

# Shades of a dream: String Theory, Mathematics and the Unity of Science

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The laws of Nature are written in the language of mathematics ... the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word.

*Galileo Galilei*

Whence arises all that order and beauty we see in the world?

*Isaac Newton, "Opticks"*

The eternal mystery of the world is its comprehensibility [...] The fact that it is comprehensible is a miracle.

*Albert Einstein, "Physics and Reality" (1936)*

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

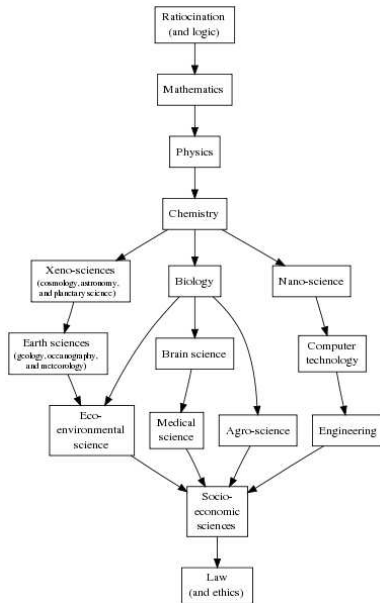
*Eugene Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", Comm. in Pure Appl. Math. 13 (1960)*



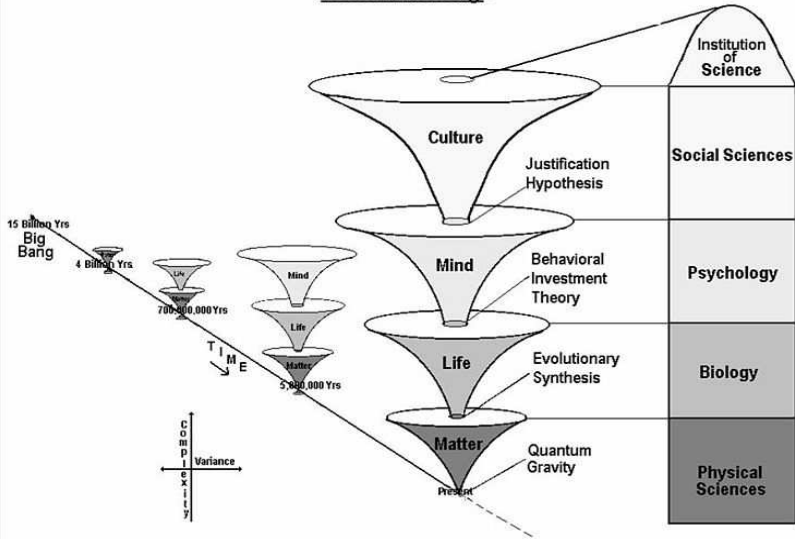
And although learned men have long since thought of some kind of language or universal characteristic by which all concepts and things can be put into beautiful order [...] yet no one has attempted a language or characteristic which includes at once both the arts of discovery and judgement, that is, one whose signs and characters serve the same purpose that arithmetical signs serve for numbers, and algebraic signs for quantities taken abstractly. Yet it does seem that since God has bestowed these two sciences on mankind, he has sought to notify us that a far greater secret lies hidden in our understanding, of which these are but the shadows.

*Gottfried Wilhelm Leibniz, "Zur allgemeinen Charakteristik" (1679)*

# Reductionism ?



# Tree of Knowledge

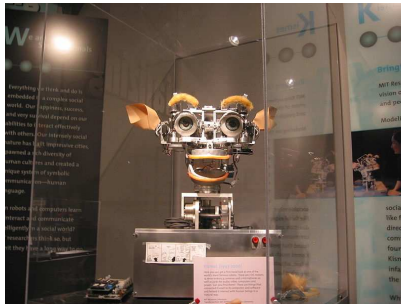


# Modeling the mind ?



The Blue Brain Project is an attempt to create a synthetic brain by reverse-engineering the mammalian brain down to the molecular level. The aim of the project, founded in May 2005 by the Brain and Mind Institute of the Ecole Polytechnique Federale de Lausanne (Switzerland) is to study the brain's architectural and functional principles. Using a Blue Gene supercomputer running Michael Hines's NEURON software, the simulation does not consist simply of an artificial neural network, but involves a biologically realistic model of neurons. There are a number of sub-projects, including the Cajal Blue Brain, coordinated by the Supercomputing and Visualization Center of Madrid (CeSViMa), and others run by universities and independent laboratories.

# Creating minds ?



Kismet speaks a proto-language with a variety of phonemes, similar to baby's babbling. It uses the DECTalk voice synthesizer, and changes pitch, timing, articulation, etc. to express various emotions. Intonation is used to vary between question and statement-like utterances.

# Which language ?

If “Mathematics is the language of nature”, then *what* is the language of Mathematics ?

- Logic ?
- String Manipulation, formal systems ?
- Realism ?
- Intuitionism ? Constructivism ?
- Conventionalism ? Psychologism ?
- Platonism ?
- Structuralism ?

Frege, Russell, Whitehead, Goedel, Hilbert, Carnap, Tarski, Curry, Turing, Church, Skolem ... Explosion of models, approaches and issues.

Problems:

- Distinguish language from meta-language
- Distinguish language from interpretation
- Distinguish symbol from referent (syntax vs. semantics)
- Distinguish 'truth value' from correctness and ontology

Most issues *can* be avoided with sufficient care. *Not* an insurmountable obstacle. A reasonable (“structuralist”) basis is provided by *category theory* and its variations.



# Category theory

- Objects and morphisms. Morphisms compose.
- Common throughout Mathematics, needs no explanation.
- Formal development originated in topology (Samuel Eilenberg, Saunders MacLane)
- Provides novel perspectives on foundations of Mathematics (Grothendieck's topos theory)

**Crucial observation.** Category theory is useful and natural in topology and geometry (algebraic, non-commutative etc) and in quantum physics (especially in quantum gravity) since **it is not based on set theory and thus does not rely on the notion of a 'point', thereby avoiding the conflict between the continuum models used in classical physics and the need to quantize space-time in such theories.**

This point of view was pioneered in algebraic geometry by A. Grothendieck.

# What is a category ?

Essentially, it is a *monoid with many objects*:

- Collection (class)  $\text{Ob}\mathcal{A}$  of *objects*  $A, B, C \dots$
- Collection (class)  $\text{Mor}\mathcal{A}$  of *morphisms*  $f, g, h, \dots$
- Maps  $h, t : \text{Mor}\mathcal{A} \rightarrow \text{Ob}\mathcal{A}$
- Partially-defined composition  $\circ : \text{Mor}\mathcal{A} \times_{\text{Ob}\mathcal{A}} \text{Mor}\mathcal{A} \rightarrow \text{Mor}\mathcal{A}$  which is associative and unital (units are morphisms  $\text{id}_A : A \rightarrow A$  where  $A$  is any object).

Certain useful generalizations exist (especially useful for ‘algebraic homotopy theory’, a subject initiated by Quillen, Stasheff etc.). These are also useful in Physics, since homotopy plays an important role in physical processes:

- $A_\infty$  categories
- Higher categories

# Quantum gravity

Two fundamental insights of 20th century physics which need to be combined ('unified'):

- (Quantum mechanics): The physical world is **quantum** → formulation through  $C^*$ -algebras (starting with von Neumann).
- (General relativity): Gravity is **geometric** → formulation through pseudo-Riemannian manifolds

Combining these insights is *highly nontrivial*, since we have to satisfy many constraints imposed by experiment, mathematical consistency and sheer common sense. We don't yet have an entirely satisfactory answer, but have a good candidate in **string theory**. There is a *huge* amount of Physics and Mathematics background which enters into the construction of string theory. It has the ambition to be a 'theory of all fundamental physics'.

Einstein, Fermi, Dirac, Feynman ... Witten + many others.

# Quantum mechanics is categorical

One can formulate quantum mechanics in category-theoretic language ('categorical quantum mechanics'):

- Objects are density operators  $\rho$  on a separable Hilbert space  $\mathcal{H}$  (non-negative operators with unit trace, i.e. 'mixed states'; this is the 'density matrix formalism' of von Neumann)
- Morphisms ('quantum processes' or 'quantum operations') are linear maps  $\Phi$  between spaces of trace-class operators such that:
  - $\text{Tr}\Phi(\rho) \leq 1$  if  $\rho$  is a density operator
  - $\Phi$  is completely positive, i.e.  $\Phi \otimes I_n$  is positive for all  $n \geq 1$ .

Various frameworks recover increasingly larger parts of the formalism of quantum physics, for example 'dagger symmetric monoidal categories' (Abramsky et al). Morphisms describe both unitary operations (such as adiabatic evolution) and measurement processes. Formalism can be adapted to space-time local theories such as quantum field theory on curved space-times and thus can be used in quantum gravity.

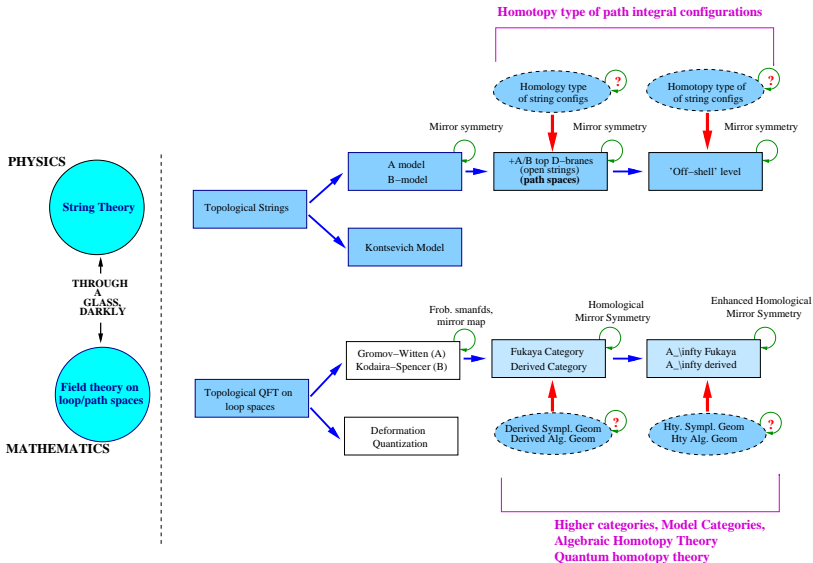
# Homotopical quantum theory

Refinement of the above which takes into account the role of symmetries and constraints, which is crucial especially in quantum field theory and string theory. This is one of the programs being pursued at IBS-CGP.

Basic idea is to construct a model category (and its infinity-category counterparts) capturing the dynamics of such systems. Deep connections with homological algebras, BV-BRST quantization, topology and algebraic geometry.

We believe that this is the fundamental theory which explains and unifies numerous phenomena which were observed before by various people within the broad realm of 'stringy geometry' – in particular, the appearance of various types of homotopy and derived categories in homological mirror symmetry, topological defect theories and the classification of D-branes (which has been one of the themes of my own work over the years).

# A snapshot of 'String Geometry'



# Scientific influences



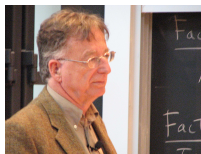
Brian Greene



John Morgan



Martin Rocek



Dennis Sullivan



Ed Witten



M. Kontsevich



Jim Stasheff



Kenji Fukaya

# Perturbative String Theory

An attempt to quantize fluctuations of gravity around a given background space-time  $M$  by postulating that:

- (a) The fundamental constituents of nature are one-dimensional objects (connected smooth curves) embedded in  $M$ , which are allowed to deform and to move inside  $M$ .
- (b) Interactions between strings are described through splitting-joining processes, with the assumption that the union of the world trajectories of all strings involved in any such process forms the image  $\varphi(\Sigma)$  of a smooth embedding  $\varphi$  of a (non-necessary connected, non-necessarily oriented) compact two-manifold with corners  $\Sigma$  into the background space-time.
- (c) The fundamental objects of interest are the *string amplitudes*  $\mathcal{A}_\alpha$  associated to the oriented diffeomorphism class  $\alpha$  of the surface  $\Sigma$  (relative to its frontier). These are constructed in a model-dependent fashion.



# String amplitudes

$h$  a Riemannian metric on  $\Sigma$

$\varphi$  the map embedding  $\Sigma$  into  $M$

$q$  fields  $(q_j)$  on  $\Sigma$ , valued in  $\varphi$ -pullbacks of 'god-given' fiber bundles over  $M$

$S_\alpha$  the action of this two-dimensional Euclidean field theory (the *Polyakov action*).

$\mathcal{A}_\alpha$  the partition function of this Euclidean field theory:

$$\mathcal{A}_\alpha := \int \mathcal{D}[h, q] e^{-S_\alpha[h, q]} \quad (1)$$

**sigma model** the model which result by fixing  $\alpha$  and  $h$

**conformal model** model for which  $\mathcal{D}[h, q] e^{-S_\alpha[h, q]}$  is invariant with respect to the conformal group of  $(\Sigma, h)$ . Partial gauge-fixing gives the description **CFT (matter) + Liouville + ghosts**, the last two giving an integration measure for the matter sector over the moduli space  $\mathcal{M}_\alpha$  of conformal structures on  $\Sigma$ .

**topological model** model in which  $\mathcal{D}[h, q] e^{-S_\alpha[h, q]}$  is invariant under the diffeomorphism group of  $\Sigma$  (then  $S_\alpha[h, q]$  is also described as a *topological sigma model coupled to topological gravity*). Partial gauge fixing gives description as **topological sigma model + integration measure over  $\mathcal{M}_\alpha$** . Connection to various versions of Mumford's compactification of the moduli space of closed Riemann surfaces.

# Theory types

The topologies allowed for  $\Sigma$  restrict the theory. One has a cobordism category with objects given by segments and circles and morphisms given by isotopy classes of Riemann surfaces with corners, possibly considered together with certain involutive diffeomorphisms (“orientifolds”).

**Oriented strings** restrict to *oriented* surfaces  $\Sigma$  (otherwise, theory is called *unoriented*)

**Closed strings** restrict to *bordered* surfaces  $\Sigma$  (no corners allowed)

**Open strings** restrict to *planar* surfaces  $\Sigma$

**D-branes** can be described by decoration data on the segments and circles of the cobordism category (each decoration corresponding to a different type of boundary condition for the worldsheet fields).

# Superconformal worldsheet models

Models for which  $\mathcal{D}[h, q]e^{-S[h, q]}$  is invariant with respect to a 'local superextension' of the conformal group of  $(\Sigma, h)$ . There exists a correlation between the amount of worldsheet supersymmetry and the amount of supersymmetry of the resulting spacetime effective action (Friedan, Martinec & Shenker, 1984). The conformal anomaly cancels in the quantized two-dimensional model for a specific dimension known as the critical dimension.

$0_{d=2}$  bosonic string; critical dimension  $D = 26$ .

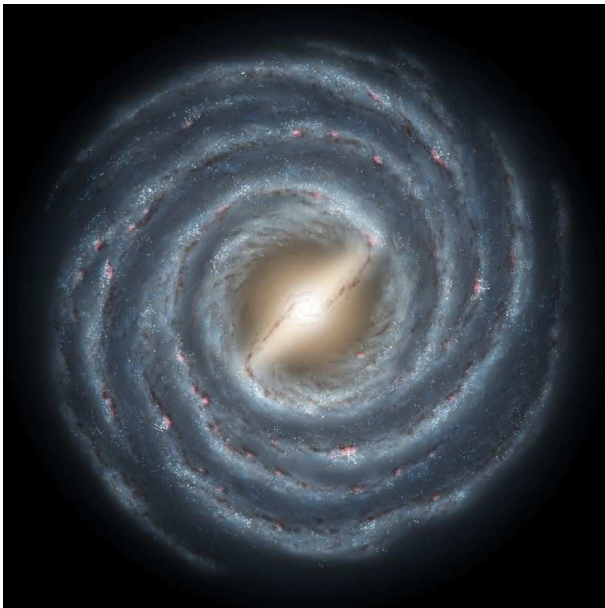
$(2, 2)_{d=2}$  superstring (open, closed; oriented or unoriented); critical dimension  $D = 10$ . Via GSO projections and orientifolding, this gives the type IIA, type IIB and type I string theories.

$2_{d=2}$  **het** heterotic string (with a special construction using a bosonic string/superstring for left/right movers respectively, most generally allowing  $(0, 2)$  worldsheet SUSY]: versions:  $E_8 \times E_8$  and  $\text{Spin}(32)/\mathbb{Z}_2$ .

**other** Allowed in non-critical space-time dimensions (e.g.  $D = 2$ ), when the Liouville sector makes sense on its own; various slight variants also allowed by including fluxes, massive space-time backgrounds etc.

# Topological A/B strings

- Obtained by 'twisting' the usual oriented superstring model (open or closed), which amounts to the modification  $T \rightarrow T(z) \pm \frac{1}{2}J(z)$  for the conformal stress-energy tensor. Using the same sign on left and right movers gives the A-model, while opposite signs on the two types of movers gives the B-model.
- The anomaly of the twisted  $N = 2$  superconformal algebra cancels when the target space is a Calabi-Yau threefold  $X$  (though some slight extensions are possible).
- The perturbation expansion is similar to that of the bosonic string. This allows one to construct A and B-type *topological string field theories*. In the open string sector, these admit a cubic formulation (Witten).
- Localization arguments show that Witten's string field action reduces to an action of generalized Chern-Simons type, but including all boundary sectors (all possible D-branes) is non-trivial.
- The result is that topological B-type branes are the objects of the bounded derived category  $D^b(X)$  of  $X$  while topological A-type branes are objects of the bounded derived version  $D^b(\text{Fuk}(X))$  of a certain  $A_\infty$  category  $\text{Fuk}(X)$  (the Fukaya category) associated with  $X$ .
- Physics predicts that the string field action of the A-twisted string on  $X$  is BV-equivalent with the string field action of the twisted B-twisted string on the mirror  $\check{X}$  of  $X$ , which implies that a certain  $A_\infty$  enhancement of  $D^b(X)$  (built using Dolbeault resolutions) is quasi-equivalent with  $\text{Fuk}(\check{X})$  (strong homological mirror symmetry). This implies an exact equivalence between  $D^b(X)$  and  $D^b(\text{Fuk}(\check{X}))$  (weak homological mirror symmetry).
- The HMS conjecture can be extended to non-compact cases and to non-Calabi-Yau models, by considering linear sigma models. Some very simple cases of this were proved during the past 4 years.



As time permits: higher cats, Igusa, Lurie, Chen iterated integrals ...