The geometric algebra of metric cones and supersymmetry

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Setting

- (M, g) = pseudo-Riemannian manifold of *even* dimension d
- $\hat{M}$ =manifold diffeomorphic with $\mathbb{R} \times M$ (on which we shall consider the cylinder and cone metrics, respectively).
- (p, q)=the signature type of the metric g on M;
- $\dim \hat{M} = d + 1$; both cone and cylinder metric on $\hat{M}$ have signature type $(p + 1, q)$.
- Also assume that $\mathrm{Cl}_{\mathbb{K}}(p + 1, q)$ is *non-simple* and that its Schur algebra equals $\mathbb{K}$, i.e.:

  (A) $\mathbb{K} = \mathbb{C}$,
  or
  (B) $\mathbb{K} = \mathbb{R}$ and $p - q \equiv_8 0$.

  Then $\mathrm{Cl}_{\mathbb{K}}(p, q)$ is simple and its Schur algebra also equals $\mathbb{K}$. We further assume that M is oriented and on $\hat{M}$ we choose the orientation compatible with that of M.

Preparations

On $\hat{M}$, consider the cylinder metric $g_{\rm cyl}$ whose squared line element takes the form:

$$\mathrm{d}s^2_{\rm cyl} = \mathrm{d}u^2 + \mathrm{d}s^2 \qquad (u \in \mathbb{R}) \quad ,$$

where $\mathrm{d}s^2$ is the squared line element of g. This is related by a conformal transformation to the cone metric $g_{\rm cone}$ on $\hat{M}$, whose squared line element is given by:

$$\mathrm{d}s^2_{\rm cone} = \mathrm{d}r^2 + r^2\mathrm{d}s^2 = r^2\mathrm{d}s^2_{\rm cyl} \quad (r \stackrel{\rm def.}{=} e^u \in (0, +\infty)) \quad .$$

We have $g_{\rm cone} = r^2 g_{\rm cyl}$ and $\hat{g}_{\rm cone} = \frac{1}{r^2}\hat{g}_{\rm cyl}$, where we view u and $r = e^u$ as smooth functions defined on $\hat{M}$, namely $u \in \mathcal{C}^\infty(\hat{M}, \mathbb{R})$ and $r \in \mathcal{C}^\infty(\hat{M}, (0, +\infty)) \subset \mathcal{C}^\infty(\hat{M}, \mathbb{R})$. The transformation $u \to r$ maps the limit $u \to -\infty$ to the limit $r \to 0$. Unless M is a sphere, the cone metric is not complete due to the conical singularity which arises when one attempts to add the point at $r = 0$. For any vector field $V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$ and any one-form $\eta \in \Gamma(\hat{M}, T^*_{\mathbb{K}}\hat{M}) = \Omega^1_{\mathbb{K}}(\hat{M})$, we have $V_{\#_{\rm cone}} = r^2 V_{\#_{\rm cyl}}$ and $\eta^{\#_{\rm cone}} = \frac{1}{r^2}\eta^{\#_{\rm cyl}}$, where $\#_{\rm cyl}$ and $\#_{\rm cone}$ are the musical isomorphisms of the cylinder and cone, respectively.

Preparations

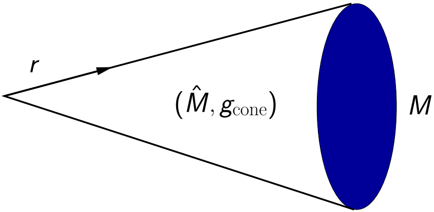


Figure: Metric cone over M

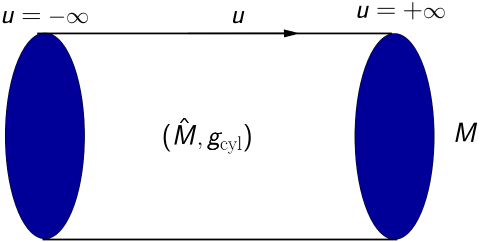


Figure: Metric cylinder over M

The ring $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K})$

We let $\Pi : \hat{M} \to M$ be the projection on the second factor. For later reference, consider the following unital subring of the commutative ring $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$:

$$\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K}) \stackrel{\mathrm{def.}}{=} \{f \circ \Pi | f \in \mathcal{C}^\infty(M, \mathbb{K})\} \subset \mathcal{C}^\infty(\hat{M}, \mathbb{K}) \quad .$$

It coincides with the image $\Pi^*(\mathcal{C}^\infty(M, \mathbb{K}))$ through the pullback map Π^*, which acts as follows on smooth functions defined on M:

$$\Pi^*(f) = f \circ \Pi \in \mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K}) \quad , \quad \forall f \in \mathcal{C}^\infty(M, \mathbb{K}) \quad .$$

In fact, Π^* corestricts to a unital isomorphism of rings:

$$\mathcal{C}^\infty(M, \mathbb{K}) \stackrel{\Pi^*|^{\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K})}}{\longrightarrow} \mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K}) \quad ,$$

which allows us to identify $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K})$ with $\mathcal{C}^\infty(M, \mathbb{K})$.

The canonical normalized one-forms

The one-form:

$$\psi = \mathrm{d}u = \frac{1}{r}\mathrm{d}r$$

has unit norm with respect to the cylinder metric, being dual to the unit norm vector field $\psi^{\#_{\mathrm{cyl}}} = \partial_u = r\partial_r$ with respect to the metric g_{cyl}:

$$\psi = \partial_u \lrcorner g_{\mathrm{cyl}} \quad .$$

Similarly, the one-form:

$$\theta = \mathrm{d}r = r\psi$$

has unit norm with respect to the cone metric, being dual to the unit norm vector field $\theta^{\#_{\mathrm{cone}}} = \partial_r$ with respect to the metric g_{cone}:

$$\theta = \partial_r \lrcorner g_{\mathrm{cone}} \quad .$$

The Euler operator

The Euler operator $\mathcal{E} = \oplus_{k=0}^{d+1} k \ \mathrm{id}_{\Omega^k_{\mathbb{K}}(\hat{M})}$ acts as follows on a general inhomogeneous form:

$$\mathcal{E}(\omega) = \sum_{k=0}^{d+1} k\omega^{(k)} \ , \ \ \forall \omega = \sum_{k=0}^{d+1} \omega^{(k)} \in \Omega_{\mathbb{K}}(\hat{M}) \ \ \mathrm{with} \ \ \omega^{(k)} \in \Omega^k_{\mathbb{K}}(\hat{M}) \ .$$

The scaling operators $\lambda^{\mathcal{E}}$ $(\lambda > 0)$ act as:

$$\lambda^{\mathcal{E}}(\omega) = \sum_{k=0}^{d+1} \lambda^k \omega^{(k)} \ , \ \ \forall \omega = \sum_{k=0}^{d+1} \omega^{(k)} \in \Omega_{\mathbb{K}}(\hat{M}) \ \ \mathrm{with} \ \ \omega^{(k)} \in \Omega^k_{\mathbb{K}}(\hat{M}) \ .$$

Using the definition of generalized products, we find:

$$\triangle_p^{\mathrm{cone}} = \frac{1}{r^{2p}} \triangle_p^{\mathrm{cyl}} \quad , \quad \forall p = 0 \ldots d+1 \quad .$$

These identities imply:

$$r^{\mathcal{E}} \circ \triangle_p^{\mathrm{cyl}} = \frac{1}{r^{2p}} \triangle_p^{\mathrm{cyl}} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow r^{\mathcal{E}}(\omega \triangle_p^{\mathrm{cyl}} \eta) = \frac{1}{r^{2p}} [r^{\mathcal{E}}(\omega) \triangle_p^{\mathrm{cyl}} r^{\mathcal{E}}(\eta)] \ , \ \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M}) \ ,$$
$$r^{\mathcal{E}} \circ \triangle_p^{\mathrm{cone}} = \frac{1}{r^{2p}} \triangle_p^{\mathrm{cone}} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow r^{\mathcal{E}}(\omega \triangle_p^{\mathrm{cone}} \eta) = \frac{1}{r^{2p}} [r^{\mathcal{E}}(\omega) \triangle_p^{\mathrm{cone}} r^{\mathcal{E}}(\eta)] \ , \ \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M}) \ .$$

and

$$r^{\mathcal{E}} \circ \diamond^{\mathrm{cyl}} = \diamond^{\mathrm{cone}} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow r^{\mathcal{E}}(\omega \diamond^{\mathrm{cyl}} \eta) = r^{\mathcal{E}}(\omega) \diamond^{\mathrm{cone}} r^{\mathcal{E}}(\eta) \ , \ \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})$$

The Kähler-Atiyah algebra

Proposition. The maps $r^{\mathcal{E}}$ and $r^{-\mathcal{E}}$ are mutually inverse $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the Kähler-Atiyah algebras of the cylinder and cone:

$$(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}}{\overset{r^{\mathcal{E}}}{\rightleftarrows}} (\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}}) \quad .$$

Corollary. The maps $r^{\mathcal{E}}$ and $r^{-\mathcal{E}}$ restrict to mutually inverse unital isomorphisms between the algebras $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$ and $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})$:

$$(\Omega^{\perp}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}} (\Omega^{\perp}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}}) \quad .$$

The special and vertical subalgebras

One can show that $\mathcal{L}_{\partial_u}$ is an even $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear derivation of the Kähler-Atiyah algebra $(\Omega_\mathbb{K}(\hat{M}),\diamond^{\mathrm{cyl}})$:

$$\mathcal{L}_{\partial_u}\circ\diamond^{\mathrm{cyl}}=\diamond^{\mathrm{cyl}}\circ\left(\mathcal{L}_{\partial_u}\otimes\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}+\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}\otimes\mathcal{L}_{\partial_u}\right)~~.$$

This implies that the operator $\mathcal{L}_{\partial_u}-\mathcal{E}$ is a degree zero $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear derivation of all generalized products of the cone:

$$(\mathcal{L}_{\partial_u}-\mathcal{E})\circ\triangle_p^{\mathrm{cone}}=\triangle_p^{\mathrm{cone}}\circ\left[(\mathcal{L}_{\partial_u}-\mathcal{E})\otimes\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}+\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}\otimes(\mathcal{L}_{\partial_u}-\mathcal{E})\right]$$

and hence of the Kähler-Atiyah algebra $(\Omega_\mathbb{K}(\hat{M}),\diamond^{\mathrm{cone}})$:

$$(\mathcal{L}_{\partial_u}-\mathcal{E})\circ\diamond^{\mathrm{cone}}=\diamond^{\mathrm{cone}}\circ\left[(\mathcal{L}_{\partial_u}-\mathcal{E})\otimes\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}+\mathrm{id}_{\Omega_\mathbb{K}(\hat{M})}\otimes(\mathcal{L}_{\partial_u}-\mathcal{E})\right]~~.$$

In particular, the following subspaces of $\Omega_\mathbb{K}(\hat{M})$:

$$\Omega_\mathbb{K}^{\mathrm{cyl}}(\hat{M})\stackrel{\mathrm{def.}}{=}\mathcal{K}(\mathcal{L}_{\partial_u})~~,~~\Omega_\mathbb{K}^{\mathrm{cone}}(\hat{M})\stackrel{\mathrm{def.}}{=}\mathcal{K}\left(\mathcal{L}_{\partial_u}-\mathcal{E}\right)$$

are unital $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras of the Kähler-Atiyah algebras $(\Omega_\mathbb{K}(\hat{M}),\diamond^{\mathrm{cyl}})$ and $(\Omega_\mathbb{K}(\hat{M}),\diamond^{\mathrm{cone}})$, which we shall call the *special subalgebras* of the cylinder and cone, respectively.

The special and vertical subalgebras

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the special subalgebras of the cylinder and cone:

$$(\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}}) \quad \underset{r^{-\mathcal{E}}|_{\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}} \quad (\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}}) \quad .$$

The subspace:

$$\Omega^{\perp}_{\mathbb{K}}(\hat{M}) \overset{\mathrm{def.}}{=} \{\omega\in\Omega_{\mathbb{K}}(\hat{M})|\partial_u\lrcorner\omega=0\}=\{\omega\in\Omega_{\mathbb{K}}(\hat{M})|\partial_r\lrcorner\omega=0\}$$

is a unital $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-subalgebra of both $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})$. Therefore, the intersections:

$$\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \overset{\mathrm{def.}}{=} \Omega^{\perp}_{\mathbb{K}}(\hat{M})\cap\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \;\;,\;\; \Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \overset{\mathrm{def.}}{=} \Omega^{\perp}_{\mathbb{K}}(\hat{M})\cap\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})$$

are unital $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})$ respectively (the *vertical subalgebras* of the cylinder and cone). The operator $r^{\mathcal{E}}$ satisfies:

$$r^{\mathcal{E}}(\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}))=\Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \quad .$$

The special and vertical subalgebras

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the vertical subalgebras of the cylinder and cone:

$$(\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}),\diamond^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}|_{\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})}}{\rightleftarrows}} (\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M}),\diamond^{\mathrm{cone}}) \quad .$$

Special twisted (anti)selfdual forms

Definition. The subalgebras of *special twisted (anti-)selfdual forms* are the following $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras of the Kähler-Atiyah algebras of the cylinder and of the cone:

$$\Omega^{\pm,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega^{\pm}_{\mathbb{K},\mathrm{cyl}}(\hat{M})\cap\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M})~,~~\Omega^{\pm,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega^{\pm}_{\mathbb{K},\mathrm{cone}}(\hat{M})\cap\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})~.$$

These algebras have units $p^{\mathrm{cyl}}_\pm=\frac{1}{2}(1\pm\nu^{\mathrm{cyl}})$ and $p^{\mathrm{cone}}_\pm=\frac{1}{2}(1\pm\nu^{\mathrm{cone}})$, respectively. Combining the observations above gives:

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the subalgebras of special twisted selfdual/anti-selfdual forms of the cylinder and cone:

$$(\Omega^{\pm,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}|_{\Omega^{\pm,\mathrm{cone}}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\pm,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}} (\Omega^{\pm,\mathrm{cone}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})~.$$

The $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebra $(\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$ can be identified with the Kähler-Atiyah algebra $(\Omega_{\mathbb{K}}(M),\diamond)$ as follows. Let $\Pi:\hat{M}\rightarrow M$ be the projection on the second factor.

Proposition. The pullback map $\Pi^\ast:\Omega_{\mathbb{K}}(M)\rightarrow\Omega_{\mathbb{K}}(\hat{M})$ has image equal to $\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})$. Furthermore, its corestriction to this image (which we again denote by $\Pi^\ast$) is a unital $\mathcal{C}^\infty(M,\mathbb{K})$-linear isomorphism of algebras from $(\Omega_{\mathbb{K}}(M),\diamond)$ to the vertical subalgebra $(\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$ of the cylinder, provided that we identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})\approx\mathcal{C}^\infty(M,\mathbb{K})$. The inverse of this isomorphism is the pullback map $j^\ast$, where $j:M\hookrightarrow\hat{M}$ is the embedding of M as the section $r=1$ of $\hat{M}$. Thus, we have mutually inverse unital isomorphisms of $\mathcal{C}^\infty(M,\mathbb{K})\approx\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras:

$$(\Omega_{\mathbb{K}}(M),\diamond)\ \underset{j^\ast|_{\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\overset{\Pi^\ast|^{\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}}\ (\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})\ .$$

Recovering the Kähler-Atiyah algebra of (M, g)

Proposition. We have mutually-inverse unital isomorphisms of $\mathbb{K}$-algebras:

$$(\Omega_{\mathbb{K}}(M), \diamond) \quad \underset{j^* \circ\, r^{-\mathcal{E}}|_{\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})}}{\overset{r^{\mathcal{E}} \circ\, \Pi^*|^{\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})}}{\rightleftarrows}} \quad (\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M}), \diamond^{\mathrm{cone}}) ~.$$

Thus $\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})$ consists of those inhomogeneous forms on $\hat{M}$ which are Π-pullbacks of inhomogeneous forms ω on M; this pullback will be called the *cylinder lift* ω_{cyl} of ω:

$$\omega_{\mathrm{cyl}} \overset{\mathrm{def.}}{=} \Pi^*(\omega) \in \Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}) ~~,~~ \forall \omega \in \Omega_{\mathbb{K}}(M) ~.$$

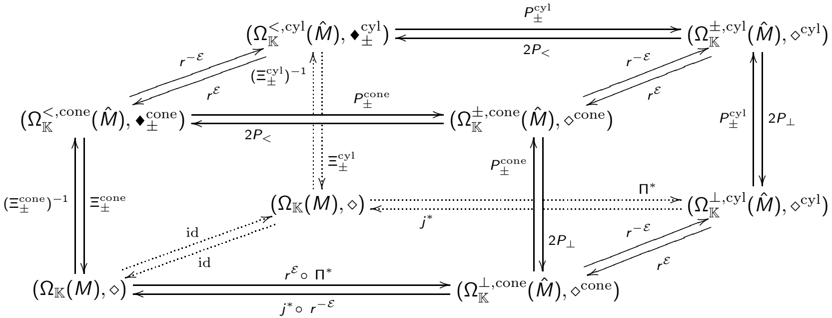
Similarly, $\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(M)$ consists of *cone lifts*:

$$\omega_{\mathrm{cone}} \overset{\mathrm{def.}}{=} r^{\mathcal{E}}(\Pi^*(\omega)) \in \Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M}) ~~,~~ \forall \omega \in \Omega_{\mathbb{K}}(M) ~~,$$

which are inhomogeneous forms of the type:

$$\omega_{\mathrm{cone}} = r^{\mathcal{E}}(\Pi^*(\omega)) = \sum_{k=0}^{d} r^k \Pi^*(\omega^{(k)}) ~,~ \forall \omega = \sum_{k=0}^{d} \omega^{(k)} ~,~ \omega^{(k)} \in \Omega_{\mathbb{K}}^{k}(M) ~.$$

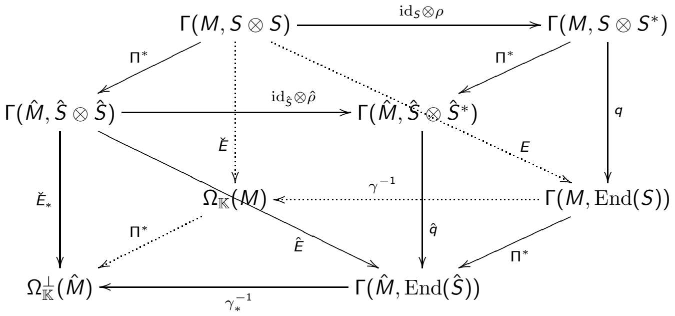
The full collection of isomorphic models of the Kähler-Atiyah algebra of (M, g) (viewed as a $\mathbb{K}$-algebra) which arise from the cone and cylinder constructions is summarized in the commutative diagram below:

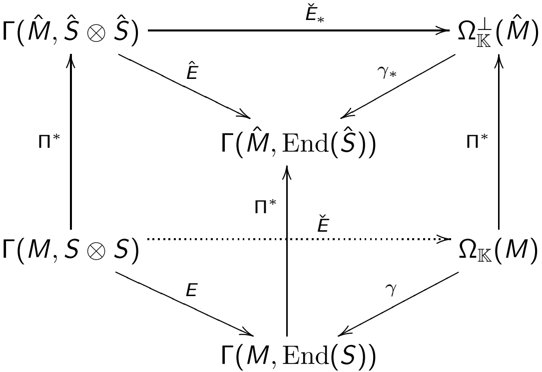


$$
\begin{array}{ccccc}
\Omega_{\mathbb{K}}(\hat{M}) & & \xrightarrow{\quad r^{\mathcal{E}} \quad} & & \Omega_{\mathbb{K}}(\hat{M}) \\
& \searrow^{\gamma_{\mathrm{cyl}}} & & \swarrow^{\gamma_{\mathrm{cone}}} & \\
\Big\downarrow \varphi_{\epsilon}^{\mathrm{cyl}} & & \Gamma(\hat{M}, \mathrm{End}(\hat{S})) & & \Big\downarrow \varphi_{\epsilon}^{\mathrm{cone}} \\
& \nearrow^{\gamma_{*}} & & \nwarrow^{\gamma_{*}\circ\, r^{-\mathcal{E}}} & \\
\Omega_{\mathbb{K}}^{\perp}(\hat{M}) & & \xrightarrow{\quad r^{\mathcal{E}} \quad} & & \Omega_{\mathbb{K}}^{\perp}(\hat{M})
\end{array}
$$

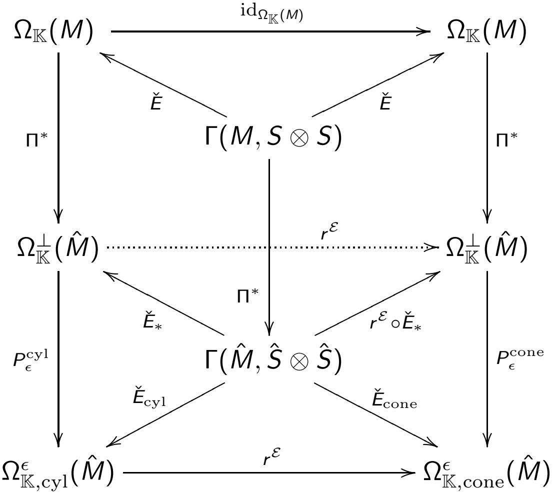
$$
\begin{array}{ccc}
\Gamma(\hat{M}, \mathrm{End}(\hat{S})) & \xrightarrow{\gamma_{\mathrm{cone}}^{-1}} & \Omega_{\mathbb{K},\mathrm{cone}}^{\epsilon}(\hat{M}) \\
\Big\downarrow \gamma_{*}^{-1} & \searrow^{\gamma_{\mathrm{cyl}}^{-1}} & \Big\uparrow r^{\mathcal{E}} \\
\Omega_{\mathbb{K}}^{\perp}(\hat{M}) & \xrightarrow[P_{\epsilon}^{\mathrm{cyl}}]{} & \Omega_{\mathbb{K},\mathrm{cyl}}^{\epsilon}(\hat{M})
\end{array}
$$

The Fierz Isomorphism of cylinders and cones

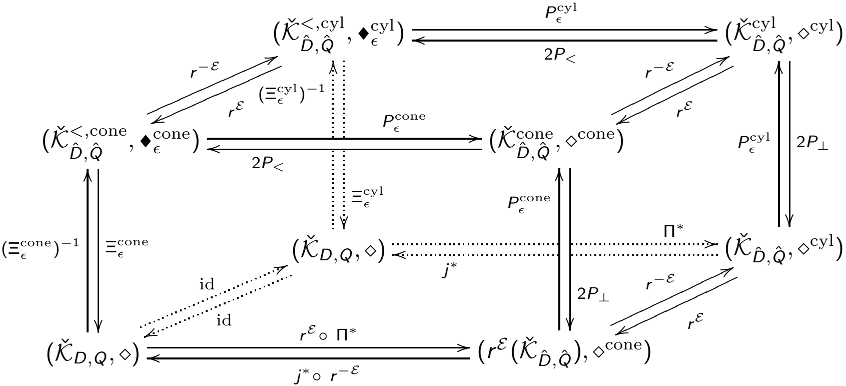


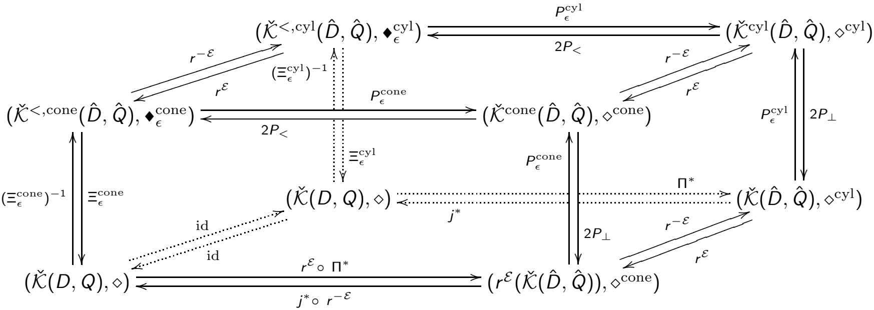

$\Gamma(\hat{M}, \hat{S} \otimes \hat{S})$
$\check{E}_*$
$\Omega^{\perp}_{\mathbb{K}}(\hat{M})$
$\hat{E}$
γ_*
$\Gamma(\hat{M}, \mathrm{End}(\hat{S}))$
Π^*
Π^*
Π^*
$\check{E}$
$\Gamma(M, S \otimes S)$
$\Omega_{\mathbb{K}}(M)$
E
γ
$\Gamma(M, \mathrm{End}(S))$

The pull-back of pinors



The supersymmetry conditions (CGK pinor equations)




$(\check{\mathcal{K}}^{<,\mathrm{cyl}}(\hat{D},\hat{Q}),\blacklozenge_\epsilon^{\mathrm{cyl}})$
$P_\epsilon^{\mathrm{cyl}}$
$2P_<$
$(\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D},\hat{Q}),\diamond^{\mathrm{cyl}})$
$r^{-\mathcal{E}}$
$r^{\mathcal{E}}$
$(\Xi_\epsilon^{\mathrm{cyl}})^{-1}$
$P_\epsilon^{\mathrm{cone}}$
$r^{-\mathcal{E}}$
$r^{\mathcal{E}}$
$(\check{\mathcal{K}}^{<,\mathrm{cone}}(\hat{D},\hat{Q}),\blacklozenge_\epsilon^{\mathrm{cone}})$
$(\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D},\hat{Q}),\diamond^{\mathrm{cone}})$
$P_\epsilon^{\mathrm{cyl}}$
$2P_\perp$
$2P_<$
$\Xi_\epsilon^{\mathrm{cyl}}$
$P_\epsilon^{\mathrm{cone}}$
Π^*
$(\Xi_\epsilon^{\mathrm{cone}})^{-1}$
$\Xi_\epsilon^{\mathrm{cone}}$
$(\check{\mathcal{K}}(D,Q),\diamond)$
$(\check{\mathcal{K}}(\hat{D},\hat{Q}),\diamond^{\mathrm{cyl}})$
j^*
id
$r^{-\mathcal{E}}$
$2P_\perp$
id
$r^{\mathcal{E}}$
$r^{\mathcal{E}}\circ\Pi^*$
$(\check{\mathcal{K}}(D,Q),\diamond)$
$(r^{\mathcal{E}}(\check{\mathcal{K}}(\hat{D},\hat{Q})),\diamond^{\mathrm{cone}})$
$j^*\circ r^{-\mathcal{E}}$