $\mathscr{B}$-compatible locally nondegenerate subspaces of dimension $s$
- $\mathrm{Trivf}_s(M, S, \mathscr{B})$ = the subset of $\mathrm{Trivf}_s(M, S)$ consisting of those pairs $(K, \mathbf{D}) \in \mathrm{Trivf}_s(M, S)$ for which $\mathbf{D}$ is a $\mathscr{B}$-compatible connection.

**Proposition.** There exists a natural bijection $\Phi_s : \mathrm{Grn}_s(M, S) \xrightarrow{\sim} \mathrm{Trivf}_s(M, S)$ such that $\Phi_s(\mathcal{K})_p = \mathrm{ev}_p(\mathcal{K})$, whose inverse is given by $\Phi_s^{-1}(K, \mathbf{D}) = \Gamma_{\mathrm{flat}}(K, \mathbf{D})$, where:

$$\Gamma_{\mathrm{flat}}(K, \mathbf{D}) \stackrel{\mathrm{def.}}{=} \{\xi \in \Gamma(M, K) | \mathbf{D}\xi = 0\}$$

is the space of all $\mathbf{D}$-flat sections of $K$. Moreover, $\Phi_s$ restricts to a bijection between $\mathrm{Grn}_s(M, S, \mathscr{B})$ and $\mathrm{Trivf}_s(M, S, \mathscr{B})$.

From now on, we work with the flat bundle $K \subset S$ defined by a locally non-degenerate subspace $\mathcal{K} \subset \Gamma(M, S)$.

**Definition.** A finite-dimensional subspace $\mathcal{K}$ of $\Gamma(M, S)$ is called a *virtual CGK space* if there exists a connection $D$ on $S$ and a globally-defined endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ such that $\mathcal{K} = \mathcal{K}(D, Q)$. A virtual CGK space $\mathcal{K}$ is called *$\mathscr{B}$-compatible* if there exists a $\mathscr{B}$-compatible connection $D$ on $S$ and a global endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ such that $\mathcal{K} = \mathcal{K}(D, Q)$.

**Proposition.** Let $\mathcal{K}$ be an $s$-dimensional subspace of $\Gamma(M, S)$. Then the following statements are equivalent:

(a) $\mathcal{K}$ is a ($\mathscr{B}$-compatible) virtual CGK space.

(b) $\mathcal{K}$ is a ($\mathscr{B}$-compatible) locally non-degenerate subspace.

Let $\mathcal{K} \subset \Gamma(M, S)$ be a $\mathscr{B}$-compatible Ind subspace and $(K, \mathbf{D}) = \Phi_s(\mathcal{K})$. Let:

$$P_\pm \stackrel{\mathrm{def.}}{=} \frac{1}{2}(1 \pm \gamma(\nu)) \in \Gamma(M, \mathrm{Hom}(S, S^\pm))$$

be the $\mathscr{B}$-orthogonal projectors of $S$ onto the chirality sub-bundles $S^\pm$.

**Definition.** The *chiral projections* of $K$ are the smooth generalized sub-bundles of $S^\pm$ defined through:

$$K_\pm \stackrel{\mathrm{def.}}{=} P_\pm K \subset S^\pm \quad .$$

The *chiral rank functions* $r_\pm$ of $K$ are the rank functions of $K_\pm$:

$$r_\pm \stackrel{\mathrm{def.}}{=} \mathrm{rk} K_\pm : M \to \mathbb{N} \quad .$$

The *chiral slices* of $K$ are the following cosmooth generalized sub-bundles of $K$:

$$K^\pm \stackrel{\mathrm{def.}}{=} S^\pm \cap K \quad .$$

The functions $r_\pm$ are lower semicontinuous and satisfy:

$$r_\pm \leq s \quad , \quad r_+ + r_- \geq s \quad .$$

We have exact sequences of generalized sub-bundles of $S$:

$$0 \to K^{\mp} \hookrightarrow K \stackrel{P_\pm|_K}{\longrightarrow} K_\pm \to 0 \ ,$$

which give the relations:

$$\sigma_\pm \stackrel{\mathrm{def.}}{=} \mathrm{rk}K^\pm = s - r_\mp \ .$$

**Definition.** A point $p \in M$ is $K$*-special* if $(r_-(p), r_+(p)) \neq (s,s)$. The $K$*-special locus* is the subset:

$$\mathcal{S} \stackrel{\mathrm{def.}}{=} \{p \in M | p \text{ is } K\text{−special}\} \ .$$

The open complement:

$$\mathcal{G} \stackrel{\mathrm{def.}}{=} M \setminus \mathcal{S} = \{p \in M | r_-(p) = r_+(p) = s\}$$

is the *non-special locus* of $K$; its elements are the *non-special points*. The special locus admits a stratification induced by the chiral rank functions:

$$\mathcal{S} = \sqcup_{\substack{0 \leq k,l \leq s \\ k+l \geq s \\ (k,l) \neq (s,s)}} \mathcal{S}_{kl} \ ,$$

where:

$$\mathcal{S}_{kl} \stackrel{\mathrm{def.}}{=} \{p \in \mathcal{S} | r_-(p) = k \ \& \ r_+(p) = l\} \ .$$

**Definition.** The *chirality stratification* of $M$ induced by $K$ is the decomposition:

$$M = \mathcal{G} \sqcup \sqcup_{\substack{0 \leq k,l \leq s \\ k+l \geq s \\ (k,l) \neq (s,s)}} \mathcal{S}_{kl} \ .$$

**Definition.** The *stabilizer group of $K$ at $p$* is the closed subgroup of $\mathrm{Spin}(T_pM, g_p)$ consisting of those elements which act trivially on the subspace $K_p \subset S_p$:

$$H_p \stackrel{\mathrm{def.}}{=} \{h \in \mathrm{Spin}(T_pM, g_p) |\ hu = u \ \ \forall u \in K_p\} \ \ .$$

Let $\mathcal{K} \subset \Gamma(M,S)$ be an $s$-dimensional lnd subspace. The *stabilizer stratification of $M$ induced by $\mathcal{K}$* is the stratification of $M$ given by the isomorphism type of $H_p$.

**The stratified G-structure defined by $K$.** Assuming $\mathrm{rk}K \geq 1$, let $\mathfrak{q}_p : \mathrm{Spin}(T_pM, g_p) \to \mathrm{SO}(T_pM, g_p)$ denote the double covering morphism. The image $G_p \stackrel{\mathrm{def.}}{=} \mathfrak{q}_\mathfrak{p}(H_p)$ is a subgroup of $\mathrm{SO}(T_pM, g_p)$, isomorphic with $H_p$ through the restriction of $\mathfrak{q}_p$. Let $T$ be a stratum of the connected refinement of the stabilizer stratification and $G_T$ denote the isomorphism type of $G_p \simeq H_p$ for $p \in T$. Endow $T$ with the topology induced from $M$. The restriction $\mathrm{Fr}_+(M)|_T$ of the oriented frame bundle $\mathrm{Fr}_+(M)$ of $M$ is a principal $\mathrm{SO}(8)$ bundle (in the sense of general topology) defined over the connected topological space $T$. Picking specific $G_p$-orbits inside the fibers $\mathrm{Fr}_p(M)$ for $p \in T$ specifies a $G_T$-reduction of structure group of $\mathrm{Fr}(M)|_T$ and such reductions for all connected strata $T$ fit together into a *stratified G-structure* defined on $M$.

Consider compactifications down to an $\mathrm{AdS}_3$ space of cosmological constant $\Lambda = -8\kappa^2$, where $\kappa$ is a positive parameter. The eleven-dimensional background **M** is diffeomorphic with $N \times M$, where $N$ is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the $\mathrm{AdS}_3$ metric $g_3$. The metric on **M** is a warped product:

$$\mathrm{d}\mathbf{s}^2 \quad = \quad e^{2\Delta}\mathrm{d}s^2 \quad \text{where} \quad \mathrm{d}s^2 = \mathrm{d}s_3^2 + g_{mn}\mathrm{d}x^m\mathrm{d}x^n \quad . \tag{2}$$

The warp factor $\Delta$ is a smooth real-valued function defined on $M$ while $\mathrm{d}s_3^2$ is the squared length element of the $\mathrm{AdS}_3$ metric $g_3$. The Ansatz for the field strength **G** of eleven-dimensional supergravity is:

$$\mathbf{G} = \nu_3 \wedge \mathbf{f} + \mathbf{F} \quad , \quad \text{with} \;\; \mathbf{F} \stackrel{\mathrm{def.}}{=} e^{3\Delta}F \quad , \quad \mathbf{f} \stackrel{\mathrm{def.}}{=} e^{3\Delta}f$$

= where $f \in \Omega^1(M)$, $F \in \Omega^4(M)$ and $\nu_3$ is the volume form of $(N, g_3)$. The Ansatz for the supersymmetry generator is:

$$\boldsymbol{\eta} = e^{\frac{\Delta}{2}} \sum_{i=1}^{s} \zeta_i \otimes \xi_i \quad ,$$

where $\xi_i \in \Gamma(M,S)$ are Majorana spinors of spin $1/2$ on the internal space $(M,g)$ and $\zeta_i$ are Majorana spinors on $(N, g_3)$ which satisfy the Killing equation with positive Killing constant.

Assuming that $\zeta_i$ are Killing spinor on the $\mathrm{AdS}_3$ space $(N, g_3)$, the supersymmetry condition is satisfied if $\xi_i$ satisfies the CGK equations with:

$$D_X = \nabla^S_X + \frac{1}{4}\gamma(X \lrcorner F) + \frac{1}{4}\gamma((X_\sharp \wedge f)\nu) + \kappa\gamma(X \lrcorner \nu) \quad , \quad X \in \Gamma(M, TM)$$

and:

$$Q = \frac{1}{2}\gamma(\mathrm{d}\Delta) - \frac{1}{6}\gamma(\iota_f \nu) - \frac{1}{12}\gamma(F) - \kappa\gamma(\nu) \quad .$$

Here $\nabla^S$ is the connection induced on $S$ by the Levi-Civita connection of $(M, g)$, while $\nu$ is the volume form of $(M, g)$. Neither $Q$ nor the connection $D$ preserve the chirality decomposition $S = S^+ \oplus S^-$ of $S$ when $\kappa \neq 0$:

$$D(S^\pm) \not\subseteq T^*M \otimes S^\pm \quad , \quad Q(S^\pm) \not\subseteq S^\pm \quad .$$

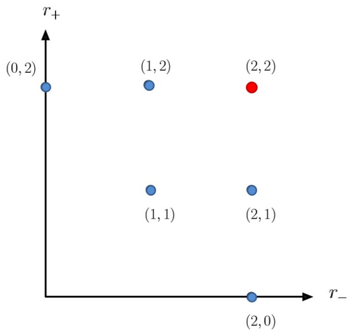
It is not hard to check that $D$ is $\mathscr{B}$-compatible:

$$\mathrm{d}\mathscr{B}(\xi', \xi'') = \mathscr{B}(D\xi', \xi'') + \mathscr{B}(\xi', D\xi'') \quad , \quad \forall \xi', \xi'' \in \Gamma(M, S) \quad .$$

This implies that any $\xi, \xi' \in \mathcal{K}(D, Q)$ satisfy $\mathscr{B}(\xi, \xi') = \mathrm{constant}$, i.e. $\mathcal{K}$ is a $\mathscr{B}$-compatible flat subspace of $\Gamma(M, S)$. The restriction $\mathbf{D} = D|_K$ is a $\mathscr{B}$-compatible trivial flat connection on $\mathcal{K}(D, Q)$.

Let $\mathcal{K}$ be a two-dimensional $\mathscr{B}$-compatible locally-nondegenerate subspace of $\Gamma(M, S)$ and $(K, \mathbf{D})$ be the associated trivial flat sub-bundle of $S$. We have:

$$(r_-(p), r_+(p)) \in \{(0,2), (2,0), (1,1), (1,2), (2,1), (2,2)\} \quad , \quad \forall p \in M \quad .$$




Allowed values for the pair $(r_-(p), r_+(p))$. The values corresponding to $K$-special points are shown in blue, while the remaining value is shown as a red dot.

A point $p \in M$ is $K$-special if $(r_-(p), r_+(p)) \neq (2,2)$ (the blue dots in the figure).

The special locus decomposes as:

$$\mathcal{S} = \mathcal{S}_{12} \sqcup \mathcal{S}_{21} \sqcup \mathcal{S}_{11} \sqcup \mathcal{S}_{02} \sqcup \mathcal{S}_{20} \quad ,$$

where $\mathcal{S}_{kl} = \{p \in M | r_-(p) = k, \ r_+(p) = l\}$, while the chirality stratification is given by:

$$M = \mathcal{G} \sqcup \mathcal{S}_{12} \sqcup \mathcal{S}_{21} \sqcup \mathcal{S}_{11} \sqcup \mathcal{S}_{02} \sqcup \mathcal{S}_{20} \quad ,$$

where $\mathcal{G}$ is the non-special locus.

Consider the compact convex body:

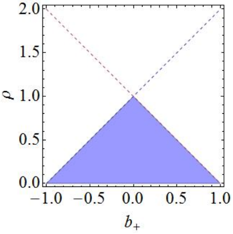
$$\boxed{\mathcal{R} = \{(b_+, b_-, b_3) \in [-1,1]^3 \mid \sqrt{b_-^2 + b_3^2} \leq 1 - |b_+|\}} \quad ,$$

which is contained in the three-dimensional compact unit ball. Setting:

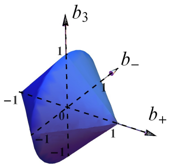
$$\rho \stackrel{\text{def.}}{=} \sqrt{b_-^2 + b_3^2} \in [0,1] \quad ,$$

one finds that $\mathcal{R}$ is the solid of revolution obtained by rotating the following isosceles right triangle around its hypothenuse:

$$\Delta \stackrel{\text{def.}}{=} \{(b_+, \rho) \in [-1,1] \times [0,1] \mid \rho \leq 1 - |b_+|\} \quad .$$




(a) The region $\Delta$ in the $(b_+, \rho)$ plane.




(b) The body $\mathcal{R}$ is the solid of revolution obtained by rotating $\Delta$ around its hypotenuse, which lies on the $b_+$ axis; it is the union of two compact right-angled cones.

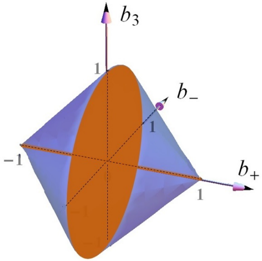
The compact interval:

$$I \stackrel{\text{def.}}{=} \{(b_+, 0, 0) | b_+ \in [-1, 1]\} = \{b \in \mathcal{R} | b_- = b_3 = 0\}$$

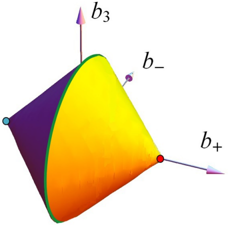
will be called the *axis* of $\mathcal{R}$ while the compact disk:

$$D \stackrel{\text{def.}}{=} \{(0, b_-, b_3) | b_-^2 + b_3^2 \leq 1\} = \{b \in \mathcal{R} | b_+ = 0\}$$

will be called the *median disk* of $\mathcal{R}$. The boundary $\partial D$ of the median disk will be called the *median circle*.




The axis $I$ and the median disk $D$, depicted in orange.




The connected refinement of the canonical Whitney stratification of $\partial\mathcal{R}$. We use green for the median circle $\partial_1\mathcal{R} = \partial D$, purple for $\partial_2^-\mathcal{R}$, yellow for $\partial_2^+\mathcal{R}$, blue for $\partial_0^-\mathcal{R}$ and red for $\partial_0^+\mathcal{R}$. The $b$-preimage of $\partial_1\mathcal{R}$ equals $\mathcal{S}_{11}$, while the $b$-preimages of $\partial_2^+\mathcal{R}$ and $\partial_2^-\mathcal{R}$ equal $\mathcal{S}_{12}$ and $\mathcal{S}_{21}$ respectively. The $b$-preimages of $\partial_0^+\mathcal{R}$ and $\partial_0^-\mathcal{R}$ are the sets $\mathcal{S}_{02}$ and $\mathcal{S}_{20}$.

| connected stratum | dimension | component of | topology | $b_+$ | $\rho$ |
|---|---|---|---|---|---|
| $\partial_0^\pm\mathcal{R}$ | 0 | $\partial_0\mathcal{R}$ | point | $\pm 1$ | 0 |
| $\partial_1\mathcal{R}$ | 1 | $\partial_1\mathcal{R}$ | circle | 0 | 1 |
| $\partial_2^\pm\mathcal{R}$ | 2 | $\partial_2\mathcal{R}$ | open annulus | $\pm(1-\rho)$ | $(0,1)$ |

Connected strata of $\partial\mathcal{R}$.

Define the function $b \in \mathcal{C}^\infty(M, \mathbb{R}^3)$ through:

$$b(p) \stackrel{\text{def.}}{=} (b_+(p), b_-(p), b_3(p)) \quad .$$

**Proposition.** The image of $b$ is a subset of $\mathcal{R}$.

**Theorem 1.** The $K$-special locus is given by:

$$\mathcal{S} = b^{-1}(\partial\mathcal{R}) = \{p \in M | b(p) \in \partial\mathcal{R}\} \quad .$$

Furthermore, we have:

- $\mathcal{S}_{11} = b^{-1}(\partial_1\mathcal{R}) = b^{-1}(\partial D)$
- $\mathcal{S}_{12} = b^{-1}(\partial_2^+\mathcal{R})$ and $\mathcal{S}_{21} = b^{-1}(\partial_2^-\mathcal{R})$
- $\mathcal{S}_{02} = b^{-1}(\partial_0^+\mathcal{R})$ and $\mathcal{S}_{20} = b^{-1}(\partial_0^-\mathcal{R})$.

Moreover, we have $\mathcal{G} = b^{-1}(\mathrm{Int}\mathcal{R})$ and hence the chirality stratification of $M$ coincides with the $b$-preimage of the connected refinement of the canonical Whitney stratification of $\mathcal{R}$.

| stratum | $\mathcal{R}$-description | $r_-(p)$ | $r_+(p)$ | $\mathrm{rk}\mathcal{D}$ | $\mathrm{rk}\mathcal{D}_0$ | $b_+$ | $\rho$ | $H_p$ |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_{02}$ | $b^{-1}(\partial_0^+\mathcal{R})$ | 0 | 2 | 8 | 8 | $+1$ | 0 | $\mathrm{SU}(4)$ |
| $\mathcal{S}_{20}$ | $b^{-1}(\partial_0^-\mathcal{R})$ | 2 | 0 | 8 | 8 | $-1$ | 0 | $\mathrm{SU}(4)$ |
| $\mathcal{S}_{11}$ | $b^{-1}(\partial_1\mathcal{R})$ | 1 | 1 | 7 | 7 | 0 | 1 | $\mathrm{G}_2$ |
| $\mathcal{S}_{12}$ | $b^{-1}(\partial_2^+\mathcal{R})$ | 1 | 2 | 6 | 6 | $1-\rho$ | $(0,1)$ | $\mathrm{SU}(3)$ |
| $\mathcal{S}_{21}$ | $b^{-1}(\partial_2^-\mathcal{R})$ | 2 | 1 | 6 | 6 | $-(1-\rho)$ | $(0,1)$ | $\mathrm{SU}(3)$ |
| $\mathcal{G}$ | $b^{-1}(\mathrm{Int}\mathcal{R})$ | 2 | 2 | $5,6,7$ | $4,6$ | $(-1,1)$ | $< 1-\lvert b_+\rvert$ | $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$ |

Chirality stratification for $s = 2$.We have $\sigma_\pm(p) = \dim K^\pm(p) = 2 - r_\mp(p)$.

Let:

$$f_\pm(b_+, b_-, b_3) = f_\pm(b_+, \rho) \stackrel{\mathrm{def.}}{=} \frac{1}{2}\left(1 - b_+^2 + \rho^2 \pm \sqrt{h(b_+, \rho)}\right) \quad .$$

The functions $f_\pm$ satisfy:

$$0 \le f_-(b) \le f_+(b) \le 1 \quad , \quad \forall b \in \mathcal{R} \quad ,$$

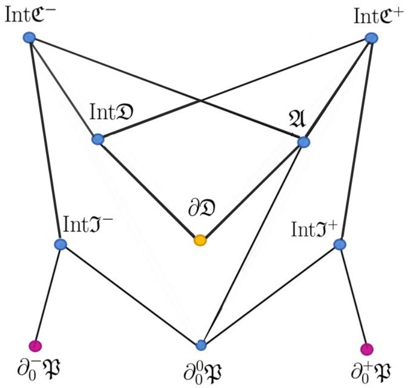
For every $b \in \mathcal{R}$, consider the closed interval:

$$J(b) = J(b_+, \rho) \stackrel{\mathrm{def.}}{=} [\sqrt{f_-(b)}, \sqrt{f_+(b)}] \subset [\sqrt{b_-^2 + b_3^2}, \sqrt{1 - b_+^2}] \quad .$$

This interval degenerates to a single point for $b \in \partial\mathcal{R}$, namely $J|_{\partial\mathcal{R}} = \{\sqrt{\rho}\}$. Finally, consider the following four-dimensional compact body:

$$\boxed{\mathfrak{P} \stackrel{\mathrm{def.}}{=} \{(b, \beta) \in \mathbb{R}^4 | b \in \mathcal{R} \ \& \ \beta \in J(b)\}} \quad ,$$

which is fibered over $\mathcal{R}$ via the projection $(b, \beta) \stackrel{\pi}{\to} b$. The fiber over $b \in \mathcal{R}$ is the segment $J(b)$, which, as mentioned above, degenerates to a point over $\partial\mathcal{R}$.




The Hasse diagram of the incidence poset of the connected refinement of the Whitney stratification of $\partial\mathfrak{P}$. The $B$-preimages of the connected components depicted as points colored in magenta, yellow and cyan are strata of $\mathrm{SU}(4)$, $G_2$ and $\mathrm{SU}(3)$ structure in $M$. The diagram depicts the covering relation of the incidence poset, namely an element of that poset covers another iff it sits above it in the diagram and there is an edge connecting the two elements. The small frontier of each connected Whitney stratum is the disjoint union of the strata covered by it in the diagram.

| connected stratum | dimension | component of | topology | $b_+$ | $\rho$ | $\beta$ |
|---|---|---|---|---|---|---|
| $\partial_0^- \mathfrak{P}$ | 0 | $\partial_0 \mathfrak{P}$ | point | $-1$ | 0 | 0 |
| $\partial_0^+ \mathfrak{P}$ | 0 | $\partial_0 \mathfrak{P}$ | point | $+1$ | 0 | 0 |
| $\partial_0^0 \mathfrak{P}$ | 0 | $\partial_0 \mathfrak{P}$ | point | 0 | 0 | 0 |
| $\mathrm{Int}\mathfrak{I}^-$ | 1 | $\partial_1 \mathfrak{P}$ | open interval | $(-1, 0)$ | 0 | 0 |
| $\mathrm{Int}\mathfrak{I}^+$ | 1 | $\partial_1 \mathfrak{P}$ | open interval | $(0, 1)$ | 0 | 0 |
| $\partial \mathfrak{D}$ | 1 | $\partial_1 \mathfrak{P}$ | circle | 0 | 1 | 1 |
| $\mathrm{Int}\mathfrak{D}$ | 2 | $\partial_2 \mathfrak{P}$ | open disk | 0 | $[0, 1)$ | 1 |
| $\mathfrak{A}$ | 2 | $\partial_2 \mathfrak{P}$ | open annulus | 0 | $(0, 1)$ | $\rho$ |
| $\mathrm{Int}\mathfrak{C}^-$ | 3 | $\partial_3 \mathfrak{P}$ | open full cone | $-g(\rho, \beta)$ | $(0, 1)$ | $(\rho, 1)$ |
| $\mathrm{Int}\mathfrak{C}^+$ | 3 | $\partial_3 \mathfrak{P}$ | open full cone | $+g(\rho, \beta)$ | $(0, 1)$ | $(\rho, 1)$ |

Connected refinement of the Whitney stratification of $\partial\mathfrak{P}$. The colors used in this table (magenta, yellow and cyan) correspond to loci of $\mathrm{SU}(4)$, $G_2$ and $\mathrm{SU}(3)$ structures on $M$.

An orthonormal basis $(\xi_1, \xi_2)$ of $\mathcal{K}$ induces three smooth functions $b_i \in \mathcal{C}^\infty(M, \mathbb{R})$ $(i = 1, 2, 3)$, namely:

$$b_1 =_U \mathscr{B}(\xi_1, \gamma(\nu)\xi_1) \ , \ b_2 =_U \mathscr{B}(\xi_2, \gamma(\nu)\xi_2) \ , \ b_3 =_U \mathscr{B}(\xi_1, \gamma(\nu)\xi_2) \ .$$

It is convenient to work with the combinations:

$$b_\pm \stackrel{\mathrm{def.}}{=} \frac{1}{2}(b_1 \pm b_2) \ .$$

Also consider the one-forms $V_i, V_3, W \in \Omega^1(M)$ (with $i = 1, 2$) given by:

$$V_i =_U \mathscr{B}(\xi_i, \gamma_a \xi_i) e^a \ , \ V_3 \stackrel{\mathrm{def.}}{=}_U \mathscr{B}(\xi_1, \gamma_a \xi_2) e^a \ , \ W \stackrel{\mathrm{def.}}{=}_U \mathscr{B}(\xi_1, \gamma_a \gamma(\nu) \xi_2) e^a \ ,$$

It is convenient to work with the linear combinations:

$$V_\pm \stackrel{\mathrm{def.}}{=} \frac{1}{2}(V_1 \pm V_2) \ , \ V_3^\pm = \frac{1}{2}(V_3 \pm W) \ .$$

We have:

$$V_1 = V_+ + V_- \ , \ V_2 = V_+ - V_- \ , \ V_3 = V_3^+ + V_3^- \ , \ W = V_3^+ - V_3^- \ .$$

Decomposing $\xi_i$ into their positive and negative chirality parts gives:

$$V_1 =_U 2\mathscr{B}(\xi_1^-, \gamma_a \xi_1^+) e^a \ , \ V_2 =_U 2\mathscr{B}(\xi_2^-, \gamma_a \xi_2^+) e^a \ , \ V_3^\pm =_U \mathscr{B}(\xi_1^\mp, \gamma_a \xi_2^\pm) e^a \ .$$

Consider the cosmooth generalized distributions:

$$\mathcal{D} \stackrel{\mathrm{def.}}{=} \ker V_1 \cap \ker V_2 \cap \ker V_3 = \ker V_+ \cap \ker V_- \cap \ker V_3$$

$$\mathcal{D}_0 \stackrel{\mathrm{def.}}{=} \ker V_+ \cap \ker V_- \cap \ker V_3^+ \cap \ker V_3^- = \mathcal{D} \cap \ker W \subset \mathcal{D} \quad .$$

**Remark.** In compactifications to $\mathrm{AdS}_3$, one can show that the supersymmetry conditions imply that $\mathcal{D}$ integrates to a singular foliation in the sense of Haefliger, while $\mathcal{D}_0$ may be non-holonomic.

The compact manifold $M$ decomposes into a disjoint union according to the rank of $\mathcal{D}$:

$$M = \mathcal{U} \sqcup \mathcal{W} \quad ,$$

where the open set:

$$\mathcal{U} \stackrel{\mathrm{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}(p) = 5\} = \{p \in M | V_+(p), V_-(p), V_3(p) \text{ are linearly independent}\}$$

will be called the *generic locus* while its closed complement:

$$\mathcal{W} \stackrel{\mathrm{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}(p) > 5\} = \{p \in M | V_+(p), V_-(p), V_3(p) \text{ are linearly dependent}\}$$

will be called the *degeneration locus*.

The degeneration locus stratifies according to the corank of $\mathcal{D}(p)$:

$$\mathcal{W} = \sqcup_{k=0}^{2} \mathcal{W}_k \quad ,$$

with locally closed strata:

$$\mathcal{W}_k \stackrel{\mathrm{def.}}{=} \{p \in \mathcal{W} | \dim \mathcal{V}_p = k\} = \{p \in \mathcal{W} | \mathrm{rk}\mathcal{D}(p) = 8 - k\} \quad .$$

Combining everything gives the *rank stratification of* $\mathcal{D}$:

$$M = \mathcal{U} \sqcup \mathcal{W}_2 \sqcup \mathcal{W}_1 \sqcup \mathcal{W}_0 \quad .$$

**Definition.** $\mathcal{K}$ is called *generic* if $\mathcal{U} \neq \emptyset$ and *non-generic* otherwise.

Notice that $\mathcal{K}$ is non-generic iff $\mathrm{rk}\mathcal{D}(p) \geq 6$ for all $p \in M$, i.e. iff $V_1(p)$, $V_2(p)$ and $V_3(p)$ are linearly dependent for all $p \in M$.

Define:

$$\beta \stackrel{\mathrm{def.}}{=} \sqrt{b_3^2 + ||V_3||^2} = \sqrt{b_-^2 + ||V_-||^2} \ .$$

**Theorem 2.** The image of the map $B \stackrel{\mathrm{def.}}{=} (b, \beta)$ is contained in $\mathfrak{P}$. Furthermore, the following hold for $p \in M$:

- $\mathrm{rk}\mathcal{D}(p) = 5$ iff $B(p) \in \mathrm{Int}\mathfrak{P}$
- $\mathrm{rk}\mathcal{D}(p) = 6$ iff $B(p) \in \partial_2\mathfrak{P} \cup \partial_3\mathfrak{P} = \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^-$
- $\mathrm{rk}\mathcal{D}(p) = 7$ iff $B(p) \in \partial_0^0\mathfrak{P} \sqcup \partial_1\mathfrak{P} = \partial\mathfrak{D} \sqcup \mathrm{Int}\mathfrak{I}$
- $\mathrm{rk}\mathcal{D}(p) = 8$ iff $B(p) \in \partial_0^+\mathfrak{P} \sqcup \partial_0^-\mathfrak{P} = \partial\mathfrak{I}$.

In particular, the rank stratification of $\mathcal{D}$ is given by:

$$\mathcal{U} = B^{-1}(\mathrm{Int}\mathfrak{P}) \ , \ \mathcal{W}_2 = B^{-1}(\partial_2\mathfrak{P}\cup\partial_3\mathfrak{P}) \ , \ \mathcal{W}_1 = B^{-1}(\partial\mathfrak{D}\sqcup\mathrm{Int}\mathfrak{I}) \ , \ \mathcal{W}_0 = B^{-1}(\partial\mathfrak{I})$$

and we have $\mathcal{W} = B^{-1}(\partial\mathfrak{P})$.

**Theorem 3.** The stabilizer stratification coincides with the rank stratification of $\mathcal{D}_0$.

**Theorem 4.** For $p \in M$, we have:

- $\mathrm{rk}\mathcal{D}_0(p) = 4$ iff $B(p) \in \mathrm{Int}\mathfrak{P}$ i.e. iff $p \in \mathcal{U}$
- $\mathrm{rk}\mathcal{D}_0(p) = 6$ iff $B(p) \in \mathrm{Int}\mathfrak{I} \sqcup \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^- = \mathrm{Int}\mathfrak{I} \sqcup \partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P}$
- $\mathrm{rk}\mathcal{D}_0(p) = 7$ iff $B(p) \in \partial\mathfrak{D}$
- $\mathrm{rk}\mathcal{D}_0(p) = 8$ (i.e. $\mathcal{D}(p) = T_pM$) iff $B(p) \in \partial\mathfrak{I}$.

Hence the rank stratification of $\mathcal{D}_0$ is given by:

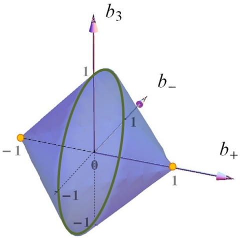
$$\mathcal{U}_0 = \mathcal{U} \ , \ \mathcal{Z}_3 = \emptyset \ , \ \mathcal{Z}_2 = B^{-1}(\mathrm{Int}\mathfrak{I} \sqcup \partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P}) \ , \ \mathcal{Z}_1 = B^{-1}(\partial\mathfrak{D}) \ , \ \mathcal{Z}_0 = B^{-1}(\partial\mathfrak{I}) = \mathcal{W}_0$$

and the stabilizer group $H_p$ is given by:

- $H_p \simeq \mathrm{SU}(2)$ if $p \in \mathcal{U}_0 = \mathcal{U}$
- $H_p \simeq \mathrm{SU}(3)$ if $p \in \mathcal{Z}_2$
- $H_p \simeq \mathrm{G}_2$ if $p \in \mathcal{Z}_1$
- $H_p \simeq \mathrm{SU}(4)$ if $p \in \mathcal{Z}_0$ .

| $\mathfrak{P}$-description | $\mathcal{D}$-stratum | $\mathcal{D}_0$-stratum | $\mathrm{rk}\mathcal{D}$ | $\mathrm{rk}\mathcal{D}_0$ | $H_p$ |
|---|---|---|---|---|---|
| $B^{-1}(\partial\mathfrak{I})$ | $\mathcal{W}_0$ | $\mathcal{Z}_0$ | 8 | 8 | $\mathrm{SU}(4)$ |
| $B^{-1}(\partial\mathfrak{D})$ | $\mathcal{W}_1^1$ | $\mathcal{Z}_1$ | 7 | 7 | $\mathrm{G}_2$ |
| $B^{-1}(\mathrm{Int}\mathfrak{I})$ | $\mathcal{W}_1^0$ | $\subset \mathcal{Z}_2$ | 7 | 6 | $\mathrm{SU}(3)$ |
| $B^{-1}(\partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P})$ | $\mathcal{W}_2$ | $\subset \mathcal{Z}_2$ | 6 | 6 | $\mathrm{SU}(3)$ |
| $\mathrm{Int}\mathfrak{P}$ | $\mathcal{U}$ | $\mathcal{U}_0$ | 5 | 4 | $\mathrm{SU}(2)$ |

The ranks of $\mathcal{D}$ and $\mathcal{D}_0$ on various loci and the isomorphism type of $H_p$.




The $b$-image of the $\mathrm{SU}(4)$ locus is contained in $\partial I$ (orange). The $b$-image of the $\mathrm{G}_2$ locus is contained in $\partial D$ (green). The $b$-image of the $\mathrm{SU}(3)$ locus is contained in $\mathcal{R} \setminus (\partial I \cup \partial D)$ (blue), while the $b$-image of the $\mathrm{SU}(2)$ locus is contained in $\mathrm{Int}\mathcal{R}$ (blue).

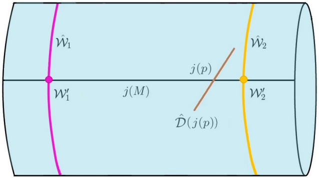
| | $\mathfrak{P}$-description | $b$-image | $\mathcal{D}$-stratum | $\mathcal{D}_0$-stratum | $b_+$ | $\rho$ | $\beta$ | $H_p$ |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{W}_0^+$ | $B^{-1}(\partial_0^+\mathfrak{P})$ | $\partial_0^+\mathcal{R}$ | $\mathcal{W}_0$ | $\mathcal{Z}_0$ | $+1$ | $0$ | $0$ | $\mathrm{SU}(4)$ |
| $\mathcal{W}_0^-$ | $B^{-1}(\partial_0^-\mathfrak{P})$ | $\partial_0^-\mathcal{R}$ | $\mathcal{W}_0$ | $\mathcal{Z}_0$ | $-1$ | $0$ | $0$ | $\mathrm{SU}(4)$ |
| $\mathcal{W}_1^1$ | $B^{-1}(\partial\mathfrak{D})$ | $\partial_1\mathcal{R} = \partial D$ | $\mathcal{W}_1$ | $\mathcal{Z}_1$ | $0$ | $1$ | $1$ | $\mathrm{G}_2$ |
| $\mathcal{W}_1^{0+}$ | $B^{-1}(\mathrm{Int}\mathfrak{I}^+)$ | $\mathrm{Int}(I^+)$ | $\mathcal{W}_1$ | $\mathcal{Z}_2$ | $(0, +1)$ | $0$ | $0$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_1^{0-}$ | $B^{-1}(\mathrm{Int}\mathfrak{I}^-)$ | $\mathrm{Int}(I^-)$ | $\mathcal{W}_1$ | $\mathcal{Z}_2$ | $(-1, 0)$ | $0$ | $0$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_1^{00}$ | $B^{-1}(\partial_0^0\mathfrak{P})$ | $\{0_{\mathbb{R}^3}\}$ | $\mathcal{W}_1$ | $\mathcal{Z}_2$ | $0$ | $0$ | $0$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_2^{2+}$ | $B^{-1}(\mathrm{Int}\mathfrak{D})$ | $\mathrm{Int}D$ | $\mathcal{W}_2$ | $\mathcal{Z}_2$ | $0$ | $[0, 1)$ | $1$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_2^{2-}$ | $B^{-1}(\mathfrak{A})$ | $\mathrm{Int}D \setminus \{0\}$ | $\mathcal{W}_2$ | $\mathcal{Z}_2$ | $0$ | $(0, 1)$ | $\rho$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_2^{3+}$ | $B^{-1}(\mathrm{Int}\mathfrak{C}^+)$ | $\mathrm{Int}(\mathcal{R}^+)$ | $\mathcal{W}_2$ | $\mathcal{Z}_2$ | $+g(\rho, \beta)$ | $[0, 1)$ | $(\rho, 1)$ | $\mathrm{SU}(3)$ |
| $\mathcal{W}_2^{3-}$ | $B^{-1}(\mathrm{Int}\mathfrak{C}^-)$ | $\mathrm{Int}(\mathcal{R}^-)$ | $\mathcal{W}_2$ | $\mathcal{Z}_2$ | $-g(\rho, \beta)$ | $[0, 1)$ | $(\rho, 1)$ | $\mathrm{SU}(3)$ |
| $\mathcal{U}$ | $B^{-1}(\mathrm{Int}\mathfrak{P})$ | $\mathrm{Int}\mathcal{R}$ | $\mathcal{U}$ | $\mathcal{U}_0$ | $(-1, 1)$ | $[0, 1)$ | $J(b_+, \rho)$ | $\mathrm{SU}(2)$ |

Preimage of the connected refinement of the canonical Whitney stratification of $\mathfrak{P}$.

| $\mathfrak{P}$-description | $\mathcal{S}$-stratum | $\mathcal{D}$-stratum | $\mathcal{D}_0$-stratum | $\mathrm{rk}\mathcal{D}$ | $\mathrm{rk}\mathcal{D}_0$ | $H_p$ |
|---|---|---|---|---|---|---|
| $B^{-1}(\partial_0^+\mathfrak{P})$ | $\mathcal{S}_{02}$ | $\mathcal{W}_0^+$ | $\mathcal{Z}_0^+$ | 8 | 8 | $\mathrm{SU}(4)$ |
| $B^{-1}(\partial_0^-\mathfrak{P})$ | $\mathcal{S}_{20}$ | $\mathcal{W}_0^-$ | $\mathcal{Z}_0^-$ | 8 | 8 | $\mathrm{SU}(4)$ |
| $B^{-1}(\partial\mathfrak{D})$ | $\mathcal{S}_{11}$ | $\mathcal{W}_1^1$ | $\mathcal{Z}_1$ | 7 | 7 | $\mathrm{G}_2$ |
| $B^{-1}(\mathfrak{S}^+)$ | $\mathcal{S}_{12}$ | $\subset \mathcal{W}_2^{3+}$ | $\subset \mathcal{Z}_2$ | 6 | 6 | $\mathrm{SU}(3)$ |
| $B^{-1}(\mathfrak{S}^-)$ | $\mathcal{S}_{21}$ | $\subset \mathcal{W}_2^{3-}$ | $\subset \mathcal{Z}_2$ | 6 | 6 | $\mathrm{SU}(3)$ |

Description of the special strata of the chirality stratification. The table does not show the non-special locus $\mathcal{G}$.

Some aspects of $\mathcal{N} = 2$ compactifications of eleven-dimensional supergravity down to $\mathrm{AdS}_3$ were approached in previous work using a nine-dimensional formalism based on the auxiliary 9-manifold $\hat{M} \stackrel{\mathrm{def.}}{=} M \times S^1$. That work makes intensive use of an assumption that holds only in the highly non-generic case when the $\mathrm{SU}(2)$ locus $\mathcal{U}$ of $M$ is empty. Failure of that assumption is related to the transversal vs. non-transversal character of the intersection of a certain distribution $\hat{\mathcal{D}}$ defined on $\hat{M}$ with the pullback to $\hat{M}$ of the tangent bundle of $M$.




The decomposition of $M$ induced by $\hat{\mathcal{D}}$. The figure shows the particular case when each of the loci $\hat{\mathcal{W}}_1$ and $\hat{\mathcal{W}}_2$ (depicted in magenta and yellow respectively) is connected. The open stratum $\hat{\mathcal{U}}$ (depicted in cyan) defined by $\hat{\mathcal{D}}$ is the complement of $\hat{\mathcal{W}} = \hat{\mathcal{W}}_1 \sqcup \hat{\mathcal{W}}_2$ inside $\hat{M}$. The intersection of $\hat{\mathcal{W}}_k$ with $j(M)$ determines loci $\mathcal{W}'_k \subset M$, which in this low-dimensional rendering are depicted as dots. The intersection of $\hat{\mathcal{U}}$ with $j(M)$ determines the locus $\mathcal{U}' \subset M$, which is the complement of the union $\mathcal{W}' = \mathcal{W}'_1 \sqcup \mathcal{W}'_2$ in $M$. In brown, we depicted the space $\hat{\mathcal{D}}(j(p)) \subset T_{j(p)}\hat{M}$ for a point $p \in M$.

# References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