Integrable Systems: Past-Present-Future
Integrability of dynamical systems and rational elliptic surfaces

Adrian-Stefan Carstea

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Research Group on Geometry and Physics
Department of Theoretical Physics
IFIN-HH

- A. S. Carstea, T. Takenawa *"Classification of discrete integrable mappings and rational elliptic surface (I)"*, [J. Phys. A: Math. Theor. 45, 155206, (2012)]
- A. S. Carstea, *"On the geometry of Q4 mapping"*, [to appear in Contemporary Mathematics]
- A. S. Carstea, T. Takenawa, *"A note on minimization of rational elliptic surfaces from birational dynamics"*, [to appear in Journal of Nonlinear Mathematical Physics - special issue Geometry of discrete equations]
- C. Babalic, A. S. Carstea, *"On two integrable discretisations of generalised Volterra system"* [J. Phys. A: Math. Theor. 46, 145205, (2013)]
- C. Babalic, A. S. Carstea, *"On some new forms of lattice integrable equations"* submitted to [J. Math. Phys.]

Papers in work:

- A. S. Carstea, C. Babalic *"Lax pairs of higher order lattice soliton equations"*

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and the "butterfly effect" is seen from::

$$\frac{dy_n}{dc_0} = \frac{1}{2} 2^n \cos(2^n c_0) \tag{2}$$

So the equation is solvable but chaotic (more precisely is in the ergodic region)

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$$w' + w^2 = 0, \quad w = (z - c_0)^{-1}$$

$$2w' + w^3 = 0, \quad w = (z - c_0)^{-1/2}$$

$$ww'' - w' + 1 = 0, \quad w = (z - c_0)\log(z - c_0) + \alpha(z - c_0)$$

$$\sqrt{3}ww'' - (1 - \sqrt{3})w'^2 = 0, \quad w = \alpha(z - c_0)^{\sqrt{3}}$$

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Integrability is not compatible with any kind of branching proliferation or essential singularities (Painleve property)

Historical evolution can be roughly divided into some periods:

- 1970-1985: Soliton equations - developing of IST, bilinear formalism, bi-hamiltonian structure, culminating with Jimbo-Miwa theory of integrable hierarchies.
- 1980-1985: Quantum IST - Faddev group, discovery of quantum groups (Jimbo-Drinfeld)
- 1990-2002: Discrete mappings(nonlinear ordinary difference equations) : developing of main tools for discrete integrability, culminating with Sakai classification of discrete Painleve equations
- 2002-present: Discrete geometry, tropical geometry, ultradiscrete equations, new views on lattice soliton equations

They triggered the development of other fundamental results in String Theory: WDVV equations and TFT as integrable systems (Dubrovin 1993), KdV-hierarchy describing the intersection numbers on the moduli stack of algebraic curves (Witten-Kontsevich 1990), Seiberg-Witten theory etc.

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Eliminating y, z one gets Painleve III equation (solvable by inverse monodromy method)

$$x''x - x'^2 + x^4/4 - Ke^{-4t/3} = 0 \Longleftrightarrow X'' = \frac{X'^2}{X} - \frac{X'}{T} + X^3 - \frac{1}{X} \qquad (3)$$

by means of new variables:

$$x(t) = \frac{2ic}{3}e^{-t/3}X(T), T = ce^{-t/3}, c^4 - (3K)^4 = 0$$

2. PDE case - computing multisoliton solution: Celebrated Korteweg de Vries equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4}$$

Substitution:

$$u(x,t) = 2\partial_x^2 \log \tau(x,t)$$

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Existence of general N-soliton solution equivalent with integrability:

$$\tau(x,t) = \sum_{\{\mu_1,\ldots\mu_N\}\in\{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i(k_i x - k_i^3 t) + \sum_{i<j} A_{ij}(k_i,k_j)\mu_i\mu_j\right)$$

Deep thing - Virasoro structure

$$D_x e^{nx} \bullet e^{mx} = (m-n)e^{(m+n)x}$$

3. Inverse Spectral Method - the most powerfull; gives both integrals and solutions

$$L = \partial_x^2 + u(x,t), \quad B = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x \tag{5}$$

$$\partial_t L - [L, B] = u_t + 6uu_x + u_{xxx} = 0 \tag{6}$$

It gives integrals:

$$I_n = Tr(L^n), \forall n$$

And the hierarchy:

$$u_{t_n} = \partial_x \frac{\delta I_n}{\delta u}$$

Example: KdV-equation:

$$u_{t_3} + 6uu_x + u_{xxx} = 0$$

Lax-5 equation:

$$u_{t_5} + (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x = 0$$

...

Lax-5 has the following Lax pair:

$$L = \partial_x^2 + u$$

$$B = 16\partial_x^5 + 40u\partial_x^3 + 60u_x\partial_x^2 + (50u_{xx} + 30u^2)\partial_x + 15u_{xxx} + 30uu_x$$

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- Perhaps more fundamental than continuous equations
- Easier to implement numerically
- can simulate many continuous system in the same time (i.e. a discrete equations can have many continuum limits

Example discrete KdV:

$$x_{n+1,m+1} - x_{n,m} = \alpha \left(\frac{1}{x_{n+1,m}} - \frac{1}{x_{n,m+1}} \right)$$

Continuum limit:

$$\xi = \epsilon(n-m), \tau = \epsilon^3 m, x_{n,m} = 1 + \epsilon^5 u(\xi, \tau)$$

When ϵ goes to zero the discrete KdV goes to

$$u_\tau + auu_\xi + bu_{\xi\xi\xi} = 0$$

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- multi-soliton solution
- hard/impossible to define symmetry and hamiltonian structure
- singularity analysis (algebraic geometry techniques)
- complexity growth, algebraic entropy
- polyhedral consistency

Main results:

- Construction of discrete Painleve equations (Ramani-Grammaticos)
- Sakai classification - all integrable 2D nonlinear nonautonomous ordinary difference as automorphisms of rational surfaces obtained by blowing up projective plane in 9 points
- Adler-Bobenko-Suris classification of lattice soliton equations

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How to simplify a discrete equation such that to be almost linear but to retain nonlinear dynamical behaviour? For this what is the simplest nonlinear function?
Lattice mKdV:

$$x_{n+1,m+1} = x_{n,m}\frac{x_{n,m+1} + ax_{n+1,m}}{ax_{n,m+1} + x_{n+1,m}}$$

Use this:

$$\lim_{\epsilon\to 0} \epsilon \log(e^{X_1/\epsilon} + e^{X_2/\epsilon} + \ldots + e^{X_k/\epsilon}) = \max(X_1, \ldots, X_k)$$

Consider:

$$x_{n,m} = \exp(X_{n,m}/\epsilon), a = \exp(A/\epsilon)$$

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- $x_{n,m} = \exp(X_{n,m}/\epsilon) \Leftrightarrow \psi = A\exp(\frac{i}{\hbar} S)$
- what is integrability here? (equations are piecewise linear, and may have new solutions with no discrete correspondent

Future directions:

- higher order Painleve equations
- higher order discrete mappings (extension of Sakai classification)
- ultradiscrete integrability - arithmetic integrability
- supersymmetric integrability
- noncommutative integrability (open-closed TFT, noncommutative WDVV etc.)
- ....