M-theory foliated backgrounds and non-commutative geometry

Mirela Babalic

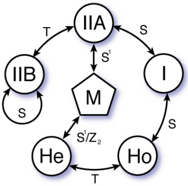
(joint work with Calin Lazaroiu)

Geometry and Physics group
(http://events.theory.nipne.ro/gap/)
Department of Theoretical Physics, IFIN-HH

April 24, 2015

Outline

Motivation



We want to describe the geometry and topology of the most general 8-dimensional backgrounds with fluxes which preserve a certain amount of supersymmetry when compactifying M-theory (11-dim SUGRA) to AdS_3 manifolds.

$$\mathbf{M} = M_3 \times M_8$$

Compactifications down to AdS_3

- SUGRA action in 11 dimensions (involving the SUGRA fields **g**, **C**, **Ψ**):

$$\boxed{S_{11} = \int d^{11}y \left[\mathbf{R}\nu - \frac{1}{2}\mathbf{G}\wedge\star\mathbf{G} - \frac{1}{6}\mathbf{G}\wedge\mathbf{G}\wedge\mathbf{C}\right] + \text{terms involving } \boldsymbol{\Psi}}$$

- The metric on $\mathbf{M} = M_3 \times M$ is a warped product:

$$\mathrm{d}\mathbf{s}_{11}^2 = e^{2\Delta}(\mathrm{d}s_3^2 + \mathrm{d}s_8^2) \quad , \quad \boldsymbol{\Delta} \in C^\infty(M,\mathbb{R}) \quad .$$

- $$\boxed{\mathbf{G} = \mathrm{d}\mathbf{C} = e^{3\Delta}G \quad , \quad G = \mathrm{vol}_3 \wedge f + F \ , \quad f \in \Omega^1(M), \quad F \in \Omega^4(M)}$$

- **Susy conditions**: $\boxed{\delta_{\boldsymbol{\eta}}\boldsymbol{\Psi} = \mathbf{D}\boldsymbol{\eta} = 0}$

$$\boldsymbol{\eta} = e^{\frac{\Delta}{2}}\eta \quad \text{with} \quad \eta = \zeta\otimes\xi \quad , \quad \zeta\in\Gamma(M_3,S_3) \ , \quad \xi\in\Gamma(M,S) \ ,$$

For ζ a Killing spinor on M_3, susy conditions $\Longrightarrow$ CGKS equations on M_8:

$$\boxed{D_m\xi = Q\xi = 0} \ , \quad D_m = \nabla_m + A_m$$

The chiral and nonchiral loci on the internal manifold

$$\xi = \xi^+ + \xi^- \quad , \quad \xi^\pm \stackrel{\text{def.}}{=} \frac{1}{2}(1 \pm \gamma(\nu))\xi \;\in \Gamma(M, S^\pm) \quad , \quad S = S^+ \oplus S^-$$

$$||\xi||^2 = ||\xi^+||^2 + ||\xi^-||^2 = 1 \quad , \quad b = ||\xi^+||^2 - ||\xi^-||^2 \quad \Longleftrightarrow \quad \boxed{||\xi^\pm||^2 = \frac{1}{2}(1 \pm b)}$$

When $\mathcal{N}=1$ supersymmetry is preserved on the external space, one may have:

- ξ is everywhere chiral $\Longrightarrow$ no fluxes at the classical level, M has Spin(7) holonomy
- ξ is everywhere non-chiral $\Longrightarrow$ regular foliation with leafwise G_2 structure
- ξ is chiral somewhere but not everywhere $\Longrightarrow$ singular foliation with leafwise G_2 structure

In a **general** mathematical framework for supersymmetric flux compactifications a global reduction of structure group does not exist.

- **The purely non-chiral locus** $\mathcal{U}$ (ξ is Majorana, but not Weyl, $b \neq \pm 1$):

$$\mathcal{U} \stackrel{\text{def.}}{=} \{p \in M | \xi \notin S_p^+ \cup S_p^-\} = \{p \in M | \xi_p^+ \neq 0 \text{ and } \xi_p^- \neq 0\} = \{p \in M || b(p)| < 1\}$$

- **The chiral loci** $\mathcal{W}^+$, $\mathcal{W}^-$:

$$\mathcal{W}^+ \stackrel{\text{def.}}{=} \{p \in M | \xi_p \in S_p^+ \text{ , } i.e.\ b(p) = +1\} \;,\quad \mathcal{W}^- \stackrel{\text{def.}}{=} \{p \in M | \xi_p \in S_p^- \text{ , } i.e.\ b(p) = -1\}$$

A topological no-go theorem

Bianchi identity and e.o.m. for **G**:

$$\mathrm{d}\mathbf{G} = 0 \iff \mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{f} = 0 \ , \ \ \mathrm{d} \star \mathbf{G} + \frac{1}{2}\mathbf{G} \wedge \mathbf{G} = 0 \ . \tag{1}$$

Theorem. Assume the susy conditions, the Bianchi identity, the e.o.m. for G and the Einstein equations are satisfied. There exist only the following 4 possibilities:

1. $\mathcal{U} = M \implies \mathcal{W}^+ = \mathcal{W}^- = \emptyset$.
2. $\mathcal{W}^+ = M \implies \mathcal{W}^- = \mathcal{U} = \emptyset$. Then, $\xi \in S^+$ and is covariantly constant on M, $f = F = 0$ while $\Delta = \mathrm{constant\ on\ M}$. Furthermore, M_3 becomes Minkowski.
3. $\mathcal{W}^- = M \implies \mathcal{W}^+ = \mathcal{U} = \emptyset$. Then, $\xi \in S^-$ and is covariantly constant on M, $f = F = 0$ while $\Delta = \mathrm{constant\ on\ M}$. Furthermore, M_3 becomes Minkowski.
4. $\mathcal{W}^+ \neq \emptyset$ and/or $\mathcal{W}^- \neq \emptyset$ but both of them have empty interior. In this case, $\mathcal{U}$ is dense in M and $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^- = \mathrm{Fr}\,\mathcal{U}$.

Formulation through Kähler-Atiyah bundles

There is an isomorphic realization of the Clifford bundle $\mathrm{Cl}(T^*M)$ of T^*M as the Kahler-Atiyah bundle $(\wedge T^*M, \diamond)$, where the geometric product $\diamond : \wedge T^*M \times \wedge T^*M \to \wedge T^*M$ is an associative (but non-commutative) fiberwise composition which makes the exterior bundle into a bundle of unital associative algebras and satisfies *Chevalley's formulas* for $\omega \in \Omega^k(M)$ and $X \in \Gamma(M, TM)$:

$$\begin{aligned} X_\sharp \diamond \omega &= X_\sharp \wedge \omega + X \lrcorner \omega \\ \omega \diamond X_\sharp &= (-1)^k (X_\sharp \wedge \omega - X \lrcorner \omega) \end{aligned}$$

The geometric product expands as:

$$\boxed{\omega \diamond \eta = \sum_{m=0}^{\min(k,l)} (-1)^{\left[\frac{m+1}{2}\right]} \pi^m(\omega) \bigtriangleup_m \eta \ \ ,}$$

where $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ and:

$$\omega \bigtriangleup_0 \eta = \omega \wedge \eta \ \ , \ \ \omega \bigtriangleup_{k+1} \eta = \frac{1}{k+1} g^{mn} (\partial_m \lrcorner \omega) \bigtriangleup_k (\partial_n \lrcorner \eta) \ \ , \ \ \bigtriangleup_m \overset{\text{def.}}{=} \frac{1}{m!} \wedge_m .$$

The $\mathcal{N} = 1$ supersymmetry case

Theorem: Giving a globally-defined smooth pinor $\xi \in \Gamma(M, S)$ satisfying the susy conditions is equivalent to giving a globally-defined inhomogeneous form:

$$\boxed{\check{E} = \frac{1}{16}\sum_{k=1}^{8}\frac{1}{k!}\mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi)e^{a_1\ldots a_k} = \frac{1}{16}(1 + V + Y + Z + b\nu)} \in \Omega(M)$$

such that:

$$\boxed{\nabla_m \check{E} = -[\check{A}_m, \check{E}]_- \ , \quad \check{Q}\check{E} = 0} \qquad (2)$$

where

$$||\xi||^2 = 1 \ , \ b \in \mathcal{C}^\infty(\mathbb{R}, M) \ , \ V \in \Omega^1(M) \ , \ Y \in \Omega^4(M) \ , \ Z \in \Omega^5(M)$$

$$\check{A}_m = \gamma^{-1}(A_m) = \frac{1}{4}e_m \lrcorner F + \frac{1}{4}(e_{m_\sharp} \wedge f)\nu + \kappa e_{m_\sharp}\nu \ ,$$

$$\check{Q} = \gamma^{-1}(Q) = \frac{1}{2}\mathrm{d}\Delta - \frac{1}{6}f\nu - \frac{1}{12}F - \kappa\nu$$

$$(\gamma_a^t = \gamma_a \ , \ \gamma^t_{a_1\ldots a_k} = (-1)^{\frac{k(k-1)}{2}}\gamma_{a_1\ldots a_k} \ , \ \mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi) = (-1)^{\frac{k(k-1)}{2}}\mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi) \ )$$

The non-chiral $\mathcal{N} = 1$ case

When ξ is everywhere non-chiral on M ($\xi^+ \neq 0$, $\xi^- \neq 0$, thus $|b| < 1$ everywhere), the **Fierz identities** are encoded by the relations:

$$\boxed{\check{E}^2 = \check{E} \quad , \quad \mathcal{S}(\check{E}) = 1 \quad , \quad \tau(\check{E}) = \check{E} \quad , \quad |\mathcal{S}(*\check{E})| = |b| < 1} \qquad (3)$$

and are equivalent with:

$$||V||^2 = 1 - b^2 > 0$$
$$\iota_V * Z = 0 \quad , \quad \iota_V Z = Y - b * Y$$
$$(\iota_u(*Z)) \wedge (\iota_v(*Z)) \wedge (*Z) = -6 < u \wedge V, v \wedge V > \iota_V \nu \quad , \quad \forall u, v \in \Omega^1(M)$$

The above system also implies: $||Z||^2 = 7(1 - b^2)$, $||Y||^2 = 7(1 + b^2)$.

($\tau(\omega^k) = (-1)^{\frac{k(k-1)}{2}} \omega^k, \ \ \forall \omega^k \in \Omega^k(M)$)

The Frobenius distribution defined by V

Since V is nowhere-vanishing, it determines a corank one Frobenius distribution $\mathcal{D}=\ker V\subset TM$. We introduce the normalized vector field:

$$n \stackrel{\mathrm{def.}}{=} \hat{V}^\sharp = \frac{V^\sharp}{||V||} \quad , \quad ||n||=1 \quad ,$$

which is everywhere orthogonal to $\mathcal{D}$ and generates another integrable distribution $\mathcal{D}^\perp$ (since it has rank1). This provides an orthogonal direct sum decomposition:

$$TM=\mathcal{D}\oplus\mathcal{D}^\perp$$

$\mathcal{D}$ is transversely oriented by n. Since M itself is oriented, we define the longitudinal volume form $\nu_\top=\iota_{\hat{V}}\nu=n\lrcorner\nu\in\Omega^7(\mathcal{D})$:

$$\hat{V}\wedge\nu_\top=\nu \quad .$$

Let $*_\perp:\Omega(\mathcal{D})\to\Omega(\mathcal{D})$ be the Hodge operator along $\mathcal{D}$:

$$*_\perp\omega=*(\hat{V}\wedge\omega)=(-1)^{\mathrm{rk}\omega}\iota_{\hat{V}}(*\omega)=\tau(\omega)\nu_\top \quad , \quad \forall\omega\in\Omega(\mathcal{D}) \quad .$$

Non-redundand parametrization and the G_2 structure

Proposition. The Fierz identities (3) are equivalent with the following conditions:

$$V^2 = 1 - b^2 \quad , \quad Y = (1 + b\nu)\psi \quad , \quad Z = V\psi \quad ,$$
$$(\iota_\alpha\varphi) \wedge (\iota_\beta\varphi) \wedge \varphi = -6\langle\alpha, \beta\rangle\nu_\top \quad , \quad \forall\alpha, \beta \in \Omega^1(\mathcal{D}) \quad ,$$

where $\psi \in \Omega^4(\mathcal{D})$ is the canonically normalized coassociative form of a G_2 structure on $\mathcal{D}$ compatible with the metric $g|_\mathcal{D}$ induced by g and with the orientation of $\mathcal{D}$, while $\varphi \stackrel{\text{def.}}{=} *_\perp\psi \in \Omega^3(\mathcal{D})$ is the associative form of the G_2 structure.

From now on we shall use a new parametrization, in terms of b, V, ψ, which is non-redundant:

$$\boxed{\check{E} = \frac{1}{16}(1 + V + b\nu)(1 + \psi) = P\Pi} \quad ,$$

$$P \stackrel{\text{def.}}{=} \frac{1}{2}(1 + V + b\nu) \quad \text{and} \quad \Pi \stackrel{\text{def.}}{=} \frac{1}{8}(1 + \psi)$$

are commuting idempotents in the Kähler-Atiyah algebra.

Parametrization of the 4-form fluxes

Since any form can be decomposed into parallel and orthogonal parts to any one-form, we have:

$$F = F_\perp + \hat{V} \wedge F_\top \quad , \quad f = f_\perp + f_\top \hat{V}$$

with components $F_\perp, F_\top, f_\perp, f_\top \in \Omega_7(M, \mathcal{D})$ living on the 7-dim. distribution.

The G_2 structure gives decompositions:

$$F_\perp = F_\perp^{(1)} + F_\perp^{(7)} + F_\perp^{(27)} \equiv F_\perp^{(7)} + F_\perp^{(\mathcal{S})} \quad \in \Omega^4(M, \mathcal{D})$$
$$F_\top = F_\top^{(1)} + F_\top^{(7)} + F_\top^{(27)} \equiv F_\top^{(7)} + F_\top^{(\mathcal{S})} \quad \in \Omega^3(M, \mathcal{D}) \quad , \quad \mathcal{D} = T\mathcal{F}$$

with the parametrization:

$$\boxed{F_\perp^{(7)} = \alpha_1 \wedge \varphi \quad , \quad F_\perp^{(S)} = -\hat{h}_{kl} e^k \wedge \iota_{e^l} \psi = -\frac{4}{7} \mathrm{tr}_g(\hat{h}) \psi - h_{kl}^{(0)} e^k \wedge \iota_{e^l} \psi}$$

$$\boxed{F_\top^{(7)} = -\iota_{\alpha_2} \psi \quad , \quad F_\top^{(S)} = \chi_{kl} e^k \wedge \iota_{e^l} \varphi = \frac{3}{7} \mathrm{tr}_g(\chi) \varphi + \chi_{kl}^{(0)} e^k \wedge \iota_{e^l} \varphi}$$

$\alpha_1, \alpha_2 \in \Omega^1(M, \mathcal{D})$ and $\hat{h}, \chi$ are symmetric tensors.

Solving the supersymmetry conditions

Theorem 1. Let $||V|| = \sqrt{1-b^2}$. Then the $\check{Q}$-constraints ($Q\xi = 0 \Longleftrightarrow \check{Q}\check{E} = 0$) are *equivalent* with the following relations, which determine (in terms of Δ, b, ψ and f) the components of $F^{(1)}_{\top}$, $F^{(1)}_{\perp}$ and $F^{(7)}_{\top}$, $F^{(7)}_{\perp}$:

$$\boxed{\begin{aligned} \alpha_1 &= \frac{1}{2||V||}(f - 3b\mathrm{d}\Delta)_{\perp}~, \\ \alpha_2 &= -\frac{1}{2||V||}(bf - 3\mathrm{d}\Delta)_{\perp}~, \\ \mathrm{tr}_g(\hat{h}) &= -\frac{3}{4}\mathrm{tr}_g(h) = \frac{1}{2||V||}(bf - 3\mathrm{d}\Delta)_{\top}~, \\ \mathrm{tr}_g(\hat{\chi}) &= -\frac{3}{4}\mathrm{tr}_g(\chi) = 3\kappa - \frac{1}{2||V||}(f - 3b\mathrm{d}\Delta)_{\top}~. \end{aligned}} \tag{4}$$

Notice that the $\check{Q}$-constraints do *not* determine the components $F^{(27)}_{\top}$ and $F^{(27)}_{\perp}$.

$$F^{(27)}_{\perp} = -h^{(0)}_{kl} e^k \wedge \iota_{e^l}\psi \quad , \quad F^{(27)}_{\top} = -\chi^{(0)}_{kl} e^k \wedge \iota_{e^l}\varphi$$

Solving the supersymmetry conditions

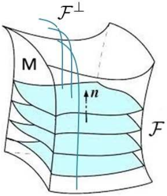
The differential constraints ($D_m\xi = 0 \Longleftrightarrow \nabla_m \check{E} = -[\check{A}_m, \check{E}]_-$) imply (using also the results of the algebraic constraints), among many other relations :

$$\mathrm{d}V = 3V \wedge (\mathrm{d}\Delta)_\perp$$

Since M is compact and connected and V is nowhere vanishing, it follows that $\mathcal{D}$ is Frobenius integrable and hence it determines a codimension one foliation $\mathcal{F}$ of M, $\mathcal{D} = T\mathcal{F}$.

Thus, the G_2 structure of $\mathcal{D}$ becomes a leafwise G_2 structure on $\mathcal{F}$.

All leaves of $\mathcal{F}$ are diffeomorphic with each other. The complementary distribution $\mathcal{D}^\perp$ determines a foliation $\mathcal{F}^\perp$, $\mathcal{D}^\perp = T\mathcal{F}^\perp$, whose leaves are integral curves of $n = \hat{V}^\sharp$.



Intrinsic and extrinsic geometry of the foliation

The **fundamental equations** of the foliation are given by:

$$\begin{aligned}
&\nabla_n n = H \quad (\perp n)\ ,\\
&\nabla_{X_\perp} n = -AX_\perp \quad (\perp n)\ ,\\
&\nabla_n(X_\perp) = -g(H, X_\perp)n + D_n(X_\perp)\ ,\\
&\nabla_{X_\perp}(Y_\perp) = \nabla^\perp_{X_\perp}(Y_\perp) + g(AX_\perp, Y_\perp)n\ .
\end{aligned}$$

Also:

$$D_n\varphi = 3\iota_\vartheta\psi \quad , \quad D_n\psi = -3\vartheta\wedge\varphi \ , \ \vartheta\in\Omega^1(\mathcal{D})\ . \tag{5}$$

The **torsion forms** $\tau_k \in \Omega^k(M, \mathcal{D})$ of the longitudinal G_2 structure are uniquely determined by the definitions:

$$\boxed{\mathrm{d}_\perp\psi = 4\tau_1\wedge\psi + *_\perp\tau_2\ , \quad \mathrm{d}_\perp\varphi = \tau_0\psi + 3\tau_1\wedge\varphi + *_\perp\tau_3}$$

Theorem 2. For $||V||=\sqrt{1-b^2}$, **the supersymmetry constraints are equivalent with** the conditions:

1. The function $b\in\mathcal{C}^\infty(M,(-1,1))$ satisfies:

$$\boxed{e^{-3\Delta}\mathrm{d}(e^{3\Delta}b)=f-4\kappa\sqrt{1-b^2}\hat{V}} \tag{6}$$

2. The fundamental tensors H and A of $\mathcal{F}^\perp$ and $\mathcal{F}$ are given by expressions in terms of b,Δ,f,F:

$$\boxed{\begin{aligned}H_\sharp &= -\frac{1}{||V||^2}(bf_\perp-3(\mathrm{d}\Delta)_\perp)\ ,\\ AX_\perp &= \frac{1}{||V||}\Big[(b\chi^{(0)}_{ij}-h^{(0)}_{ij})X^j_\perp e^i+\frac{1}{7}(14\kappa b-8\mathrm{tr}_g(\hat{h})-6b\ \mathrm{tr}_g(\hat{\chi}))X_\perp\Big]\end{aligned}} \tag{7}$$

3. The one-form $\vartheta\in\Omega(\mathcal{D})$ is given by the following relation in terms of Δ, b and f:

$$\boxed{\vartheta=\frac{1}{6||V||^2}\left[-(1+b^2)f_\perp+6b(\mathrm{d}\Delta)_\perp\right]} \tag{8}$$

4. The torsion classes of the leafwise G_2 structure are given by expressions in terms of b,Δ,f,F:

$$\boxed{\begin{aligned}&\boldsymbol{\tau}_0=\frac{4}{7||V||}\left[4\kappa+\frac{(1+b^2)f_\top-6b(\mathrm{d}\Delta)_\top}{2||V||}\right]\ ,\quad \boldsymbol{\tau}_1=-\frac{3}{2}(\mathrm{d}\Delta)_\perp\ ,\quad \boldsymbol{\tau}_2=0\ ,\\ &\boldsymbol{\tau}_3=\frac{1}{||V||}(F^{(27)}_\top-b*_\perp F^{(27)}_\perp)\ .\end{aligned}} \tag{9}$$

Eliminating the fluxes

Theorem 3. The following statements are equivalent:

(A) $\exists\, f \in \Omega^1(M)$ and $F \in \Omega^4(M)$ such that the susy equations admit at least one non-trivial solution ξ which is everywhere non-chiral (and which we can take to be everywhere of norm one).

(B) $\exists\, \Delta \in \mathcal{C}^\infty(M,\mathbb{R})$, $b \in \mathcal{C}^\infty(M,(-1,1))$, $\hat{V} \in \Omega^1(M)$ and $\varphi \in \Omega^3(M)$ such that:

1. these conditions are satisfied:
$$||\hat{V}|| = 1 \ \ , \ \ \iota_{\hat{V}}\varphi = 0 \ \ . \tag{10}$$
The Frobenius distribution $\mathcal{D} \stackrel{\mathrm{def.}}{=} \ker \hat{V}$ is integrable and we let $\mathcal{F}$ be the foliation which integrates it.
2. The quantities H, $\mathrm{tr}A$ and ϑ of the foliation $\mathcal{F}$ are given by:
$$\boxed{\begin{aligned} H_\sharp &= -\frac{b}{1-b^2}(\mathrm{d}b)_\perp + 3(\mathrm{d}\Delta)_\perp \ , \\ \mathrm{tr}A &= 12(\mathrm{d}\Delta)_\top - \frac{b(\mathrm{d}b)_\top}{1-b^2} - 8\kappa\frac{b}{\sqrt{1-b^2}} \ , \\ \vartheta &= -\frac{1+b^2}{6(1-b^2)}(\mathrm{d}b)_\perp + \frac{b}{2}(\mathrm{d}\Delta)_\perp \ \ . \end{aligned}} \tag{11}$$
3. φ induces a leafwise G_2 structure on $\mathcal{F}$ whose torsion classes satisfy:
$$\boxed{\begin{aligned} \boldsymbol{\tau}_0 &= \frac{4}{7}\left[\frac{2\kappa(3+b^2)}{\sqrt{1-b^2}} - \frac{3b}{2}(\mathrm{d}\Delta)_\top + \frac{1+b^2}{2(1-b^2)}(\mathrm{d}b)_\top\right] \ , \\ \boldsymbol{\tau}_1 &= -\frac{3}{2}(\mathrm{d}\Delta)_\perp \ , \quad \boldsymbol{\tau}_2 = 0 \ \ . \end{aligned}} \tag{12}$$

The explicit solution for the fluxes

Thus F and f are uniquely determined by b, Δ, V and φ (or ψ):

$$\boxed{f = 4\kappa V + e^{-3\Delta}\mathrm{d}(e^{3\Delta}b)} \ ,$$

(a) $F_\perp^{(1)} = -\frac{4}{7}\mathrm{tr}_g(\hat{h})\psi \ , \quad F_\top^{(1)} = \frac{3}{7}\mathrm{tr}_g(\chi)\varphi = -\frac{4}{7}\mathrm{tr}_g(\hat{\chi})\varphi$ with:

$$\boxed{\mathrm{tr}_g(\hat{h}) = -\frac{3||V||}{2}(\mathrm{d}\Delta)_\top + 2\kappa b + \frac{b}{2||V||}(\mathrm{d}b)_\top \ , \quad \mathrm{tr}_g(\hat{\chi}) = \kappa - \frac{1}{2||V||}(\mathrm{d}b)_\top}$$

(b) $F_\perp^{(7)} = \alpha_1 \wedge \varphi \ , \quad F_\top^{(7)} = -\iota_{\alpha_2}\psi$ with:

$$\boxed{\alpha_1 = \frac{1}{2||V||}(\mathrm{d}b)_\perp \ , \quad \alpha_2 = -\frac{b}{2||V||}(\mathrm{d}b)_\perp + \frac{3||V||}{2}(\mathrm{d}\Delta)_\perp}$$

(c) $F_\perp^{(27)} = -h_{kl}^{(0)} e^k \wedge \iota_{e^l}\psi \ , \quad F_\top^{(27)} = \chi_{kl}^{(0)} e^k \wedge \iota_{e^l}\varphi \ ,$ with:

$$\boxed{\begin{aligned} h_{ij}^{(0)} &= -\frac{b}{4||V||}[\langle e_i \lrcorner \varphi, e_j \lrcorner \boldsymbol{\tau}_3\rangle + (i \leftrightarrow j)] - \frac{1}{||V||}A_{ij}^{(0)} \ , \\ \chi_{ij}^{(0)} &= -\frac{1}{4||V||}[\langle e_i \lrcorner \varphi, e_j \lrcorner \boldsymbol{\tau}_3\rangle + (i \leftrightarrow j)] - \frac{b}{||V||}A_{ij}^{(0)} \ , \end{aligned}}$$

where $||V|| = \sqrt{1-b^2}$ and $A^{(0)}$ is the traceless part of the Weingarten tensor of $\mathcal{F}$.

Topology of $\mathcal{F}$ in the everywhere non-chiral case

Having obtained, from the supersymmetry conditions, that:

$$\mathrm{d}\boldsymbol{\omega} = 0 \quad , \quad \mathrm{d}\mathbf{f} = 0 \quad , \quad \boldsymbol{\omega} = \mathbf{f} - \mathrm{d}\mathbf{b} \quad (\boldsymbol{\omega} = 4\kappa e^{3\Delta} V \ , \ \ \mathbf{f} = e^{3\Delta} f \ , \ \ \mathbf{b} = e^{3\Delta} b)$$

$\boldsymbol{\omega}$ must belong to the cohomology class of $\mathbf{f}$, $\mathfrak{f} \in H^1(M, \mathbb{R})$, which cannot be zero since V (and thus $\boldsymbol{\omega}$) are nowhere-vanishing here, thus the first Betti number must be positive, $b^1(M) > 0$, which implies that the first homotopy group $\Pi_1(M)$ is non-trivial.

Integration of any element of $\mathfrak{f}$ over closed paths provides a group morphism from the first homotopy group to the additive group of $\mathbb{R}$:

$$\mathrm{per}_{\mathfrak{f}} : \Pi_1(M) \to \mathbb{R} \quad .$$

The character of the foliation depends on the rank $\rho(\mathfrak{f})$ of the period group $\mathrm{img}(\mathrm{per}_{\mathfrak{f}})$ called the **irrationality rank** of $\mathfrak{f}$.

- When $\rho(\mathfrak{f}) = 1$, we say that $\boldsymbol{\omega}$ is **projectively rational** (all periods of $\boldsymbol{\omega}$ can be commonly rescaled to integers). The leaves of $\mathcal{F}$ are compact and coincide with the fibers of a fibration $\mathfrak{h} : M \to S^1$.
- When $\rho(\mathfrak{f}) > 1$, $\boldsymbol{\omega}$ is called **projectively irrational** and each leaf of $\mathcal{F}$ is non-compact and dense in M. Hence $\mathcal{F}$ cannot be a fibration. The case when $\mathcal{F}$ is not a fibration might also arise as a consistent background in M-theory.

Non-commutative geometry of the foliation

In the projectively irrational case, one can consider the C^* algebra $C(M/\mathcal{F})$ of the foliation, which encodes the 'noncommutative topology' of its leaf space, being a noncommutative torus of dimension equal to the irrationality rank.

Let $\Pi_\mathfrak{f} \approx \mathbb{Z}^\rho$ be the group of periods of $\mathfrak{f}$. Then $C(M/\mathcal{F})$ is separable and Morita equivalent with the crossed product algebra $C(\mathbb{R}) \rtimes \Pi_\mathfrak{f}$, which is isomorphic with $C(S^1)$ when $\rho = 1$ and with a ρ-dimensional noncommutative torus when $\rho > 1$.

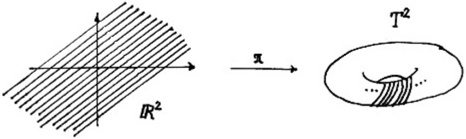


Figure: The linear foliations of T^2 model the noncommutative geometry of the leaf space of $\mathcal{F}$ in the case $\rho(\mathfrak{f}) \leq 2$.

The not everywhere non-chiral case

When ξ is allowed to become chiral on some locus $\mathcal{W}=\mathcal{W}^+\cup\mathcal{W}^-\subsetneq M$, $\mathcal{W}$ must be a set with empty interior, which is therefore negligible with respect to the Lebesgue measure of the internal space M. Thus, the behavior of geometric data along this locus can be obtained from the non-chiral locus $\mathcal{U}\stackrel{\mathrm{def.}}{=}M\setminus\mathcal{W}$ through a limiting process.

When $\emptyset\neq\mathcal{W}\subsetneq M$, the regular foliation $\mathcal{F}$ extends to a singular foliation $\bar{\mathcal{F}}$ of the whole manifold M by adding leaves which have singularities at points belonging to $\mathcal{W}$. This singular foliation $\bar{\mathcal{F}}$ "integrates" the kernel distribution $\mathcal{D}$ of a closed one-form $\boldsymbol{\omega}$, which now can vanish at some points.

$\bar{\mathcal{F}}$ carries a **longitudinal G_2 structure** which degenerates at the singular points.

Topology of the singular foliation – the foliation graph

The topology of singular foliations defined by a closed one-form can be extremely complicated in general. The situation is better understood in the case when $\boldsymbol{\omega}$ is a **Morse one-form**. The Morse case is generic, i.e. the Morse 1-forms constitute an open and dense subset of the closed one-forms belonging to the cohomology class $\mathfrak{f}$.

In the Morse case, the singular foliation $\bar{\mathcal{F}}$ can be described using the **foliation graph**, which provides a combinatorial way to encode some important aspects of the foliation's topology — up to neglecting the information contained in the so-called *minimal components* of the decomposition, components which should possess a non-commutative geometric description.

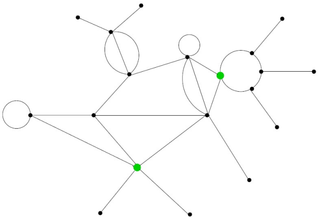


Figure: Example of a foliation graph

The foliation graph for the regular foliation

In the everywhere non-chiral case $\mathcal{U}=M$, the foliation graph is reduced to either a **circle** (when $\mathcal{F}$ has compact leaves, being a fibration over S^1) or to an **exceptional vertex** (when $\mathcal{F}$ has non-compact dense leaves, being a minimal foliation). The exceptional vertex corresponds to a noncommutative torus which encodes the noncommutative geometry of the leaf space.

(a) Foliation graph when $\mathcal{W}=\emptyset$ and $\rho(\omega)=1$.   (b) Foliation graph when $\mathcal{W}=\emptyset$ and $\rho(\omega)>1$.

Figure: The foliation graph for the $\mathcal{N}=1$ everywhere non-chiral case, i.e. when $\mathcal{U}=M$

Further directions – new insights into $\mathcal{N}=2$ case

Using the 2 Majorana spinors ξ_1, ξ_2 one can construct :

$$b_1 = \mathscr{B}(\xi_1, \gamma(\nu)\xi_1) \ , \ b_2 = \mathscr{B}(\xi_2, \gamma(\nu)\xi_2) \ , \ b_3 = \mathscr{B}(\xi_1, \gamma(\nu)\xi_2) \ ,$$

$$V_1 = \mathscr{B}(\xi_1, \gamma_a\xi_1)e^a \ , \ V_2 = \mathscr{B}(\xi_2, \gamma_a\xi_2)e^a \ , \ V_3 \overset{\mathrm{def.}}{=} \mathscr{B}(\xi_1, \gamma_a\xi_2)e^a \ ,$$

$$W \overset{\mathrm{def.}}{=}_U \mathscr{B}(\xi_1, \gamma_a\gamma(\nu)\xi_2)e^a \ ,$$

plus many higher order forms.

We use in this case the theory of **semialgebraic sets** with **Whitney stratifications**. In this case we have 2 distributions:

$$\mathcal{D} \overset{\mathrm{def.}}{=} \ker V_1 \cap \ker V_2 \cap \ker V_3 = \ker V_+ \cap \ker V_- \cap \ker V_3 \ ,$$

$$\mathcal{D}_0 \overset{\mathrm{def.}}{=} \ker V_+ \cap \ker V_- \cap \ker V_3 \cap \ker W \ \subset \mathcal{D} \ , \quad V_\pm = \frac{1}{2}(V_1 \pm V_2)$$

and three types of stratifications (which do not coincide as in the $\mathcal{N}=1$ case):

- chirality stratification
- stabilizer stratification
- rank stratification

We have 2 semialgebraic sets represented as the body $\mathcal{R}$ and the body $\mathfrak{P}$

$$\mathcal{R} = \{(b_+, b_-, b_3) \in [-1,1]^3 \mid \sqrt{b_-^2 + b_3^2} \stackrel{\text{def.}}{=} \rho \leq 1 - |b_+|\} \quad , \quad b_\pm = \frac{1}{2}(b_1 \pm b_2)$$

$$\mathfrak{P} \stackrel{\text{def.}}{=} \{(b, \beta) \in \mathbb{R}^4 | b \in \mathcal{R} \ \& \ \beta \stackrel{\text{def.}}{=} \sqrt{b_3^2 + ||V_3||^2} = \sqrt{b_-^2 + ||V_-||^2} \in [\rho, \sqrt{1 - b_+^2}]\}$$

$$b \stackrel{\text{def.}}{=} \{b_+, b_-, b_3\}$$

	$\mathcal{R}$-description	$r_-(p)$	$r_+(p)$	b_+	ρ	H_p	$\sigma_+(p)$	$\sigma_-(p)$
$\mathcal{S}_{02}$	$b^{-1}(\partial_0^+\mathcal{R})$	0	2	$+1$	0	$SU(4)$	2	0
$\mathcal{S}_{20}$	$b^{-1}(\partial_0^-\mathcal{R})$	2	0	-1	0	$SU(4)$	0	2
$\mathcal{S}_{11}$	$b^{-1}(\partial D)$	1	1	0	1	G_2	1	1
$\mathcal{S}_{12}$	$b^{-1}(\partial_2^+\mathcal{R})$	1	2	$1-\rho$	$(0,1)$	$SU(3)$	1	0
$\mathcal{S}_{21}$	$b^{-1}(\partial_2^-\mathcal{R})$	2	1	$-(1-\rho)$	$(0,1)$	$SU(3)$	0	1
$\mathcal{G}$	$b^{-1}(Int\mathcal{R})$	2	2	$(-1,1)$	$< 1 - \lvert b_+\rvert$	$SU(2)$ or $SU(3)$	0	0

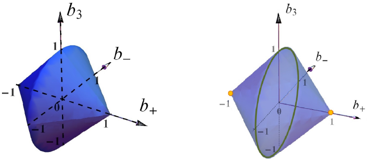


Figure: The body R

References

- [1]. E.M. Babalic, C.I. Lazaroiu, *Foliated 8-manifolds for M-theory compactifications*, JHEP 01 (2015) 140 , (60 pg);
- [2]. E.M. Babalic, C.I. Lazaroiu, *Singular foliations for M-theory compactifications*, JHEP 03 (2015) 116 , (63 pg);
- [3]. E.M. Babalic, C.I. Lazaroiu, *The landscape of G-structures in eight-manifold compatifications of M-theory*, preprint;

- [4]. D. Martelli, J. Sparks, *G-structures, fluxes and calibrations in M-theory*, Phys. Rev. D 68 (2003) 085–014.
- [5]. D. Tsimpis, *M-theory on eight-manifolds revisited: N = 1 supersymmetry and generalized* $\mathrm{Spin}(7)$ *structures*, JHEP 0604 (2006) 027.