Geometric algebra techniques in supersymmetric flux compactifications of supergravity theories - generalities and examples

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Outline

Preprints submitted to JHEP:

- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"Geometric algebra techniques in flux compactifications (I)"*, [arXiv:1212.6766 [hep-th]]
- C. I. Lazaroiu, E. M. Babalic, *"Geometric algebra techniques in flux compactifications (II)"*, [arXiv:1212.6918 [hep-th]]
- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"The geometric algebra of Fierz identities in arbitrary dimensions and signatures"*, [arXiv:1304.4403 [hep-th]]

Preprints submitted as conference proceedings:

- E. M. Babalic, C. I. Lazaroiu, *"Revisiting eight-manifold flux compactifications of M-theory using geometric algebra techniques"*, [arXiv:1301.5106 [hep-th]] (QFTHS 2012, Craiova)
- C. I. Lazaroiu, E. M. Babalic, *"Geometric algebra and M-theory compactifications"*, [arXiv:1301.5094 [hep-th]] (QFTHS 2012, Craiova)
- E. M. Babalic, I. A. Coman, C. I. Lazaroiu, *"A unified approach to Fierz identities"*, [arXiv:1303.1575 [hep-th]] (TIM-12)
- E. M. Babalic, I. A. Coman, C. Condeescu, C. I. Lazaroiu, A. Micu, *"On $\mathcal{N}=2$ compactifications of M-theory to AdS_3 using geometric algebra techniques"*, (TIM-12)
- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"The geometric algebra of supersymmetric backgrounds"*, (String-Math 2012)

Papers in work:

- E.M. Babalic, I.A. Coman, C. Condeescu, C. I. Lazaroiu, A. Micu, *"The general $N=2$ warped compactifications of M-theory to three dimensions"*
- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"The geometric algebra approach to Fierz identities for various normal cases"*

Motivation

- Our first intention was to find a generalization of F-theory, which can be interpreted as a decompactification limit of M-theory.
- For this we analysed $\mathcal{N} = 2$ warped flux compactifications of M-theory on an 8-manifold down to AdS_3 spaces, a case which was not studied in its full generality before (eliminating the Weyl condition on the spinors, which is not necessary, but was used in the literature just for simplifying calculations)
- The computations proved to be very difficult if performed in the standard way.
- We propose a new approach, using **geometric algebra**.
- This approach leads to beautiful sistematizations and very effective computations which **can be implemented in symbolic computation systems**.
- **We created a code** in Ricci-Mathematica that generalizes such computations.

Flux compactificatons - compactifications with fluxes on the internal background.

$$\tilde{M} = M_{int} \times M_{ext}$$

For any SUGRA on $\tilde{M}$ the supersymmetry conditions lead to the following constraints on sections ξ of spin $\frac{1}{2}$ (of the spin bundle S_{int} associated to the internal manifold M_{int} of dimension d), called CGK spinor equations:

$$D_m\xi = 0 \quad , \quad D_m = \nabla^S_m + A_m \quad , \quad \nabla^S_m = \partial_m + \omega_{m|np}\gamma^{np} \quad , \quad m, n, p = \overline{1, d}$$
$$Q_1\xi = \ldots = Q_\chi\xi = 0 \quad , \quad \forall\xi$$

A_m and Q_k are algebraic combinations of gamma matrices with coefficients which depend on metric and fluxes.

These **CGK**(constrained generalized Killing) **spinor equations** are equivalent with:

$$\partial_m \mathscr{B}(\xi_i, \gamma^A\xi_j) = \mathscr{B}(\xi_i, [D_m, \gamma^A]_{-,\circ}\xi_j) \ , \ \forall A = m_1\ldots m_p, \ \forall p = \overline{1, d}$$
$$\mathscr{B}(\xi_i, (\gamma^A \circ Q_k - Q_k^t \circ \gamma^A)\xi_j) = 0 \quad , \quad \forall i, j, k$$

Notations and conventions:
$(e_m)_{m=\overline{1,d}}$=local frame of (M_{int}, g), with dual local coframe (e^m), $e_m = g_{mn}e^n$,
$e^{m_1} \wedge \ldots \wedge e^{m_k} = e^{m_1\cdots m_k}$, $\gamma : \Lambda T^* M_{int} \longrightarrow \mathrm{End}(S_{int})$, $\gamma^m = \gamma(e^m)$, $\gamma^{m_1\cdots m_k} = \gamma^{m_1} \circ \ldots \circ \gamma^{m_k}$,
γ^A can be any ordered gamma matrices $\gamma^{a_1\cdots a_k}$, $\forall k = \overline{1, d}$

- A central problem in the study of flux compactifications of supergravity and string theories is finding geometric descriptions of supersymmetry conditions for various backgrounds in the presence of fluxes.

- Another issue – analyzing the Fierz identities (between bilinears in spinors) in curved backgrounds for various dimensions and signatures is a complicated task in supergravity and string theories. But the very construction of many such theories relies in crucial ways on such identities.

$\mathcal{N}=1$ warped compactifications of 11-dim SUGRA on 8-manifolds to AdS_3 spaces

$$\tilde{M} = M_8 \times M_3$$

The fields of 11-dim SUGRA: $\tilde{g}$, $\tilde{C} \in \Omega^3(\tilde{M})$, $\tilde{\Psi}_A$.

Fluxes: $\tilde{G} = d\tilde{C} \in \Omega^4(\tilde{M})$

11-dim SUGRA has $\mathcal{N}=1$ SUSY, thus it has one SUSY generator $\tilde{\eta}$ (a Majorana spinor of spin $\frac{1}{2}$).

Warp compactifications ansatz (with warp factor Δ):

$$d\tilde{s}_{11}^2 = e^{2\Delta} ds_{11}^2 \quad , \quad ds_{11}^2 = ds_3^2 + g_{mn} dx^m dx^n \quad , \quad m,n = 1,...,8$$
$$\tilde{\eta} = e^{\frac{\Delta}{2}} \eta \quad , \quad \eta = \xi \otimes \psi \quad , \quad \xi \in \Gamma(M_8, S_8) \ , \ \psi \in \Gamma(M_3, S_3)$$
$$\tilde{G} = e^{3\Delta} G \quad , \quad G = vol_3 \wedge f + F \quad , \quad f \in \Omega^1(M_8) \ , \ F \in \Omega^4(M_8)$$

SUSY condition: $\delta_{\tilde{\eta}} \tilde{\Psi}_A = \tilde{D}_A \tilde{\eta} = 0 \quad , \quad A = 0, ..., 10$
$\tilde{D}_A$ is the supercovariant connection, well known in the literature.

SUSY condition $\tilde{D}_A \tilde{\eta} = 0$ implies the following CGK spinor equations for ξ (since we chose $\mathcal{N} = 1$ on M_8 there is only one independent ξ):

$$D_m \xi = 0 \ , \ D_m = \nabla_m^S + A_m \ , \ A_m = \frac{1}{4} f_p \gamma_m{}^p \gamma^{(9)} + \frac{1}{24} F_{mpqr} \gamma^{pqr} + \kappa \gamma_m \gamma^{(9)} \quad (1)$$

$$Q\xi = 0 \quad , \quad Q = \frac{1}{2} \gamma^m \partial_m \Delta - \frac{1}{288} F_{mpqr} \gamma^{mpqr} - \frac{1}{6} f_p \gamma^p \gamma^{(9)} - \kappa \gamma^{(9)} \quad (2)$$

where $\gamma^{(9)} = \gamma^1 \circ ... \circ \gamma^8$ and κ is a positive real parameter proportional to the square root of minus the cosmological constant.

Relations (1) and (2) are equivalent to the system:

$$\partial_m \mathscr{B}(\xi, \gamma^{m_1 \cdots m_k} \xi) = \mathscr{B}(\xi, [D_m, \gamma^{m_1 \cdots m_k}]_{-,\circ} \xi) \ , \ \forall k = 1...8$$

$$\mathscr{B}(\xi, (\gamma^{m_1 \cdots m_k} \circ Q - Q^t \circ \gamma^{m_1 \cdots m_k}) \xi) = 0 \quad , \quad \forall k = 1...8$$

Due to the symmetry properties of the gamma matrices, one can built the following spinor bilinears:

$$K_m = \mathscr{B}(\xi, \gamma_m \xi) \quad , \quad Y_{m_1 \dots m_4} = \mathscr{B}(\xi, \gamma_{m_1 \dots m_4} \xi)$$
$$Z_{m_1 \dots m_5} = \mathscr{B}(\xi, \gamma_{m_1 \dots m_5} \xi) \quad , \quad W_{m_1 \dots m_8} = \mathscr{B}(\xi, \gamma_{m_1 \dots m_8} \xi)$$

These bilinears must satisfy the general Fierz identities for one spinor ξ.

The treatement of CGK spinor equations and Fierz identities is not done in a very efficient and systematic way in the literature in general.

- **Very efficient IDEA** (which goes back to Chevalley and Riesz):

  Use geometric algebra approach to spinors, which means working with differential forms (constructed as bilinears in spinors) instead of spinors.

Geometric algebra is an approach to the differential and spin geometry of (pseudo-)Riemannian manifolds (M, g) which allows for a synthetic and effective formulation of operations on forms and form-valued spinor bilinears that can be constructed using only the differential and Riemannian structure.

It employs the Kahler-Atiyah bundle $(\wedge T^*M, \diamond)$, which is isomorphic to the Clifford bundle $\mathrm{Cl}(T^*M)$, where $\diamond : \wedge T^*M \times \wedge T^*M \to \wedge T^*M$ is the *geometric product* – an associative (but non-commutative) fiberwise composition which makes the exterior bundle into a bundle of unital associative algebras.

It allows for a powerful **reformulation of the differential and spin geometry** of Riemannian manifolds, which is extremely effective in supergravity theories, especially in the presence of fluxes.

Until now we have used the geometric algebra approach for:

- automating the analysis of CGK spinor equations
- automating the construction and analysis of Fierz identities
- generalizing and reformulating Christian Bar's cone formalism

Mathematical setting

- (M, g) is a smooth, connected, paracompact and oriented pseudo-Riemannian manifold of dimension $d = p + q$ (p and q are the numbers of positive and negative eigenvalues of the metric tensor g).
- $\Lambda T^*M \stackrel{\mathrm{def.}}{=} \oplus_{k=0}^{d} \Lambda^k T^*M$ is the exterior bundle of M
- $\Omega(M) \stackrel{\mathrm{def.}}{=} \Gamma(M, \Lambda T^*M)$ is the space of inhomogeneous differential forms on M, with fixed rank components $\Omega^k(M) = \Gamma(M, \Lambda^k T^*M)$.
- $(\wedge T^*M, \diamond)$ is the Kähler-Atiyah bundle of (M, g).
- $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ is a morphism of bundles of algebras, with S the bundle of spinors over (M, g) and $\mathrm{End}(S)$ the bundle of endomorphisms of S.
- The *geometric product* $\diamond$ implements Clifford multiplication $\circ$.

The action of $\diamond : \Omega(M) \times \Omega(M) \longrightarrow \Omega(M)$ has the following expansion in terms of the *generalized products* $\triangle_k : \Omega(M) \times \Omega(M) \longrightarrow \Omega(M)$:

$$\omega \diamond \eta = \sum_{k=0}^{\left[\frac{d}{2}\right]} (-1)^k \omega \triangle_{2k} \eta + \sum_{k=0}^{\left[\frac{d-1}{2}\right]} (-1)^{k+1} \pi(\omega) \triangle_{2k+1} \eta \quad , \quad \forall \omega, \eta \in \Omega(M) \quad ,$$

where $\triangle_k$ are the homogeneous components of $\diamond$ of degree $-2k$, constructed recursively from wedge products and contractions:

$$\omega \triangle_0 \eta = \omega \wedge \eta \quad , \quad \omega \triangle_{k+1} \eta = \frac{1}{k+1} g_{ab} (\iota_{e^a} \omega) \wedge_k (\iota_{e^b} \eta) \quad .$$

Here, π is the *parity automorphism* of the Kähler-Atiyah bundle, defined through:

$$\pi(\omega) = \sum_{k=0}^{d} (-1)^k \omega^{(k)} \quad , \quad \forall \omega = \sum_{k=0}^{d} \omega^{(k)} \in \Omega(M) \quad , \quad \omega^{(k)} \in \Omega^k(M) \quad .$$

ι is the *interior product*, defined as the adjoint of the wedge product,

- **Properties of the (real) volume form** $\nu = e^1 \wedge \ldots \wedge e^d$:

$$\nu \diamond \nu = (-1)^{q+[\frac{d}{2}]} 1_M = \begin{cases} (-1)^{\frac{p-q}{2}} 1_M \ , & \text{if } d = \text{even} \\ (-1)^{\frac{p-q-1}{2}} 1_M \ , & \text{if } d = \text{odd} \end{cases} ,$$

$$\nu \diamond \omega = \pi^{d-1}(\omega) \diamond \nu \quad , \quad \forall \omega \in \Omega(M) \ .$$

Hence ν is central ($\nu \diamond \omega = \omega \diamond \nu$) when d is odd and twisted central ($\nu \diamond \omega = \pi(\omega) \diamond \nu$) when d is even.

	$\nu \diamond \nu = +1$	$\nu \diamond \nu = -1$
ν is central	**1, 5**	**3, 7**
ν is twisted central	**0, 4**	**2, 6**

In table we indicate the values of $p - q \pmod 8$ for the corresponding properties.

- Ordinary **Hodge operator** $*$: $*\omega = \iota_\omega \nu = \tau(\omega) \diamond \nu$, $\forall \omega \in \Omega(M)$
- **Twisted Hodge operator** $\tilde{*}$: $\tilde{*}\omega = \omega \diamond \nu$, $\forall \omega \in \Omega(M)$

$\tau(\omega) = (-1)^{\frac{k(k-1)}{2}} \omega$, $\forall \omega \in \Omega^k(M)$ is the main antiautomorphism, called *reversion*.

- **Twisted (anti-)selfdual forms**: $\tilde{*}\omega = \pm \omega$, $\forall \omega \in \Omega(M)$.

The effective domain of definition of γ

$$\gamma : (\wedge^{\gamma} T^* M) \longrightarrow \mathrm{End}(S)$$

$$\wedge^{\gamma} T^* M \stackrel{\text{def.}}{=} \begin{cases} \wedge T^* M \ , & \text{if } \ \gamma \ \text{ is fiberwise injective (simple case)} \ , \\ \wedge^{\epsilon_\gamma} T^* M \ , & \text{if } \ \gamma \ \text{ is fiberwise non} - \text{injective (non} - \text{simple case)} \ . \end{cases}$$

$\wedge^{\epsilon_\gamma} T^* M$ is the bundle of twisted (anti-)selfdual forms $(\epsilon_\gamma \in \{+1, -1\})$.
$\Omega^{\epsilon_\gamma}(M) = \Gamma(M, \wedge^{\epsilon_\gamma} T^* M)$ is the associated space of twisted (anti-)selfdual forms.

γ	injective	non-injective
surjective	**0, 2**	**1**
non-surjective	**3, 7, 4, 6**	**5**

The numbers appearing in the table indicate the vaue of $p - q (\mathrm{mod}\ 8)$.
When γ is injective $\mathrm{Cl}(p, q)$ is simple, for γ fiberwise *non-injective* it is non-simple.

Schur algebras and representation types

- The *Schur bundle* of γ is the commutant sub-bundle of the image of γ inside $(\mathrm{End}(S), \circ)$: $\Sigma_\gamma \stackrel{\mathrm{def.}}{=} \{T \in \mathrm{End}(S) | [T, \gamma(\omega)]_{-,\circ} = 0, \ \forall \omega \in \wedge T^* M\}$.
- When γ is fiberwise-irreducible, Σ_γ is a bundle of simple associative algebras $\mathbb{S}$, isomorphic with $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In those cases, real spin bundles S are called: **normal** $(\mathbb{R})$, **almost complex** $(\mathbb{C})$ or **quaternionic** $(\mathbb{H})$ and the Schur algebra $\mathbb{S}$ depends only on $p-q \pmod 8$.
- **Summary of spin bundle types**:

$\mathbb{S}$	$p-q$ mod 8	$\wedge T^*M \approx \mathrm{Cl}(p,q)$	Δ	N	Number of choices for γ	$\gamma(\wedge T^*M)$
$\mathbb{R}$	**0, 2**	$\mathrm{Mat}(\Delta, \mathbb{R})$	$2^{[\frac{d}{2}]} = 2^{\frac{d}{2}}$	$2^{[\frac{d}{2}]}$	1	$\mathrm{Mat}(\Delta, \mathbb{R})$
$\mathbb{H}$	**4, 6**	$\mathrm{Mat}(\Delta, \mathbb{H})$	$2^{[\frac{d}{2}]-1} = 2^{\frac{d}{2}-1}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta, \mathbb{H})$
$\mathbb{C}$	**3, 7**	$\mathrm{Mat}(\Delta, \mathbb{C})$	$2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta, \mathbb{C})$
$\mathbb{H}$	**5**	$\mathrm{Mat}(\Delta, \mathbb{H})^{\oplus 2}$	$2^{[\frac{d}{2}]-1} = 2^{\frac{d-3}{2}}$	$2^{[\frac{d}{2}]+1}$	2 $(\epsilon_\gamma = \pm 1)$	$\mathrm{Mat}(\Delta, \mathbb{H})$
$\mathbb{R}$	**1**	$\mathrm{Mat}(\Delta, \mathbb{R})^{\oplus 2}$	$2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]}$	2 $(\epsilon_\gamma = \pm 1)$	$\mathrm{Mat}(\Delta, \mathbb{R})$

$N \stackrel{\mathrm{def.}}{=} \mathrm{rk}_{\mathbb{R}} S$ (the real rank of S), $\Delta \stackrel{\mathrm{def.}}{=} \mathrm{rk}_{\Sigma_\gamma} S$ (the Schur rank of S).

The non-simple cases are indicated in table through the blue color.

In order to solve the CGK spinor equations and the Fierz identities in this approach we use the basis elements for the Kähler-Atiyah algebra written as spinor bilinears:

$$\boxed{\check{E}_{\xi_i,\xi_j} = \frac{\Delta}{2^d}\sum_{k=0}^{d}\check{\mathbf{E}}^{(k)}_{\xi_i,\xi_j} \in \Omega(M)}$$

Depending on the type of representation $(\mathbb{R},\mathbb{C},\mathbb{H})$ one may have more admissible pairings $\mathscr{B}$, with properties ($\sigma_{\mathscr{B}} \in \{+1,-1\}$ and $\epsilon_{\mathscr{B}} \in \{+1,-1\}$) given in the literature. Any choice of $\mathscr{B}$ gives equivalent results.

$$\mathscr{B}(\xi,\xi') = \sigma_{\mathscr{B}}\mathscr{B}(\xi',\xi) \ , \ \forall \xi,\xi' \in \Gamma(M,S)$$

$$\mathscr{B}(\gamma(\omega)\xi,\xi') = \mathscr{B}(\xi,\gamma(\tau_{\mathscr{B}}(\omega))\xi') \ , \ \text{where } \tau_{\mathscr{B}} \stackrel{\text{def.}}{=} \tau\circ\pi^{\frac{1-\epsilon_{\mathscr{B}}}{2}}$$

The general CGK spinor relations:

$$\partial_m\mathscr{B}(\xi_i,\gamma^A\xi_j) = \mathscr{B}(\xi_i,[D_m,\gamma^A]_{-,\circ}\xi_j) \ , \ \forall A = m_1\ldots m_p, \ \forall p = \overline{1,d}$$

$$\mathscr{B}(\xi_i,(\gamma^A\circ Q_k - Q_k^t\circ\gamma^A)\xi_j) = 0 \quad , \ \forall i,j,k$$

are equivalent to the geometric relations:

$$\nabla_m\check{E}_{\xi_i,\xi_j} = -[\check{A}_m,\check{E}_{\xi_i,\xi_j}]_{-,\diamond} \iff d\check{E}_{\xi_i,\xi_j} = -e^m\wedge[\check{A}_m,\check{E}_{\xi_i,\xi_j}]_{-,\diamond}$$

$$\check{Q}_k\diamond\check{E}_{\xi_i,\xi_j} = 0 \ , \ \forall i,j,k$$

where $\check{Q}_k = \gamma^{-1}(Q_k)$ and $\check{A_m} = \gamma^{-1}(A_m)$

- **Normal case** ($\mathbb{S} \sim \mathbb{R}$): $p - q \equiv_8 0, 1, 2$, $N = \Delta = 2^{[\frac{d}{2}]}$

$$\boxed{\check{\mathbf{E}}^{(k)}_{\xi,\xi'} = \frac{1}{k!}(\epsilon_{\mathscr{B}})^k \mathscr{B}(\xi, \gamma_{m_1 \ldots m_k} \xi') e_\gamma^{m_1 \ldots m_k} \quad , \quad \forall \xi, \xi' \in \Gamma(M, S)}$$

The general Fierz identities can be equivalently expressed as geometric Fierz identities through:

$$\check{E}_{\xi_1,\xi_2} \diamond \check{E}_{\xi_3,\xi_4} = \mathscr{B}(\xi_3, \xi_2)\check{E}_{\xi_1,\xi_4} \quad , \quad \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(M, S) \quad .$$

Summary of subcases of the normal case:

$p - q$ mod 8	$\mathrm{Cl}(p, q)$	ϵ_γ	$\gamma(\nu)$	$\nu \diamond \nu$	ν is central
0	simple	N/A	$\gamma(\nu)$	$+1$	No
1	non-simple	± 1	± 1	$+1$	Yes
2	simple	N/A	$\gamma(\nu)$	-1	No

- We have slightly different generators $\check{E}_{\xi,\xi'}$ and geometric Fierz identities for the almost complex and quaternionic cases.

The same example – One spinor in eight Euclidean dimensions ($p = 8,\ q = 0$)

- $d \equiv_8 0,\ p - q \equiv_8 0 \implies$ we are in the normal simple case, $\gamma(\nu) = \mathrm{id}_{S_8}$.
- We have 2 admissible pairing $\mathscr{B}$ on S_8, but we choose to work with the one with the properties $\sigma_{\mathscr{B}} = +1,\ \epsilon_{\mathscr{B}} = +1$).

  We can assume that $\mathscr{B}$ is a scalar product on S_8 and we denote the norm $||\ ||$.
- In the case of one spinor $\xi \in \Gamma(M_8, S_8)$ we are interested in spinor bilinears such as $\check{\mathbf{E}}^{(k)} \stackrel{\mathrm{def.}}{=} \frac{1}{k!}\mathscr{B}(\xi, \gamma_{a_1 \ldots a_k}\xi)e^{a_1 \cdots a_k} \in \Omega^k(M_8)$, $\forall k = \overline{1, 8}$.
- We choose $\mathscr{B}(\xi, \xi) = 1$. Using the properties of the bilinear pairing $\mathscr{B}$ one can construct the form-valued bilinears:

$$K \stackrel{\mathrm{def.}}{=} \check{\mathbf{E}}^{(1)} = \mathscr{B}(\xi, \gamma_m \xi)e^m \quad , \quad Y \stackrel{\mathrm{def.}}{=} \check{\mathbf{E}}^{(4)} = \frac{1}{4!}\mathscr{B}(\xi, \gamma_{m_1 \ldots m_4}\xi)e^{m_1 \cdots m_4} \quad ,$$

$$Z \stackrel{\mathrm{def.}}{=} \check{\mathbf{E}}^{(5)} = \frac{1}{5!}\mathscr{B}(\xi, \gamma_{m_1 \ldots m_5}\xi)e^{m_1 \cdots m_5} \quad , \quad W \stackrel{\mathrm{def.}}{=} \check{\mathbf{E}}^{(8)} = \frac{1}{8!}\mathscr{B}(\xi, \gamma_{m_1 \ldots m_8}\xi)e^{m_1 \cdots m_8} \sim \nu$$

- In this case ($\mathcal{N} = 1$) the Fierz algebra admits a single basis element $\check{\mathbf{E}}$:

$$\check{E} = \frac{1}{2^{[\frac{d}{2}]}}\sum_{k=0}^{8}\check{\mathbf{E}}^{(k)} = \frac{1}{16}(1 + K + Y + Z + b\nu) \ , \quad \text{where } b \in \mathcal{C}^\infty(M_8, \mathbb{R})$$

The **differential constraints, Q-constraints** and **Fierz identities** read:

$$d\check{E} = -e^m \wedge [\check{A}_m, \check{E}]_{-,\diamond} \tag{3}$$
$$\check{Q} \diamond \check{E} = 0 \tag{4}$$
$$\check{E} \diamond \check{E} = \check{E} \tag{5}$$

Using our code in Mathematica-Ricci we obtain the following 3 systems of equations.

First, the differential constraints separated on ranks:

$$\begin{aligned}
\mathrm{d}b &= 2\kappa\, K + \frac{1}{2}\iota_{*Z}F \ , \\
\mathrm{d}K &= -\frac{1}{2}F \bigtriangleup_3 Y + \iota_f * Z \ , \\
\mathrm{d}Y &= F \bigtriangleup_2 Z - 2f \wedge *Y - 2F \wedge K \ , \\
\mathrm{d}Z &= \frac{3}{2}F \bigtriangleup_1 Y + 3\iota_f * K \ , \\
\mathrm{d} * Z &= 2bF + 8\kappa\, Y - *(Y \bigtriangleup_2 F) - 2\iota_f Z \ .
\end{aligned} \tag{6}$$

The *Q*-constraints separated on ranks:

$$
\begin{aligned}
-\frac{1}{6}\iota_F Y + \iota_{\mathrm{d}\Delta} K - 2\kappa b &= 0 \ , \\
-\frac{1}{6}\iota_F Z - \frac{b}{3} f + \mathrm{d}\Delta &= 0 \ , \\
\mathrm{d}\Delta \wedge K - \frac{1}{6} F \triangle_3 Y + \frac{1}{3}\iota_f * Z &= 0 \ , \\
-\frac{1}{3}\iota_f * Y + 2\kappa \ * Z + \frac{1}{6}\iota_K F + \iota_{\mathrm{d}\Delta} - \frac{1}{6} F \triangle_3 Z &= 0 \ , \\
\frac{1}{6} Y \triangle_2 F - 2\kappa \ * Y + \frac{1}{3} f \wedge *Z - \frac{b}{6} * F + \iota_{\mathrm{d}\Delta} Z - \frac{1}{6} F &= 0 \ , \\
-\frac{1}{6} F \wedge K + \mathrm{d}\Delta \wedge Y + \frac{1}{6} F \triangle_2 Z - \frac{1}{3} f \wedge *Y &= 0 \ , \\
\mathrm{d}\Delta \wedge Z + \frac{1}{3}\iota_f * K + \frac{1}{6} F \triangle_1 Y &= 0 \ , \\
-\frac{1}{3} \ * f + 2\kappa \ * K + \frac{1}{6} F \triangle_1 Z + b \ * (\mathrm{d}\Delta) &= 0 \ , \\
2\kappa \ \nu - \frac{1}{3} f \wedge *K + \frac{1}{6} Y \wedge F &= 0 \ .
\end{aligned}
\tag{7}
$$

The rank components of the Fierz identities:

$$\begin{aligned} ||K||^2 + ||Y||^2 + ||Z||^2 + b^2 &= 15 \ , \\ \iota_Y Z &= 7K \ , \\ -Y \triangle_2 Y - Z \triangle_3 Z + 2\iota_K Z + 2b * Y &= 14Y \ , \\ -Y \triangle_2 Z + K \wedge Y &= 7Z \ , \\ Y \wedge Y + Z \triangle_1 Z &= 14b\, \nu \ . \end{aligned} \tag{8}$$

Using (7), (6) and (8) one can solve the constraints and recover the results in the literature, some of which we list below (for $b = \sin\zeta$, as chosen in the literature, [arXiv:hep-th/0306225]):

$$\begin{aligned} &\mathrm{d}(e^{3\Delta} K) = 0 \\ &K \wedge \mathrm{d}(e^{6\Delta} \iota_K Z) = 0 \\ &e^{6\Delta} \mathrm{d}(e^{-6\Delta} * Z) = - * F + F \sin\zeta - 4\kappa\ Y \\ &e^{-3\Delta} \mathrm{d}(e^{3\Delta} \sin\zeta) = f - 4\kappa\ K \\ &||K||^2 = \cos^2\zeta \quad , \quad ||Z||^2 = 7\cos^2\zeta \\ &Y = \iota_K Z - (*Z) \wedge K \sin\zeta \end{aligned}$$

This example was chosen just to illustrate the method.