Lattice supersymmetric Korteweg de Vries equation, Bilinear formalism, solutions
and reductions to super-QRT mappings

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Algebraic Geometry Seminar

IMAR, March 19, 2015

Outline

Basics of Korteweg de Vries equation

Consider the following non-linear hyperbolic partial differential equation known as *KdV*

KdV equation

Equation form:

$$KdV: \qquad u_t + 6uu_x + u_{xxx} = 0$$

Here $u = u(x,t)$ and u_t, u_x, u_{xx} are partial derivatives with respect to t and x
We are interested in integrability of this equation; because we have many faces of integrability we shall focus on bilinear/Hirota integrability

Motivation: KdV equations has *linear* and *nonlinear* part

and solutions behave *differently:*

$$KdV: \qquad \begin{cases} u_t + u_{xxx} = 0; u \approx \sum_i e^{k_i x + \omega_i t} - \text{dispersion} \\ u_t + 6uu_x = 0; u(x,0) = F(x) \Rightarrow u(x,t) = F(x - 6tu(x,t)) - \text{collapse} \end{cases}$$

If the dispersion of the wave packet is balanced by the collapse then the KdV-solution will be stable - nonlinear mode=solitary wave (happens often in nonlinear equations)

If **any** strongly localised initial condition is developing into a train of solitary waves (nonlinear modes) that interact *elastically* then we have **multisoliton solution** - a rare phenomenon occuring only in completely integrable systems.

Bilinear form of KdV

Bilinear idea is a substitution which **swallow the nonlinearity** and transform the KdV in something whcih looks like a *total dispersion* - thus the self-organisation in nonlinear modes becomes transparent

Hirota's idea = key of bilinear formalism

Consider the following substitution $u(x,t) = 2\partial_x^2 \log \tau(x,t)$ and define the following *antisymmetric* bilinear derivative acting on a *pair* of functions:

$$D_x^n f(x) \cdot g(x) = \partial_y^n f(x+y) g(x-y)|_{y=0}$$

Introducing into KdV equation we'll get an apparently more complicated expression but *bilinear*:

$$\tau_{xt}\tau - \tau_x \tau_t + \tau_{xxxx}\tau - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 = 0$$
$$D_x(D_t + D_x^3)\tau \cdot \tau = 0$$

So now looking at the bilinear equation one can write immediately solutions as superpositions of exponentials: Let $\eta_i = k_i x - k_i^3 t$

$$\tau_1 = 1 + e^{\eta_1}, \quad \tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}$$

$$\tau_3 = 1 + \sum_{i=1}^{3} e^{\eta_i} + \sum_{i<j} A_{ij} e^{\eta_i + \eta_j} + A_{12}A_{13}A_{23} e^{\eta_1 + \eta_2 + \eta_3}$$

Remarks

- The red term is crucial. It happens only in completely integrable systems. So, if a PDE can be put in a bilinear form and has at least 3-soliton solution (supports interaction of three solitons) then the PDE is a completely integrable system.

Crucial property

$$\begin{cases} \forall A_x^N(f,g) = \sum_{i=1}^{N} c_i(\partial_x^{N-i} f)(\partial_x^i g), \quad A_x^N(e^\eta f, e^\eta g) = e^{2\eta} A_x^N(f,g), \quad \text{iff} \\ A_x^N(f,g) = D_x^N f \cdot g \Leftrightarrow c_i = (-1)^i C_N^i \end{cases}$$

Novikov, Dubrovin, Krichever extended this method and consider the most general combination of exponentials together with their interaction

$$\begin{cases} \tau = \Theta[\alpha,\beta](z|B) = \sum_{n\in\mathbb{Z}^g} \exp\{i\pi[2 < (z+\alpha),(n+\beta) > + < (n+\alpha), B(n+\alpha) >]\} \\ z = (z_1,...,z_g) \equiv (k_1 x - \omega_1 t, ..., k_g x - \omega_g t), \quad B = \text{period} - \text{matrix} \end{cases}.$$

Sato, Jimbo and Miwa proved that the bilinear formulation can be done for any hierachy and the bilinear equations are nothing but Plucker quadratic equation defining an infinite dimensional grassmannian

Existence of 3-soliton (three-phase theta) solution in the bilinear form $\Leftrightarrow$ complete integrability

Supersymmetric extension of KdV equation in $N = 1$ superspace

The basic idea is to consider coupled partial differential equations containing two dependent variables (fields) $u(x,t)$ and $\xi(x,t)$ such that $u : \mathbb{R}^2 \to \Lambda_0, \xi : \mathbb{R}^2 \to \Lambda_1$ where Λ_0 and Λ_1 are the even (bosonic) and odd (fermionic) sectors of an infinite dimensional Grassmann algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$; it is a graduate modulo 2 algebra i.e. $\Lambda_0\Lambda_0 \subset \Lambda_0, \Lambda_1\Lambda_1 \subset \Lambda_0, \Lambda_0\Lambda_1 \subset \Lambda_1$. Also when $\xi = 0$ then we recover the usual KdV equation in the variable u.
Let us consider the following coupled system:

$$sKdV : \quad \begin{cases} u_t + u_{xxx} + 6uu_x - 3\xi\xi_{xx} = 0, \\ \xi_t + 3(u\xi)_x + \xi_{xxx} = 0 \end{cases}$$

invariant at the following transformations,

$$supersymmetry : \quad \begin{cases} \delta u = \gamma\xi_x, \\ \delta\xi = \gamma u \end{cases}$$

$N = 1$ Superspace: one can unify the above description by extending the $\mathbb{R}$ to a bigger one containing two variables (x, θ) where θ is an anticommuting variable (and of course $\theta^2 = 0$). In addition we unify the two fields $u(x,t)$ and $\xi(x,t)$ into a larger superfield which can be bosonic (not interesting) or fermionic and have the expression, $\Phi(x,t,\theta) = \xi(x,t) + \theta u(x,t)$. Also a superderivative is introduced namely $D = \partial_\theta + \theta\partial_x$ which is practically a square root of usual derivative $D^2 = \partial_x$. Accordingly the super-KdV equation will be written in superspace as

$$N = 1 super - KdV : \quad \begin{cases} u(x,t), \xi(x,t) \to \Phi(x,t,\theta) = \xi + \theta u, D = \partial_\theta + \theta\partial_x, \\ \Phi_t + \Phi_{xxx} + 3(\Phi D\Phi)_x = 0 \end{cases}$$

Remarks

One can see immediately that supersymmetry transformation is nothing but a symmetry at the *translation in superspace with respect to anticommuting parameter* γ

$$\Phi(x,t,\theta) \to \Phi(x-\gamma\theta, t, \theta+\gamma) = \Phi(x,t,\theta) - \gamma\theta\partial_x\Phi + \gamma\partial_\theta\Phi =$$
$$= \Phi(x,t,\theta) + \delta\Phi(x,t,\theta) = \xi + \theta u + \delta\xi + \theta\delta u$$

The super-KdV equation is a completely integrable system introduced by Manin and Radul in 1985 and has the following Lax representation

$$\partial_t L = [-4L_+^{3/2}, L], \quad L = \partial_x - \Phi D,$$
$$\forall P = \sum_{i=-\infty}^{M} \alpha_i D^i \Rightarrow P_+ = \sum_{i=0}^{M} \alpha_i D^i, \quad \mathrm{sRes} P = \alpha_{-1}$$

It has two sets of conservation laws given by the following superresidues:

$$H_{2k+1} = \int dx d\theta \mathrm{sRes} L^{(2k+1)/2}, J_{2k+1} = \int dx d\theta \mathrm{sRes} L^{(2k+1)/4}$$

where superintegration is defined through the Berezin rules

$$\int d\theta = 0, \quad \int \theta d\theta = 1$$

However Manin and Radul were not able to compute the general super-soliton solutions of super-KdV equation

Super-bilinear operator

In order to construct the bilinear form of super-KdV equation some bilinear operators are needed; if we use antisymmetric derivatives it doesn't work - accordingly we have to rely on a more important property.

So we proved the following,
Theorem: A general superbilinear operator $S_x^N f \cdot g = \sum_{i=1}^N c_i (D^{N-i} f)(D^i g)$ has the property of super-gauce invariance (i.e. $S_x^n e^a f \cdot e^a g = e^{2a} S_x^N f \cdot g, a = kx + \zeta\theta$) iff

$$c_i = (-1)^{i|f|+i(i+1)/2} \left[\begin{array}{c} N \\ i \end{array} \right].$$

where the grassmann parity $|f$ is defined through

$$|f| = \begin{cases} 1, \text{if} & f \in \Lambda_1 \\ 0, \text{if} & f \in \Lambda_0 \end{cases}$$

and the super-binomial coefficients are given by

$$\left[\begin{array}{c} N \\ i \end{array} \right] = \begin{cases} C_{[N/2]}^{[i/2]}, \text{if} & (N, i) \neq (0, 1) \text{mod} 2 \\ \quad 0, & \text{otherwise} \end{cases}$$

In this definition this opeartor is the square root of the Hirota operator $S_x^N f \cdot g = \mathbf{D}_x^{1/2} f \cdot g$

Super-bilinear for super-KdV

In order to construct the bilinear form of super-KdV equation we consider the following nonlinear substitution

$$\Phi(x,t,\theta) = 2D^3 \log \tau(x,t,\theta)$$

where τ is a commuting (even) superfunction and on the components is written as $\tau = F + \theta G, F \in \Lambda_0, G \in \Lambda_1$

The bilinear forms on the superspace and on the ordinary space of the super-KdV

$$\Phi_t + 3(\Phi D\Phi)_x + \Phi_{xxx} = 0,$$

will be,

$$(S_x \mathbf{D}_t + S_x^7)\tau \cdot \tau = 0$$

equivalent with

$$\begin{cases} (\mathbf{D}_t + \mathbf{D}_x^3) G \cdot F = 0 \\ (\mathbf{D}_x \mathbf{D}_t + \mathbf{D}_x^4) F \cdot F = 2(\mathbf{D}_t + \mathbf{D}_x^3) G \cdot G \end{cases}$$

Now one can construct super-soliton solutions. If $\eta_i = k_i x - k_i^3 t, \alpha_{ij} = (k_i + k_j)/(k_i - k_j), A_{ij} = (k_i - k_j)^2/(k_i + k_j)^2, \beta_{ij} = 2/(k_j - k_i)$

$$\tau_1 = 1 + e^{\eta + \theta\zeta}$$

$$\tau_2 = 1 + e^{\eta_1 + \zeta_1\theta} + e^{\eta_2 + \zeta_2\theta} + A_{12}(1 + 2\beta_{12}\zeta_1\zeta_2)e^{\eta_1 + \eta_2 + \theta(\alpha_{12}\zeta_1\theta + \alpha_{21}\zeta_2)}$$

General N-supersoliton solution

$$\tau_3 = 1 + \sum_{i=1}^{3} e^{\eta_i + \theta\zeta_i} + \sum_{i<j}^{3} A_{ij}(1 + 2\beta_{ij}\zeta_i\zeta_j)e^{\eta_i+\eta_j+\theta(\alpha_{ij}\zeta_i\theta+\alpha_{ji}\zeta_j)} +$$

$$+ \left(\prod_{i<j,k\neq\{i,j\}}^{3} A_{ij}(1 + 2\beta_{ij}\alpha_{ik}\alpha_{jk}\zeta_i\zeta_j) \right) e^{\eta_1+\eta_2+\eta_3+\theta(\alpha_{12}\alpha_{13}\zeta_1+\alpha_{21}\alpha_{23}\zeta_2+\alpha_{31}\alpha_{32}\zeta_3)}$$

What is remarkable here is that the soliton interaction is no longer elastical. However it has a rigurous pattern and can be constructed. The same type of interaction occurs in other supersymmetric integrable PDEs. It was not obtained so far from the Lax pair and genralisation to super-Riemann theta function is not known.

Big advantage of bilinear formalim: - it provides the integrable discretisation.
Motivations:

- discrete nonlinear integrable systems are "more" fundamental having a deeper phenomenology
- all discrete nonlinear integrable systems are birational - more amenable to the tools of algebraic geometry

For the moment we introduce the notations
$\bar{A} = A(n+1, m), \tilde{A} = A(n, m+h), \underset{\sim}{\tilde{A}} = A(n-1, m+h)$ with $x = \epsilon n, t = hm$. There are three steps in discretizing a nonlinear system:

- replace Hirota bilinear operators with difference ones preserving invariance with respect to e^{linears}
- check the soliton solution
- recover the nonlinear form

In the case of ordinary KdV we have $u(x,t) = 2\partial_x^2 \log \tau$ which is not nice (hard to discretize). We prefer a form like $u(x,t) = H/F$. So we transform bilinear form. One can see immediately that if $\partial_x H = F$ then

$$(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}_x^4)F \cdot F \Leftrightarrow \begin{cases} (\mathbf{D}_t + \mathbf{D}_x^3)H \cdot F = 0 \\ \mathbf{D}_x^2 F \cdot F - \mathbf{D}_x H \cdot F = 0 \end{cases}$$

so we prefer to work with the second bilinear system. The idea of discretisation of $\mathbf{D}$ is straightforward, $\mathbf{D}_x f \cdot g = f_x g - f g_x \to (f(n+1) - f(n))g(n) - f(n)(g(n+1) - g(n)) = \bar{f} g - f \bar{g}$.

$$\begin{aligned} (\mathbf{D}_t + \mathbf{D}_x^3)H \cdot F &= 0 \\ \mathbf{D}_x^2 F \cdot F - \mathbf{D}_x H \cdot F &= 0 \end{aligned} \Rightarrow \begin{cases} \tilde{H}F - H\tilde{F} + h(\bar{\tilde{H}}\underline{F} - \underline{H}\bar{\tilde{F}}) = 0 \\ \bar{\tilde{F}}\underline{F} - F\tilde{F} - (\bar{\tilde{H}}\underline{F} - \underline{H}\bar{\tilde{F}}) = 0 \end{cases}$$

Now if $u = H/F$ then from the above bilinear system we get the integrable discrete KdV (in a birational form):

$$\tilde{u} - u = h\frac{\bar{\tilde{u}} - \underline{u}}{1 - \bar{\tilde{u}} + \underline{u}}$$

Reductions

From the above KdV equation we can find the reduction to ordinary difference equation. If $u(n,m) = y(n,m) + (h+1)n/2 - hm$,then we have

$$\tilde{u} - u = h\frac{\tilde{\bar{u}} - \underline{u}}{1 - \tilde{\bar{u}} + \underline{u}} \to (\tilde{y} - y)(\underline{y} - \tilde{\bar{y}}) = h$$

Reduction is done by considering combinations of the two independent variables in one. Accordingly the resulting equation will be no longer partial difference but ordinary difference. For instance if $\nu = n + m/h$ then we have

$$(\tilde{y} - y)(\underline{y} - \tilde{\bar{y}}) = h \Rightarrow (\bar{y} - y)(\underline{y} - \bar{\bar{y}}) = h$$

$$(\bar{y} - y)(\underline{y} - \bar{\bar{y}}) = h \underbrace{\Rightarrow}_{\bar{y} - y = q} \bar{q} + q + \underline{q} = \frac{h}{q}$$

Integrability of the mapping

$$\bar{q} + q + \underline{q} = \frac{h}{q}$$

is clear - it comes as a reduction of an integrable equation; but we need to know how to integrate i.e.constructing the invariant. We will consider it as a birational map on projective plane:

$$\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad \varphi(q,p) = (\bar{q}, \bar{p}) : \begin{cases} \bar{q} = p \\ \bar{p} = -q - p + \dfrac{h}{p} \end{cases}$$

Difference Equation as a Birational Map

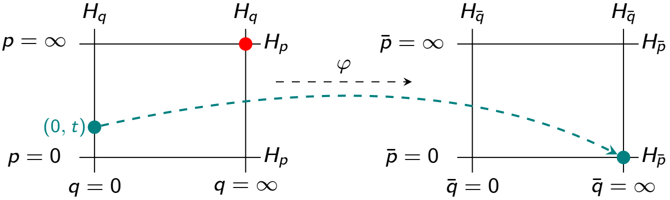
What can go wrong? Recall our map: $\bar{q} = p$, $\bar{p} = -q - p + \frac{h}{p}$.

$$\bar{q} = \frac{f(q,p)}{g(q,p)} = \begin{cases} \dfrac{0}{1} = 0 \\ \dfrac{1}{0} = \infty \text{ if we compactify } \mathbb{C} \text{ to } \mathbb{P}^1 \\ \dfrac{0}{0} =? \text{ indeterminate point, resolve using the blowup procedure} \end{cases}$$

Step 1: Compactification from $\mathbb{C}^2$ to $X = \mathbb{P}^1 \times \mathbb{P}^1$. X is a compact complex surface covered by four charts, (q,p), (Q,p), (q,P), and (Q,P), where $P = 1/p$ and $Q = 1/q$ are the charts at infinity. Also, for a compact surface, we can talk about the *Picard Lattice*:

$$\mathrm{Pic}(X) = \mathrm{Div}(X) = \mathbb{Z}\mathcal{H}_q \oplus \mathbb{Z}\mathcal{H}_p, \qquad \mathcal{H}_q \bullet \mathcal{H}_q = \mathcal{H}_p \bullet \mathcal{H}_p = 0, \mathcal{H}_q \bullet \mathcal{H}_p = 1$$

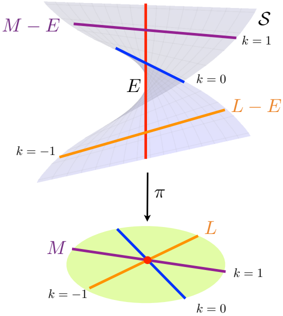
$$-K_X = 2H_q + 2H_p \quad \text{— anti-canonical divisor (dual to the symplectic area form) } \omega = dq \wedge dp$$



Extending the map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

The Blowup Procedure

Pictorially we can visualize the blowup procedure as follows:



$$L \bullet M = 1$$
$$L \bullet E = M \bullet E = 0$$
$$(L-E) \bullet (M-E) = 0 \text{ and therefore}$$
$$E \bullet E = -1.$$
$$\text{If } L^2 = L \bullet L = m, \text{ then}$$
$$(L-E)^2 = m-1.$$

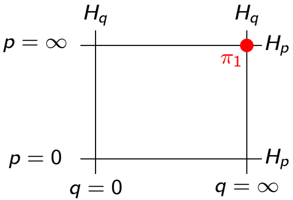
The blowup (local picture)

Coordinate description of a blowup at an arbitrary point $\pi(q_0, p_0)$ is as follows:

- The two standard complex charts on $\mathbb{P}^1$ given by $v = \zeta_0/\zeta_1$ when $\zeta_1 \neq 0$ and $V = \zeta_1/\zeta_0$ when $\zeta_0 \neq 0$;
- they induce two charts on $\mathcal{S}$: (u, v), where $(q, p, [\zeta_0 : \zeta_1]) = (u + q_0, uv + p_0, [u : 1])$, and (U, V), where $(q, p, [\zeta_0 : \zeta_1]) = (UV + q_0, V + p_0, [1 : V])$;
- the *exceptional divisor* E is given by $u = 0$ or $V = 0$;
- the transition functions are given by $U = 1/v$, $V = uv$.

Step 2: Find the Indeterminate points of both the *forward* and the *backward* dynamic.

$$\begin{cases} \bar{q} = p \\ \bar{p} = -q - p + \dfrac{h}{p} \end{cases}$$



In the coordinates (P, Q):

$$\begin{cases} \bar{q} = \dfrac{1}{P} \\ \bar{p} = \dfrac{-P - Q + hP^2 Q}{QP} = \dfrac{0}{0} \text{ when } Q = P = 0 \end{cases}$$

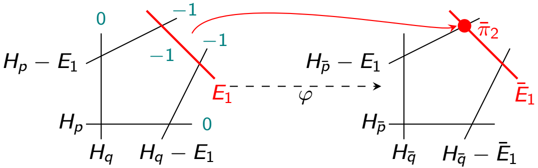
So $(Q, P) = (0, 0)$ (or $(q, p) = (\infty, \infty)$) is the indeterminate point of the dynamic. Resolve it using the blowup procedure. In the blowup coordinates (u_1, v_1), $Q = u_1$, $P = u_1 v_1$:

$$\begin{cases} \bar{q} = \dfrac{1}{u_1} = \infty, \\ \bar{p} = \dfrac{-1 - v_1 + h u_1^2 v_1}{u_1 v_1} = \dfrac{-1 - v_1}{0} = \infty \qquad \text{on } E_1\colon\ u_1 = 0\,. \end{cases}$$

Thus, $\varphi(E_1) = \bar{\pi}_1 = (\infty, \infty)$. In the blowup coordinates $\bar{u}_1 = \frac{1}{\bar{q}}$ and $\bar{v}_1 = \frac{\bar{q}}{\bar{p}}$ we get

$$\begin{cases} \bar{u}_1 = u_1 = 0 \\ \bar{v}_1 = \dfrac{v_1}{-1 - v_1 + h u_1^2 v_1} = \dfrac{v_1}{-1 - v_1} \end{cases} \qquad \text{on } E_1\colon\ u_1 = 0\,.$$

which is a *line* so it is OK.
If, φ *collapses* E_1 on a point with general coordinates $\bar{\pi}_2(\bar{u}_1 = a, \bar{v}_1 = b)$. (as in the picture) the point would appear as an indeterminate point for the inverse map. Thus, we have to blowup again at π_2 and $\bar{\pi}_2$ and continue the process:



Extending the map to $\varphi : X \dashrightarrow X$

We do this until we resolve all the singular points and thus we construct the so called space of initial conditions

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{H}_q \oplus \mathbb{Z}\mathcal{H}_p \oplus (\oplus_{i=1}^{8} \mathbb{Z}\mathcal{E}_i), \qquad \begin{aligned} &\mathcal{H}_q \bullet \mathcal{H}_q = \mathcal{H}_p \bullet \mathcal{H}_p = \mathcal{H}_q \bullet \mathcal{E}_1 = \mathcal{H}_p \bullet \mathcal{E} = 0, \\ &\mathcal{H}_q \bullet \mathcal{H}_p = 1, \quad \mathcal{E} \bullet \mathcal{E}_1 = -1. \end{aligned}$$

Invariants

The complete resolution of indeterminacies gives the following bundle mapping

$$\varphi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X) : \begin{cases} \bar{E}_1 = 2H_q + H_p - E_2 - E_3 - E_6 - E_7 - E_8 \\ \bar{E}_2 = H_q + H_p - E_3 - E_6 - E_8 \\ \bar{E}_3 = H_q + H_p - E_2 - E_6 - E_8 \\ \bar{E}_4 = H_q - E_8 \\ \bar{E}_5 = 2H_q + H_p - E_2 - E_3 - E_4 - E_6 - E_8 \\ \bar{E}_6 = E_1 \\ \bar{E}_7 = H_q - E_6 \\ \bar{E}_8 = E_5 \\ \bar{H}_q = 3H_q + 2H_p - E_2 - E_3 - E_4 - 2E_6 - E_7 - 2E_8 \\ \bar{H}_p = 2H_q + H_p - E_2 - E_3 - E_6 - E_8 \end{cases}$$

One can see immediately that

$$\varphi^*(-K_X) = -K_X, \quad -K_X = 2H_q + 2H_p - E_1 - E_2 - ... - E_8$$

If the proper transform of $-K_X$ is $f(q,p) \equiv \sum_{i,j=0}^{2} a_{ij} q^{2-i} p^{2-j}$, $f(E_i) = 0, i = 1...8$ and at least one of a_{ij} contains a *free* parameter then we have a *pencil of biquadratic curves*, the free parameter being the **integral of motion**

Discretizing super-KdV

We have seen that the bilinear form of super-KdV is given by:

$$(S_x\mathbf{D}_t + S_x^7)\tau \cdot \tau = 0$$

equivalent with

$$\begin{cases} (\mathbf{D}_t + \mathbf{D}_x^3)G \cdot F = & \!\!\!\! 0 \\ (\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)F \cdot F = & \!\!\!\! 2(\mathbf{D}_t + \mathbf{D}_x^3)G \cdot G \end{cases}$$

Since in the discrete context the supersymmetry is broken we will focus on the second bilinear system and apply the same procedure as in the ordinary KdV case. To make the long story short we propose the following (remember G is an anticommuting function so $G^2 = 0$):

$$\begin{aligned} (\mathbf{D}_t + \mathbf{D}_x^3)G \cdot F =& 0 \\ (\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)F \cdot F =& 2(\mathbf{D}_t + \mathbf{D}_x^3)G \cdot G \end{aligned} \Rightarrow \begin{cases} \tilde{G}F - G\tilde{F} + h(\tilde{\tilde{G}}\underline{F} - \underline{G}\tilde{\tilde{F}}) = 0 \\ \tilde{H}F - H\tilde{F} + h(\tilde{\tilde{H}}\underline{F} - \underline{H}\tilde{\tilde{F}}) = m_1\tilde{G}G + hm_2\tilde{\tilde{G}}\underline{G} \\ h(\tilde{\tilde{F}}\underline{F} - F\tilde{F}) - h(\tilde{\tilde{H}}\underline{F} - \underline{H}\tilde{\tilde{F}}) = m_3\tilde{G}G + hm_4\tilde{\tilde{G}}\underline{G} \end{cases}$$

For $m_1 = m_2, m_3 = m_2 + m_4$ the system admits maximum 2-supersoliton solution (nonintegrability). But for $m_2 = -2, m_4 = 1$ or $m_1 = -1, m_3 = 0$ the system admits 3-supersoliton solution. So we have **two** different integrable discretisations (and the soliton interaction is different as well) having the same continuum limit

The two integrable discretisations are the following. Considering $\psi = G/F, u = H/F$ then

- $m_2 = -2, m_4 = 1$

$$\widetilde{\psi} - \psi = h\frac{\widetilde{\overline{\psi}} - \underline{\psi}}{1 - \widetilde{\overline{u}} + \underline{u}}$$

$$\widetilde{u} - u = h\frac{\widetilde{\overline{u}} - \underline{u}}{1 - \widetilde{\overline{u}} + \underline{u}} + \frac{2 - \widetilde{\overline{u}} + \underline{u}}{(\widetilde{\overline{u}} - \underline{u} - 1)^2}(\widetilde{\overline{\psi}} - \underline{\psi})(\psi - \underline{\psi})$$

- $m_1 = -1, m_3 = 0$

$$\widetilde{\psi} - \psi = h\frac{\widetilde{\overline{\psi}} - \underline{\psi}}{1 - \widetilde{\overline{u}} + \underline{u}}$$

$$\widetilde{u} - u = h\frac{\widetilde{\overline{u}} - \underline{u}}{1 - \widetilde{\overline{u}} + \underline{u}} + \frac{(1 - \widetilde{\overline{u}} + \underline{u})\psi(\widetilde{\overline{\psi}} - \underline{\psi}) + \widetilde{\overline{\psi}}\underline{\psi}}{(\widetilde{\overline{u}} - \underline{u} - 1)^2}$$

Super-reductions

We can do the same trick of reducing the partiaol discrete super-partial difference equation to a nonlinear super-diference equation. In the case of first discretisation we obtian the following system ($q = q(n)$ is commuting and $\xi = \xi(n)$ is anticommuting):

$$\bar{q} + q + \underline{q} = \frac{h}{q} - \frac{h+q}{q^2}\xi\underline{\xi}$$

$$\bar{\xi} + \xi + \underline{\xi} = -\frac{\xi}{q}$$

Now the problem is but constructing the invariants. It is an order 4 mapping so we have two invariants and moreover is defined on the *projective space over a grassmann algebra¿* However it can be simplyfied as follows. Multiply on the left the second equation with ξ and taking into account that it is nilpotent we get

$$\xi\bar{\xi} + \xi\underline{\xi} = 0.$$

Defining $I_n = \xi_n\xi_{n-1} \equiv \xi\underline{\xi}$ then we get $I_n = I_{n+1}, \forall n$ so I_n is the *first invariant*. If from the initial condition we put $I_n = \gamma$ - a commuting (nilpotent) we will end up the the following

$$\varphi : \Lambda_0\mathbb{P}^1 \times \Lambda_0\mathbb{P}^1 \to \Lambda_0\mathbb{P}^1 \times \Lambda_0\mathbb{P}^1, \quad \varphi(q,p) = (\bar{q},\bar{p}) : \begin{cases} \bar{q} = p \\ \bar{p} = -q - p + \dfrac{h}{p} - \gamma\dfrac{h+p}{p^2} \end{cases}$$

Blow up analysis failed inasmuch as singularities appear in points with non-invertible coordinates. Fortunaltely we have the Lax pair of the above system and the second invariant is $Tr(Lax)$

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