Invariants of non-QRT mappings and rational elliptic surfaces of higher index

A. S. CARSTEA
Department of Theoretical Physics, NIPNE Bucharest
in collaboration with
TOMOYUKI TAKENAWA
University of Marine Technology, Tokyo

April 25, 2013

MOTIVATION

- Singularities play a very important role in the characterisation of a discrete dynamical systems
- Here we address the problem of finding invariants from the study of singularities.

MOTIVATION

- Singularities play a very important role in the characterisation of a discrete dynamical systems
- Here we address the problem of finding invariants from the study of singularities.

MOTIVATION

- Singularities play a very important role in the characterisation of a discrete dynamical systems
- Here we address the problem of finding invariants from the study of singularities.

OUTLINE

- Basic things about blow-ups and integrability
- Generalities on Halphen surfaces.
- Classification on mappings
- Examples

OUTLINE

- Basic things about blow-ups and integrability
- Generalities on Halphen surfaces.
- Classification on mappings
- Examples

OUTLINE

- Basic things about blow-ups and integrability
- Generalities on Halphen surfaces.
- Classification on mappings
- Examples

OUTLINE

- Basic things about blow-ups and integrability
- Generalities on Halphen surfaces.
- Classification on mappings
- Examples

Generalities on resolution of singularities, integrability

The systems under consideration have the rational reversible form:

$$(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (\overline{x}, \overline{y}) \in \mathbb{P}^1 \times \mathbb{P}^1$$

$$\overline{x} = F(x, y)$$

$$\overline{y} = G(x, y)$$

and also the inverse (F, G, Φ, Γ are rational functions of x, y)

$$\underline{x} = \Phi(x, y)$$

$$\underline{y} = \Gamma(x, y)$$

The projective space $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by the following coordinate systems ($X = 1/x$, $Y = 1/y$):

$$\mathbb{P}^1 \times \mathbb{P}^1 = (x, y) \cup (X, y) \cup (x, Y) \cup (X, Y)$$

In order to make the resolution of singularities we have to check first the nonedeterminate points for the direct and inverse mappings in any of the coordinate branches of $\mathbb{P}^1 \times \mathbb{P}^1$

Blow-up at $(x, y) = (a, b) \in \mathbb{C}^2$ by the following procedure

$$\{(x, y) : x, y \in \mathbb{C}\} \overset{\pi_{(a,b)}}{\leftarrow} \{(x-a, y-b; \zeta_1 : \zeta_2) | x, y, \zeta_1, \zeta_2 \in \mathbb{C}$$

$$, |\zeta_1| + |\zeta_2| \neq 0, (x-a)\zeta_2 = (y-b)\zeta_1\}$$

given by

$$(x, y) \overset{\pi_{(a,b)}}{\leftarrow} \underbrace{(x-a, \frac{y-b}{x-a}) \cup (\frac{x-a}{y-b}, y-b)}_{\mathbb{P}^1-line} \equiv$$

After the blowing ups projective space is transformed into a surface X and the mapping is lifted to a birational mapping

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Blow-up at $(x, y) = (a, b) \in \mathbb{C}^2$ by the following procedure

$$\{(x, y) : x, y \in \mathbb{C}\} \overset{\pi_{(a,b)}}{\leftarrow} \{(x - a, y - b; \zeta_1 : \zeta_2) | x, y, \zeta_1, \zeta_2 \in \mathbb{C}$$

$$, |\zeta_1| + |\zeta_2| \neq 0, (x - a)\zeta_2 = (y - b)\zeta_1\}$$

given by

$$(x, y) \overset{\pi_{(a,b)}}{\leftarrow} \underbrace{(x - a, \frac{y - b}{x - a}) \cup (\frac{x - a}{y - b}, y - b)}_{\mathbb{P}^1 - line} \equiv$$

After the blowing ups projective space is transformed into a surface X and the mapping is lifted to a birational mapping

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Blow-up at $(x, y) = (a, b) \in \mathbb{C}^2$ by the following procedure

$$\{(x, y) : x, y \in \mathbb{C}\} \overset{\pi_{(a,b)}}{\longleftarrow} \{(x-a, y-b; \zeta_1 : \zeta_2) | x, y, \zeta_1, \zeta_2 \in \mathbb{C}$$

$$, |\zeta_1| + |\zeta_2| \neq 0, (x-a)\zeta_2 = (y-b)\zeta_1\}$$

given by

$$(x, y) \overset{\pi_{(a,b)}}{\longleftarrow} (x-a, \underbrace{\frac{y-b}{x-a}) \cup (\frac{x-a}{y-b}}_{\mathbb{P}^1 - line}, y-b) \equiv$$

After the blowing ups projective space is transformed into a surface X and the mapping is lifted to a birational mapping

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Blow-up at $(x, y) = (a, b) \in \mathbb{C}^2$ by the following procedure

$$\{(x, y) : x, y \in \mathbb{C}\} \overset{\pi_{(a,b)}}{\leftarrow} \{(x-a, y-b; \zeta_1 : \zeta_2) | x, y, \zeta_1, \zeta_2 \in \mathbb{C}$$

$$, |\zeta_1| + |\zeta_2| \neq 0, (x-a)\zeta_2 = (y-b)\zeta_1\}$$

given by

$$(x, y) \overset{\pi_{(a,b)}}{\leftarrow} (x-a, \underbrace{\frac{y-b}{x-a}) \cup (\frac{x-a}{y-b}}_{\mathbb{P}^1 - line}, y-b) \equiv$$

After the blowing ups projective space is transformed into a surface X and the mapping is lifted to a birational mapping

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability = Weyl group of affine type
- linearisability = infinite number of blow ups, analytical stability, ruled surface S

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability = Weyl group of affine type
- linearisability = infinite number of blow ups, analytical stability, ruled surface S

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability = Weyl group of affine type
- linearisability = infinite number of blow ups, analytical stability, ruled surface S

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability = Weyl group of affine type
- linearisability = infinite number of blow ups, analytical stability, ruled surface S

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability = Weyl group of affine type
- linearisability = infinite number of blow ups, analytical stability, ruled surface S

IST "philosophy"

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity.
- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ whose action on the Picard group is linear.
- in the $\mathrm{Pic}(S)$ where the dynamics is linear one can find invariants, type of surface, and Weyl group (as the orthogonal complement of the surface Dynkin diagram)
- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above
- integrability $=$ Weyl group of affine type
- linearisability $=$ infinite number of blow ups, analytical stability, ruled surface S

Generalized Halphen surfaces and classification of the mappings

Rational elliptic surface: A complex surface X is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
- π is not birational to the projection : $E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve E
- no fibers contains exceptional curves of first kind.

Generalized Halphen surface: A rational surface X is called a *generalized Halphen surface* if the anticanonical divisor class $-K_X$ is decomposed into effective divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Generalized Halphen surfaces can be obtained from $\mathbb{P}^2$ by succesive 9 blow-ups. They can be classified by D in elliptic, multiplicative and additive type.

Generalized Halphen surfaces and classification of the mappings

Rational elliptic surface: A complex surface X is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
- π is not birational to the projection : $E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve E
- no fibers contains exceptional curves of first kind.

Generalized Halphen surface: A rational surface X is called a *generalized Halphen surface* if the anticanonical divisor class $-K_X$ is decomposed into effective divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Generalized Halphen surfaces can be obtained from $\mathbb{P}^2$ by succesive 9 blow-ups. They can be classified by D in elliptic, multiplicative and additive type.

Generalized Halphen surfaces and classification of the mappings

Rational elliptic surface: A complex surface X is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
- π is not birational to the projection : $E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve E
- no fibers contains exceptional curves of first kind.

Generalized Halphen surface: A rational surface X is called a *generalized Halphen surface* if the anticanonical divisor class $-K_X$ is decomposed into effective divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Generalized Halphen surfaces can be obtained from $\mathbb{P}^2$ by succesive 9 blow-ups. They can be classified by D in elliptic, multiplicative and additive type.

Halphen surfaces of higher index:

- a rational surface X is called a *Halphen surface of index m* if the dimension of the linear system $|-kK_X| = 0, k = \overline{1, m-1}$ and $|-kK_X| = 1, k = m$. A Halphen surface of index m is also referred to be a rational elliptic surface of index m.
- the linear system $|-kK_X|$ is the set of curves in $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) of degree $3k$ (resp. $4k$) passing through each pont of blow-up with multiplicity k. It is known that any Halphen surface of index m contains a unique cubic curve with multiplicity m

Halphen surfaces of higher index:

- a rational surface X is called a *Halphen surface of index m* if the dimension of the linear system $|-kK_X| = 0, k = \overline{1, m-1}$ and $|-kK_X| = 1, k = m$. A Halphen surface of index m is also referred to be a rational elliptic surface of index m.
- the linear system $|-kK_X|$ is the set of curves in $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) of degree $3k$ (resp. $4k$) passing through each pont of blow-up with multiplicity k. It is known that any Halphen surface of index m contains a unique cubic curve with multiplicity m

Halphen surfaces of higher index:

- a rational surface X is called a *Halphen surface of index m* if the dimension of the linear system $|-kK_X| = 0, k = \overline{1, m-1}$ and $|-kK_X| = 1, k = m$. A Halphen surface of index m is also referred to be a rational elliptic surface of index m.
- the linear system $|-kK_X|$ is the set of curves in $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) of degree $3k$ (resp. $4k$) passing through each pont of blow-up with multiplicity k. It is known that any Halphen surface of index m contains a unique cubic curve with multiplicity m

Classification

The main result of our talk is the following classification:
Let X be a rational elliptic surface obtained from 9 blow-ups of $\mathbb{P}^2$ (resp. 8 blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$)and let there be $\phi : X \to X$ an automorphism of X preserving the elliptic fibration $\alpha f_0(x, y, z) + \beta g_0(x, y, z) = 0$ with $(\alpha : \beta) \in \mathbb{P}^1$. Then ϕ falls in one of the following classes:

- (i-m) ϕ preserves $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)
- (ii-m) ϕ does not preserve $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)

Remark: QRT mappings belong to case $(i - 1)$

Classification

The main result of our talk is the following classification:
Let X be a rational elliptic surface obtained from 9 blow-ups of $\mathbb{P}^2$ (resp. 8 blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$)and let there be $\phi : X \to X$ an automorphism of X preserving the elliptic fibration $\alpha f_0(x, y, z) + \beta g_0(x, y, z) = 0$ with $(\alpha : \beta) \in \mathbb{P}^1$. Then ϕ falls in one of the following classes:

- (i-m) ϕ preserves $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)
- (ii-m) ϕ does not preserve $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)

Remark: QRT mappings belong to case $(i-1)$

Classification

The main result of our talk is the following classification:
Let X be a rational elliptic surface obtained from 9 blow-ups of $\mathbb{P}^2$ (resp. 8 blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$)and let there be $\phi : X \to X$ an automorphism of X preserving the elliptic fibration $\alpha f_0(x, y, z) + \beta g_0(x, y, z) = 0$ with $(\alpha : \beta) \in \mathbb{P}^1$. Then ϕ falls in one of the following classes:

- (i-m) ϕ preserves $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)
- (ii-m) ϕ does not preserve $(\alpha : \beta)$ and the degree of fibers is $3m$ (resp. $(2m, 2m)$)

Remark: QRT mappings belong to case $(i - 1)$

Examples:

First we start with a mapping which preserves elliptic fibration of degree $(2,2)$ but exchanges the fibers.

$$x_{n+1} = -x_{n-1}\frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \tag{1}$$

$$\begin{aligned} \overline{x} &= y \\ \overline{y} &= -x\frac{(y-a)(y-1/a)}{(y+a)(y+1/a)} \end{aligned} \tag{2}$$

Indeterminate points for ϕ and ϕ^{-1}:

$$\begin{aligned} P_1 &: (x,y) = (0,-a), & P_2 &: (x,y) = (0,-1/a), \\ P_3 &: (X,y) = (0,a), & P_4 &: (X,y) = (0,1/a), \\ P_5 &: (x,y) = (a,0), & P_6 &: (x,y) = (1/a,0), \\ P_7 &: (x,Y) = (-a,0), & P_8 &: (x,Y) = (-1/a,0). \end{aligned}$$

Examples:

First we start with a mapping which preserves elliptic fibration of degree $(2,2)$ but exchanges the fibers.

$$x_{n+1} = -x_{n-1}\frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \tag{1}$$

$$\begin{aligned} \overline{x} &= y \\ \overline{y} &= -x\frac{(y - a)(y - 1/a)}{(y + a)(y + 1/a)} \end{aligned} \tag{2}$$

Indeterminate points for ϕ and ϕ^{-1}:

$$\begin{aligned} P_1 : (x, y) = (0, -a), \quad & P_2 : (x, y) = (0, -1/a), \\ P_3 : (X, y) = (0, a), \quad & P_4 : (X, y) = (0, 1/a), \\ P_5 : (x, y) = (a, 0), \quad & P_6 : (x, y) = (1/a, 0), \\ P_7 : (x, Y) = (-a, 0), \quad & P_8 : (x, Y) = (-1/a, 0). \end{aligned}$$

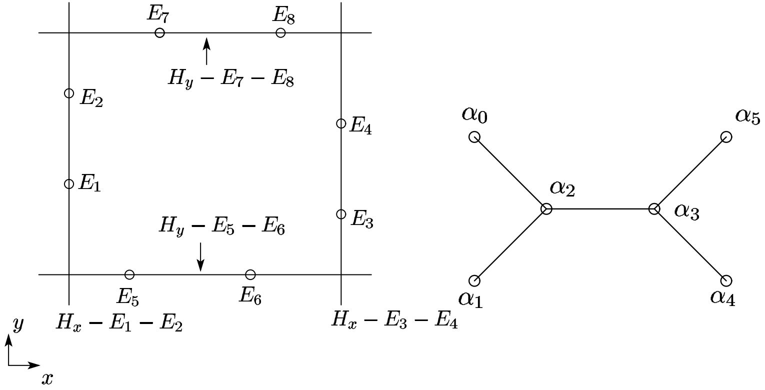

Figure: Space of initial conditions and orthogonal complement

The Picard group of X is a **Z**-module

$$\mathrm{Pic}(X) = \mathbb{Z}\, H_x \oplus \mathbb{Z}\, H_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}\, E_i,$$

H_x, H_y are the total transforms of the lines $x = \mathrm{const.}$, $y = \mathrm{const.}$
E_i are the total transforms of the eight blowing up points.
The intersection form:

$$H_z \cdot H_w = 1 - \delta_{zw}, \quad E_i \cdot E_j = -\delta_{ij}, \quad H_z \cdot E_k = 0$$

for $z, w = x, y$. Anti-canonical divisor of X:

$$-K_X = 2H_x + 2H_y - \sum_{i=1}^{8} E_i.$$

The Picard group of X is a **Z**-module

$$\mathrm{Pic}(X) = \mathbb{Z}\, H_x \oplus \mathbb{Z}\, H_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}\, E_i,$$

H_x, H_y are the total transforms of the lines $x = \mathrm{const.}$, $y = \mathrm{const.}$
E_i are the total transforms of the eight blowing up points.
The intersection form:

$$H_z \cdot H_w = 1 - \delta_{zw}, \quad E_i \cdot E_j = -\delta_{ij}, \quad H_z \cdot E_k = 0$$

for $z, w = x, y$. Anti-canonical divisor of X:

$$-K_X = 2H_x + 2H_y - \sum_{i=1}^{8} E_i.$$

If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice $(h_i, e_j \in \mathbf{Z})$ the induced bundle mapping is acting on it as

$$\phi_*(h_0, h_1, e_1, ..., e_8)$$
$$=(h_0, h_1, e_1, ..., e_8) \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$:

$$D_0 = H_x - E_1 - E_2,\ D_1 = H_y - E_5 - E_6$$
$$D_2 = H_x - E_3 - E_4,\ D_3 = H_y - E_7 - E_8$$

If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice $(h_i, e_j \in \mathbf{Z})$ the induced bundle mapping is acting on it as

$$\phi_*(h_0, h_1, e_1, ..., e_8)$$

$$=(h_0, h_1, e_1, ..., e_8) \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$:

$$D_0 = H_x - E_1 - E_2, \ D_1 = H_y - E_5 - E_6$$

$$D_2 = H_x - E_3 - E_4, \ D_3 = H_y - E_7 - E_8$$

If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice $(h_i, e_j \in \mathbf{Z})$ the induced bundle mapping is acting on it as

$$\phi_*(h_0, h_1, e_1, ..., e_8)$$

$$=(h_0, h_1, e_1, ..., e_8) \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$:

$$D_0 = H_x - E_1 - E_2,\ D_1 = H_y - E_5 - E_6$$
$$D_2 = H_x - E_3 - E_4,\ D_3 = H_y - E_7 - E_8$$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

$$F \equiv \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0$$
$$\Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.$$

- this family of curves defines a rational elliptic surface.
- anti-canonical class is preserved by the mapping, the linear system is not. More precisely the action changes k in $-k$ (the mapping exchange fibers of the elliptic fibration)
- in such cases the conserved quantity becomes higher degree as $(f/g)^\nu$ for some $\nu > 1$. In our case $\nu = 2$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

$$F \equiv \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0$$
$$\Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.$$

- this family of curves defines a rational elliptic surface.
- anti-canonical class is preserved by the mapping, the linear system is not. More precisely the action changes k in $-k$ (the mapping exchange fibers of the elliptic fibration)
- in such cases the conserved quantity becomes higher degree as $(f/g)^\nu$ for some $\nu > 1$. In our case $\nu = 2$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

$$F \equiv \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0$$
$$\Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.$$

- this family of curves defines a rational elliptic surface.
- anti-canonical class is preserved by the mapping, the linear system is not. More precisely the action changes k in $-k$ (the mapping exchange fibers of the elliptic fibration)
- in such cases the conserved quantity becomes higher degree as $(f/g)^\nu$ for some $\nu > 1$. In our case $\nu = 2$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

$$\begin{aligned} F \equiv & \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0 \\ & \Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0. \end{aligned}$$

- this family of curves defines a rational elliptic surface.
- anti-canonical class is preserved by the mapping, the linear system is not. More precisely the action changes k in $-k$ (the mapping exchange fibers of the elliptic fibration)
- in such cases the conserved quantity becomes higher degree as $(f/g)^\nu$ for some $\nu > 1$. In our case $\nu = 2$

Weyl group symmetries are related to the orthogonal complement of the space of initial conditions $A_3^{(1)}$. In order to see this we note that $\mathit{rank}\,\mathrm{Pic}(X) = \mathit{rank} < H_x, H_y, E_1, ...E_8 >_{\mathbb{Z}} = 10$ Now we define:

$$< D > = \bigoplus_{i=0}^{3} \mathbb{Z} D_i$$

$$< D >^{\perp} = \{\alpha \in \mathrm{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3\}$$

which have 6-generators:

$$< D >^{\perp} = < \alpha_0, \alpha_1, ..., \alpha_5 >_{\mathbb{Z}}$$

$$\alpha_0 = E_1 - E_2, \alpha_1 = E_3 - E_4, \alpha_2 = H_y - E_1 - E_3$$

$$\alpha_3 = H_x - E_5 - E_7, \alpha_4 = E_5 - E_6, \alpha_5 = E_7 - E_8$$

Weyl group symmetries are related to the orthogonal complement of the space of initial conditions $A_3^{(1)}$. In order to see this we note that $rank\,\mathrm{Pic}(X) = rank < H_x, H_y, E_1, ...E_8 >_{\mathbb{Z}} = 10$ Now we define:

$$< D > = \bigoplus_{i=0}^{3} \mathbb{Z} D_i$$

$$< D >^{\perp} = \{\alpha \in \mathrm{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3\}$$

which have 6-generators:

$$< D >^{\perp} = < \alpha_0, \alpha_1, ..., \alpha_5 >_{\mathbb{Z}}$$

$$\alpha_0 = E_1 - E_2, \alpha_1 = E_3 - E_4, \alpha_2 = H_y - E_1 - E_3$$

$$\alpha_3 = H_x - E_5 - E_7, \alpha_4 = E_5 - E_6, \alpha_5 = E_7 - E_8$$

we define elementary reflections:

$$w_i : \mathrm{Pic}(x) \to \mathrm{Pic}(X), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

$c_{ji} = 2(\alpha_j \cdot \alpha_i)/(\alpha_i \cdot \alpha_i)$. One can easily see that c_{ij} is a Cartan matrix of $D_5^{(1)}$-type for the root lattice $Q = \bigoplus_{i=0}^{5} \mathbb{Z}\,\alpha_i$. We introduce also permutation of roots:

$$\sigma_{10}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$\sigma_{tot}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$$

we define elementary reflections:

$$w_i : \mathrm{Pic}(x) \to \mathrm{Pic}(X), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

$c_{ji} = 2(\alpha_j \cdot \alpha_i)/(\alpha_i \cdot \alpha_i)$. One can easily see that c_{ij} is a Cartan matrix of $D_5^{(1)}$-type for the root lattice $Q = \bigoplus_{i=0}^{5} \mathbb{Z}\,\alpha_i$. We introduce also permutation of roots:

$$\sigma_{10}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$\sigma_{tot}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$$

Extended Weyl group

$$\widetilde{W}(D_5^{(1)}) =< w_0, w_1, ..., w_5, \sigma_{10}, \sigma_{tot} >$$

Mapping has the following decomposition in elementary reflections:

$$\begin{aligned}\phi_* = &\sigma_{10}\sigma_{tot}\sigma_{10}\sigma_{tot} w_2 w_1 w_0 w_2 w_1 w_0 : \\ &(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &\mapsto (-\alpha_5, -\alpha_4, -\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \alpha_0, \alpha_1).\end{aligned}$$

- **hence, ϕ^4 is a translation of the extended Weyl group, while ϕ itself is not, even though it is an automorphism of an elliptic surface.**
- by deautonomisation and transformations (up to nonautonomous factors):

$$x_{4n-1} = 1/y_{4n-1}, \quad x_{4n} = y_{4n}, \quad x_{4n+1} = y_{4n+1}, \quad x_{4n+2} = 1/y_{4n+2}$$

$$X_n = y_{2n}, \quad Y_n = y_{2n+1}$$

our mapping is equivalent with Jimbo-Sakai $q - P_{VI}$ in X_n, Y_n

Extended Weyl group

$$\widetilde{W}(D_5^{(1)}) =< w_0, w_1, ..., w_5, \sigma_{10}, \sigma_{tot} >$$

Mapping has the following decomposition in elementary reflections:

$$\phi_* = \sigma_{10}\sigma_{tot}\sigma_{10}\sigma_{tot} w_2 w_1 w_0 w_2 w_1 w_0 : \\ (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ \mapsto (-\alpha_5, -\alpha_4, -\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \alpha_0, \alpha_1).$$

- **hence, ϕ^4 is a translation of the extended Weyl group, while ϕ itself is not, even though it is an automorphism of an elliptic surface.**
- by deautonomisation and transformations (up to nonautonomous factors):

$$x_{4n-1} = 1/y_{4n-1}, \quad x_{4n} = y_{4n}, \quad x_{4n+1} = y_{4n+1}, \quad x_{4n+2} = 1/y_{4n+2}$$

$$X_n = y_{2n}, \quad Y_n = y_{2n+1}$$

our mapping is equivalent with Jimbo-Sakai $q - P_{VI}$ in X_n, Y_n

Extended Weyl group

$$\widetilde{W}(D_5^{(1)}) =< w_0, w_1, ..., w_5, \sigma_{10}, \sigma_{tot} >$$

Mapping has the following decomposition in elementary reflections:

$$\begin{aligned}\phi_* =& \sigma_{10}\sigma_{tot}\sigma_{10}\sigma_{tot} w_2 w_1 w_0 w_2 w_1 w_0 : \\ & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ & \mapsto (-\alpha_5, -\alpha_4, -\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \alpha_0, \alpha_1).\end{aligned}$$

- **hence, ϕ^4 is a translation of the extended Weyl group, while ϕ itself is not, even though it is an automorphism of an elliptic surface.**
- by deautonomisation and transformations (up to nonautonomous factors):

$$x_{4n-1} = 1/y_{4n-1}, \quad x_{4n} = y_{4n}, \quad x_{4n+1} = y_{4n+1}, \quad x_{4n+2} = 1/y_{4n+2}$$

$$X_n = y_{2n}, \quad Y_n = y_{2n+1}$$

our mapping is equivalent with Jimbo-Sakai $q-P_{VI}$ in X_n, Y_n

The HKY case

Symmetric reduction of q-P_V for $q = -1$.

$$\bar{x} = \frac{(x-t)(x+t)}{y(x-1)}$$
$$\bar{y} = x \tag{3}$$

We define the phase space as a rational surface obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points

$$\begin{aligned} &P_1 : (x,y) = (t,0), && P_2 : (x,y) = (-t,0) \\ &P_3 : (x,y) = (0,t), && P_4 : (x,y) = (0,-t) \\ &P_5 : (x,Y) = (1,0), && P_6 : (X,y) = (0,1) \\ &P_7 : (X,Y) = (0,0), && P_8 : (X,x/y) = (\infty,1) \end{aligned}$$

The HKY case

Symmetric reduction of q-P_V for $q = -1$.

$$\bar{x} = \frac{(x-t)(x+t)}{y(x-1)}$$
$$\bar{y} = x \tag{3}$$

We define the phase space as a rational surface obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points

$$\begin{aligned} &P_1 : (x, y) = (t, 0), && P_2 : (x, y) = (-t, 0) \\ &P_3 : (x, y) = (0, t), && P_4 : (x, y) = (0, -t) \\ &P_5 : (x, Y) = (1, 0), && P_6 : (X, y) = (0, 1) \\ &P_7 : (X, Y) = (0, 0), && P_8 : (X, x/y) = (\infty, 1) \end{aligned}$$

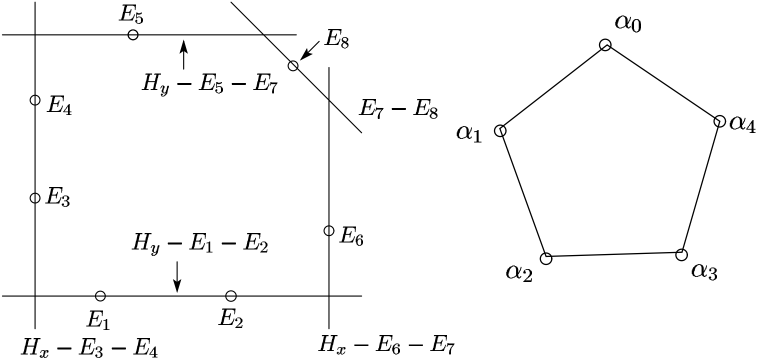

Figure: Space of initial conditions and the ortogonal complement of the same type $A_4^{(1)}$

The anti-canonical divisor $-K_X$ is decomposed as $-K_X = \sum_{i=0}^{4} D_i$:

$$D_0 = H_y - E_1 - E_2,\ D_1 = H_x - E_6 - E_7$$
$$D_2 = E_7 - E_8,\ D_3 = H_y - E_5 - E_6,\ D_4 = H_x - E_3 - E_4$$

which represents an $A_4^{(1)}$ surface. The orthogonal complement of D_i's is generated by

$$\alpha_0 = H_x + H_y - E_1 - E_3 - E_7 - E_8,\ \alpha_1 = E_1 - E_2$$
$$\alpha_2 = H_x - E_1 - E_5, \alpha_3 = H_y - E_3 - E_6,\ \alpha_4 = E_3 - E_4$$

which form the Dynkin diagram of the same type $A_4^{(1)}$.

The anti-canonical divisor $-K_X$ is decomposed as $-K_X = \sum_{i=0}^{4} D_i$:

$$D_0 = H_y - E_1 - E_2,\ D_1 = H_x - E_6 - E_7$$
$$D_2 = E_7 - E_8,\ D_3 = H_y - E_5 - E_6,\ D_4 = H_x - E_3 - E_4$$

which represents an $A_4^{(1)}$ surface. The orthogonal complement of D_i's is generated by

$$\alpha_0 = H_x + H_y - E_1 - E_3 - E_7 - E_8,\ \alpha_1 = E_1 - E_2$$
$$\alpha_2 = H_x - E_1 - E_5, \alpha_3 = H_y - E_3 - E_6,\ \alpha_4 = E_3 - E_4$$

which form the Dynkin diagram of the same type $A_4^{(1)}$.

Anti-canonical divisor class: $-K_X = 2H_x + 2H_y - E_1 - \cdots - E_8$.

The corresponding curve is $xy = 0$ trivial.

So, $\dim|-K_X| = 0$, but $\dim|-2K_X| = 1$. Indeed, we have

$$|-2K_X| = \alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 -$$

$$-2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv$$

$$\alpha f + \beta g$$

and

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$

$$+ \frac{x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4)}{x^2y^2}$$

is the conserved quantity. So it belongs to Case ii-1.

Anti-canonical divisor class: $-K_X = 2H_x + 2H_y - E_1 - \cdots - E_8$.
The corresponding curve is $xy = 0$ trivial.
So, $\dim|-K_X| = 0$, but $\dim|-2K_X| = 1$. Indeed, we have

$$|-2K_X| = \alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 -$$
$$-2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv$$
$$\alpha f + \beta g$$

and

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$
$$+ \frac{x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4)}{x^2y^2}$$

is the conserved quantity. So it belongs to Case ii-1.

Anti-canonical divisor class: $-K_X = 2H_x + 2H_y - E_1 - \cdots - E_8$.
The corresponding curve is $xy = 0$ trivial.
So, $\dim|-K_X| = 0$, but $\dim|-2K_X| = 1$. Indeed, we have

$$|-2K_X| = \alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 -$$

$$-2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv$$

$$\alpha f + \beta g$$

and

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$

$$+ \frac{x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4)}{x^2y^2}$$

is the conserved quantity. So it belongs to Case ii-1.

Anti-canonical divisor class: $-K_X = 2H_x + 2H_y - E_1 - \cdots - E_8$.
The corresponding curve is $xy = 0$ trivial.
So, $\dim |-K_X| = 0$, but $\dim |-2K_X| = 1$. Indeed, we have

$$|-2K_X| = \alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 -$$

$$-2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv$$

$$\alpha f + \beta g$$

and

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$

$$+\frac{x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4)}{x^2y^2}$$

is the conserved quantity. So it belongs to Case ii-1.

Case (ii-2)

We consider the mapping φ

$$\varphi : \begin{cases} \bar{x} & = \dfrac{x(-ix(x+1)+y(bx+1))}{y(x(x-b)+iby(x-1))} \\ \bar{y} & = \dfrac{x(x(x+1)+iby(x-1))}{b(x(x+1)-iy(x-1))} \end{cases} . \tag{4}$$

The inverse of φ is

$$\varphi^{-1} : \begin{cases} \underline{x} & = \dfrac{y(bxy-bx-by+1)}{xy-x+by-1} \\ \underline{y} & = \dfrac{-iy(bxy-bx-by+1))(bxy+x-by+1)}{bx(xy-x-y-1)(xy-x+by-1)} \end{cases} . \tag{5}$$

Case (ii-2)

We consider the mapping φ

$$\varphi : \begin{cases} \bar{x} &= \dfrac{x(-ix(x+1)+y(bx+1))}{y(x(x-b)+iby(x-1))} \\ \bar{y} &= \dfrac{x(x(x+1)+iby(x-1))}{b(x(x+1)-iy(x-1))} \end{cases} . \qquad (4)$$

The inverse of φ is

$$\varphi^{-1} : \begin{cases} \underline{x} &= \dfrac{y(bxy-bx-by+1)}{xy-x+by-1} \\ \underline{y} &= \dfrac{-iy(bxy-bx-by+1))(bxy+x-by+1)}{bx(xy-x-y-1)(xy-x+by-1)} \end{cases} . \qquad (5)$$

Case (ii-2)

We consider the mapping φ

$$\varphi : \begin{cases} \bar{x} &= \dfrac{x(-ix(x+1)+y(bx+1))}{y(x(x-b)+iby(x-1))} \\ \bar{y} &= \dfrac{x(x(x+1)+iby(x-1))}{b(x(x+1)-iy(x-1))} \end{cases} . \tag{4}$$

The inverse of φ is

$$\varphi^{-1} : \begin{cases} \underline{x} &= \dfrac{y(bxy-bx-by+1)}{xy-x+by-1} \\ \underline{y} &= \dfrac{-iy(bxy-bx-by+1))(bxy+x-by+1)}{bx(xy-x-y-1)(xy-x+by-1)} \end{cases} . \tag{5}$$

The phase space is obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points:

$$\begin{aligned} P_1 &: (x, y) = (-1, 0), & P_2 &: (x, y) = (0, b) \\ P_3 &: (x, Y) = (1, 0), & P_4 &: (X, y) = (0, 1) \\ P_5 &: (x, y) = (0, 0), & P_6 &: (x, y/x) = (0, i) \\ P_7 &: (X, Y) = (0, 0), & P_8 &: (X, x/y) = (0, -ib). \end{aligned}$$

The phase space is obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points:

$$
\begin{aligned}
P_1 : (x, y) &= (-1, 0), \quad & P_2 : (x, y) &= (0, b) \\
P_3 : (x, Y) &= (1, 0), \quad & P_4 : (X, y) &= (0, 1) \\
P_5 : (x, y) &= (0, 0), \quad & P_6 : (x, y/x) &= (0, i) \\
P_7 : (X, Y) &= (0, 0), \quad & P_8 : (X, x/y) &= (0, -ib).
\end{aligned}
$$

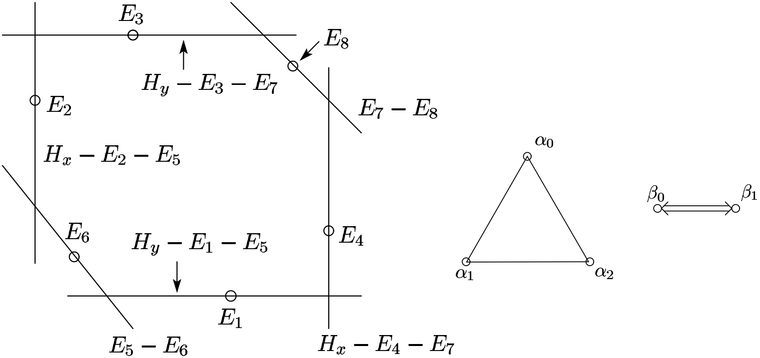

Figure: Space of initial conditions and orthogonal complement

The anti-canonical divisor consists of

$$D_0 = H_x - E_2 - E_5,\ D_1 = E_5 - E_6,\ D_2 = H_y - E_1 - E_5,$$
$$D_3 = H_x - E_4 - E_7,\ D_4 = E_7 - E_8,\ D_5 = H_y - E_3 - E_7$$

and its orthogonal complement is generated by

$$\alpha_0 = H_x + H_y - E_5 - E_6 - E_7 - E_8$$
$$\alpha_1 = H_x - E_1 - E_3$$
$$\alpha_2 = H_y - E_2 - E_4$$
$$\beta_0 = H_x + H_y - E_1 - E_2 - E_7 - E_8$$
$$(\beta_1 = H_x + H_y - E_3 - E_4 - E_5 - E_6).$$

Here $\dim|-K_X| = 0$ and $\dim|-2K_X| = 1$

$$|-2K_X| : \quad \begin{aligned} 0 = \;& kf_0(x,y) - f_1(x,y) \\ = \;& kx^2y^2 - \Big(ix(x+1)^2 - i(x+i)(x^2-1)y \\ & +b(x-1)^2y^2\Big)\Big(-ix(y-1) + y(by-1)\Big) \end{aligned}$$

and again we have exchanging of fibers:

$$k = \frac{f_1(x,y)}{f_0(x,y)} \to -k \Rightarrow k^2 = \left(\frac{f_1(x,y)}{f_0(x,y)}\right)^2$$

is the conserved quantity (case ii-2).

Here $\dim|-K_X| = 0$ and $\dim|-2K_X| = 1$

$$|-2K_X| : \quad \begin{aligned} 0 = & \; kf_0(x,y) - f_1(x,y) \\ = & \; kx^2y^2 - \Big(ix(x+1)^2 - i(x+i)(x^2-1)y \\ & +b(x-1)^2y^2\Big)\Big(-ix(y-1) + y(by-1)\Big) \end{aligned}$$

and again we have exchanging of fibers:

$$k = \frac{f_1(x,y)}{f_0(x,y)} \to -k \Rightarrow k^2 = \left(\frac{f_1(x,y)}{f_0(x,y)}\right)^2$$

is the conserved quantity (case ii-2).

Here $\dim|-K_X| = 0$ and $\dim|-2K_X| = 1$

$$|-2K_X|: \quad \begin{aligned} 0 = \ & kf_0(x,y) - f_1(x,y) \\ = \ & kx^2y^2 - \Big(ix(x+1)^2 - i(x+i)(x^2-1)y \\ & +b(x-1)^2y^2\Big)\Big(-ix(y-1) + y(by-1)\Big) \end{aligned}$$

and again we have exchanging of fibers:

$$k = \frac{f_1(x,y)}{f_0(x,y)} \to -k \Rightarrow k^2 = \left(\frac{f_1(x,y)}{f_0(x,y)}\right)^2$$

is the conserved quantity (case ii-2).

Here $\dim|-K_X| = 0$ and $\dim|-2K_X| = 1$

$$|-2K_X|: \quad \begin{aligned} 0 = \;& kf_0(x,y) - f_1(x,y) \\ = \;& kx^2y^2 - \Big(ix(x+1)^2 - i(x+i)(x^2-1)y \\ & +b(x-1)^2y^2\Big)\Big(-ix(y-1) + y(by-1)\Big) \end{aligned}$$

and again we have exchanging of fibers:

$$k = \frac{f_1(x,y)}{f_0(x,y)} \to -k \Rightarrow k^2 = \left(\frac{f_1(x,y)}{f_0(x,y)}\right)^2$$

is the conserved quantity (case ii-2).

- A. S. Carstea, T. Takenawa, *A classification of two dimensional integrable mappings and rational elliptic surfaces*, preprint, submitted to Nonlinearity, (also on arXiv)
- Sakai, H. *Rational surfaces associated with affine root systems and geometry of the Painleve equations.* Comm. Math. Phys. 220, (2001) 165–229
- Diller, J., Fravre, Ch. *Dynamics of bimeromorphic maps of surfaces*, American Journal of Mathematics 123, (2001),1135–1169
- A. S. Carstea, B. Grammaticos, A. Ramani, *Deautonomizing integrable non-QRT mappings* J. Phys. A: Math. Gen. 42, (2009) article number 485207
- Tsuda, T., Grammaticos, B., Ramani, A., Takenawa, T. *A class of integrable and nonintegrable mappings and their dynamics* Lett. Math. Phys. 82 (2007) 39–49