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Multisoliton Interactions for the Perturbed Manakov System

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I am grateful to professors A. Isar, M. Visinescu and A. Visinescu for their kind hospitality, giving me the chance to be here

PLAN

- Adiabatic N-soliton interactions of Bose-Einstein condensates in external potentials
- Derivation of generalized complex Toda chain (GCTC) for the MNLS through the variational approach
- Perturbed Manakov model and perturbed GCTC.
- Modeling Soliton Interactions of the perturbed vector nonlinear Schrödinger equation

We consider the dynamics of a train of matter - wave solitons in a one-dimensional BEC confined to a parabolic trap and optical lattice, as well as tilted periodic potentials. In the last case we demonstrate that there exist critical values of the strength of the linear potential for which one or more localized states can be extracted from a soliton train.

This is described by perturbed **scalar** NLS:

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u(x,t) = V(x)u(x,t),$$

$$V(x) = V_2 x^2 + V_1 x + V_0 + A_1 \cos(\Omega_1 x + \Omega_0).$$

Adiabatic approach to soliton interactions:
Karpman, Solov'ev, Maslov (1981) for two solitons
VSG, I. Uzunov, D. J. Kaup, ... (1995-1998) for N-soliton trains
E. Doktorov, J. Yang (2002)

The N-soliton train fixed up by the initial condition:

$$u(x,t=0)=\sum_{k=1}^{N} u_k^{1\mathrm{s}}(x,t=0), \qquad u_k^{1\mathrm{s}}(x,t)=\frac{2\nu_k e^{i\phi_k}}{\cosh z_k},$$

$$z_k(x,t)=2\nu_k(x-\xi_k(t)), \qquad \xi_k(t)=2\mu_k t+\xi_{k,0}$$

$$\phi_k(x,t)=\frac{\mu_k}{\nu_k}z_k+\delta_k(t), \qquad \delta_k(t)=W_k t+\delta_{k,0},$$

Each soliton has: amplitude ν_k, velocity μ_k, center of mass position ξ_k and phase δ_k.

The adiabatic approximation is valid provided the soliton parameters satisfy

$$|\nu_k-\nu_0|\ll\nu_0, \quad |\mu_k-\mu_0|\ll\mu_0, \qquad |\nu_k-\nu_0||\xi_{k+1,0}-\xi_{k,0}|\gg 1,$$

where $\nu_0=\frac{1}{N}\sum_{k=1}^{N}\nu_k$, and $\mu_0=\frac{1}{N}\sum_{k=1}^{N}\mu_k$. Two different scales:

$$|\nu_k-\nu_0|\simeq\varepsilon_0^{1/2}, \qquad |\mu_k-\mu_0|\simeq\varepsilon_0^{1/2}, \qquad |\xi_{k+1,0}-\xi_{k,0}|\simeq\varepsilon_0^{-1}.$$

For unperturbed NLS (i.e., for $V(x) = 0$) we get the Complex Toda chain (CTC):

$$\frac{\mathrm{d}^2 Q_j}{\mathrm{d}t^2} = 16\nu_0^2 \left(e^{Q_{j+1}-Q_j} - e^{Q_j - Q_{j-1}} \right), \qquad j = 1, \dots, N.$$

The complex-valued Q_k are expressed through the soliton parameters by:

$$Q_k(t) = 2i\lambda_0 \xi_k(t) + 2k \ln(2\nu_0) + i(k\pi - \delta_k(t) - \delta_0),$$

where $\delta_0 = 1/N \sum_{k=1}^{N} \delta_k$ and $\lambda_0 = \mu_0 + i\nu_0$. Besides we assume free-ends conditions, i.e., $e^{-Q_0} \equiv e^{Q_{N+1}} \equiv 0$.

For

$$V(x) = V_2 x^2 + V_1 x + V_0 + A_1 \cos(\Omega_1 x + \Omega_0).$$

we get perturbed CTC $\lambda_k = \mu_k + i\nu_k$:

$$\begin{aligned}
\frac{d\lambda_k}{dt} &= -4\nu_0 \left(e^{Q_{k+1}-Q_k} - e^{Q_k - Q_{k-1}}\right) - V_2\xi_k - \frac{V_1}{2} + M_k, \\
\frac{dQ_k}{dt} &= -4\nu_0(\mu_k + i\nu_k) - iD_k - \frac{i}{N}\sum_{j=1}^{N} D_j, \\
M_k &= \frac{\pi A\Omega_1^2}{8\nu_k}\frac{1}{\sinh Z_k}\sin(\Omega_1\xi_k + \Omega_0), \\
D_k &= V_2\left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2\right) - V_1\xi_k - V_0 - \frac{\pi^2 A\Omega_1^2}{16\nu_k^2}\frac{\cosh Z_k}{\sinh^2 Z_k}\cos(\Omega_1\xi_k + \Omega_0),
\end{aligned}$$

where $Z_k = \pi\Omega_1/(4\nu_k)$.

Switching on the self-consistent periodic potentials with the solitons initially located at its minima it is natural to expect that for $A > A_{\rm cr}$ the solitons will be stabilized into a bound state.

Use also that PCTC is a Hamiltonian system:

$$\begin{aligned} H_{\mathrm{PTC}} &= \sum_{k=1}^{N} \frac{1}{2} p_k^2 + \sum_{k=1}^{N-1} e^{Q_{k+1}-Q_k} + H_V, \\ H_{\mathrm{V}} &= \sum_{k=1}^{N} H_k + \sum_{k=1}^{N} (H_{k,k-1} + H_{k,k+1}), \\ H_k &= \int_{-\infty}^{\infty} dx\, V(x) |u_k^{(1\mathrm{s})}(x,t)|^2, \\ H_{k,k-1} &= \int_{-\infty}^{\infty} dx\, V(x) (u_k^{(1\mathrm{s}),*} u_{k-1}^{(1\mathrm{s})})(x,t). \end{aligned}$$

Then the condition $H_{\mathrm{V}} = 0$ gives:

$$A_{\mathrm{cr}} = -\left(1 - \frac{1}{N}\right) \frac{64\nu_0^4}{\pi\Omega} e^{-2\nu_0 r_0} \sinh \frac{\pi\Omega}{4\nu_0}.$$

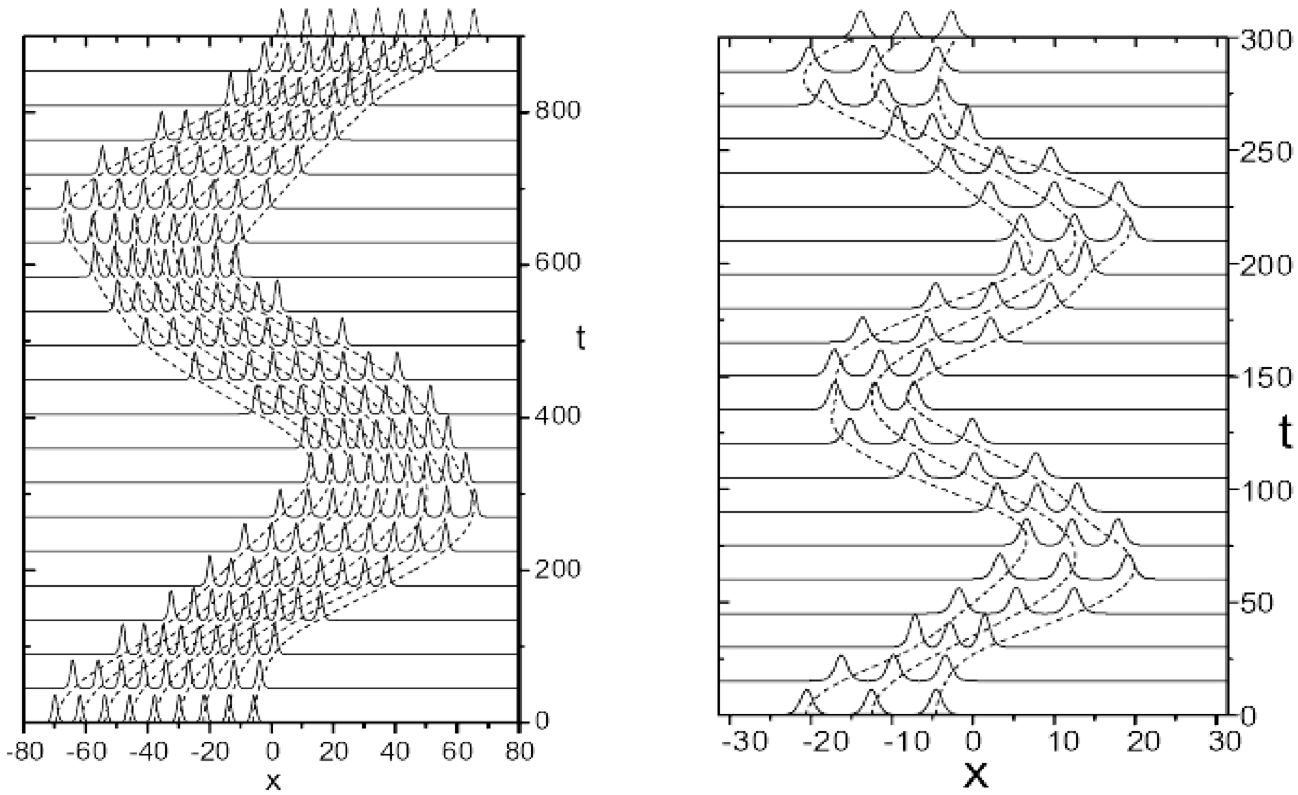


Figure 1: Harmonic oscillations of a N-soliton train initially shifted relative to the minimum of the quadratic potential $V(x) = V_2 x^2$. Left panel: 9-soliton train, $V_2 = 0.00005$. Right panel: 3-soliton train, $V_2 = 0.001$, $r_0 = 8$. Solid lines correspond to direct simulations of the NLS equation, and dashed lines to numerical solution of the PCTC.

Derivation of CTC for the Manakov equation through the variational approach

$$H = \int_{-\infty}^{-\infty} dx \, \left[\frac{1}{2} (\vec{u}_x^\dagger \vec{u}_x) - \frac{1}{2} (\vec{u} \dagger \vec{u})^2 \right], \qquad \mathcal{L} = \int_{-\infty}^{-\infty} dt \, \frac{i}{2} \left[(\vec{u} \dagger \vec{u}_t) - (\vec{u}_t^\dagger \vec{u}) \right] - H,$$

Lagrange equations of motion:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u} \dagger} = 0, \tag{1}$$

coincide with the MNLS:

$$i \frac{d\vec{u}}{dt} + \frac{1}{2} \frac{d^2 \vec{u}}{dx^2} + (\vec{u}^\dagger, \vec{u}) \vec{u}(x,t) = 0. \tag{2}$$

The initial condition is:

$$\vec{u}(x, t=0) = \sum_{k=1}^{N} \vec{u}_k(x, t=0), \tag{3}$$

where $\vec{u}_k(x,t)$ is the 1-soliton solution:

$$\begin{aligned}
\vec{u}_k(x,t) &= \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}\vec{n}_k, \qquad \vec{n}_k = \begin{pmatrix} \cos(\theta_k)e^{i\gamma_k} \\ \sin(\theta_k)e^{-i\gamma_k} \end{pmatrix}, \qquad z_k = 2\nu_k(x-\xi_k(t)), \\
\xi_k(t) &= 2\mu_k t + \xi_{k,0}, \qquad \phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \qquad \delta_k(t) = 2(\mu_k^2+\nu_k^2)t + \delta_{k,0}.
\end{aligned}$$

After some calculations I obtained:

$$
\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{\infty} dx \left(\frac{i}{2} \Big((\vec{u}^\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \Big) - \frac{1}{2} (\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2} (\vec{u}^\dagger, \vec{u})^2 \right) \\
&= \sum_{k=1}^{N} \left(\mathcal{L}_k + \sum_{n=k\pm 1} (\mathcal{X}_{k,n} - \mathcal{W}_{k,n} + \mathcal{Y}_{k,n}) \right), \\
\mathcal{L}_k &= \int_{-\infty}^{\infty} dx \left(\frac{i}{2} \Big((\vec{u}_k^\dagger, \vec{u}_{k,t}) - (\vec{u}_{k,t}^\dagger, \vec{u}_k) \Big) - \frac{1}{2} (\vec{u}_{k,x}^\dagger, \vec{u}_{k,x}) + \frac{1}{2} (\vec{u}_k^\dagger, \vec{u}_k)^2 \right) \\
\mathcal{X}_{k,n} &= \int_{-\infty}^{\infty} dx \frac{i}{2} \Big((\vec{u}_k^\dagger, \vec{u}_{n,t}) - (\vec{u}_{k,t}^\dagger, \vec{u}_n)) + (\vec{u}_n^\dagger, \vec{u}_{k,t}) - (\vec{u}_{n,t}^\dagger, \vec{u}_k) \Big), \\
\mathcal{W}_{k,n} &= \int_{-\infty}^{\infty} dx \frac{1}{2} \Big((\vec{u}_{k,x}^\dagger, \vec{u}_{n,x}) + (\vec{u}_{n,x}^\dagger, \vec{u}_{k,x}) \Big), \\
\mathcal{Y}_{k,n} &= \frac{1}{2} \int_{-\infty}^{\infty} dx (\vec{u}_k^\dagger, \vec{u}_k) \Big((\vec{u}_k^\dagger, \vec{u}_n) + (\vec{u}_n^\dagger, \vec{u}_k) \Big),
\end{aligned}
$$

We simplify $\mathcal{L}_{k,n}$ neglecting terms of order $\epsilon^{3/2}$. First we list several typical integrals which arise in evaluating the Lagrangian.

$$
\begin{aligned}
K(a,\Delta) &\equiv \int_{-\infty}^{\infty} \frac{dz\ e^{iaz}}{2\cosh z\cosh(z+\Delta)} = \frac{\pi(1-e^{-ia\Delta})}{2i\sinh(\Delta)\sinh(\pi a/2)}, \\
L(a,\Delta) &\equiv \int_{-\infty}^{\infty} \frac{dz\ e^{iaz}\sinh z}{2\cosh^2 z\cosh(z+\Delta)} \\
&= \frac{\pi i}{2\sinh^2\Delta\sinh(\pi a/2)}\left[(1-e^{-ia\Delta})\cosh\Delta - ia\sinh\Delta\right], \\
\widetilde{L}(a,\Delta) &\equiv \int_{-\infty}^{\infty} \frac{dz\ e^{iaz}\sinh(z+\Delta)}{2\cosh z\cosh^2(z+\Delta)} = e^{-ia\Delta}L(a,-\Delta), \\
M(a,\Delta) &\equiv \int_{-\infty}^{\infty} \frac{dz\ e^{iaz}\sinh z\sinh(z+\Delta)}{2\cosh^2 z\cosh^2(z+\Delta)} \\
&= \frac{\pi e^{-ia\Delta/2}}{\sinh\Delta\sinh(\pi a/2)}\left[a\cos(\frac{a\Delta}{2})\frac{\cosh\Delta}{\sinh\Delta} - \sin(\frac{a\Delta}{2})\left(1+\frac{2}{\sinh^2\Delta}\right)\right].
\end{aligned}
$$

In the adiabatic limit $a \simeq 0$ and:

$$\begin{aligned} K(0,\Delta) &= \frac{\Delta}{\sinh\Delta}, \\ L(0,\Delta) &\equiv -\widetilde{L}(0,\Delta) = \frac{1}{\sinh\Delta}\left[1 - \frac{\Delta\cosh\Delta}{\sinh\Delta}\right], \\ M(0,\Delta) &= \frac{1}{\sinh^3\Delta}\left[2\cosh(\Delta)\sinh\Delta - \Delta\sinh^2\Delta - 2\Delta\right]. \end{aligned}$$

Thus we get:

$$
\begin{aligned}
\widetilde{\mathcal{L}}_{k,n} &= \mathcal{L}_{k,n} + \mathcal{O}(\epsilon^{3/2}), \qquad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R^*_{k,n}), \\
R_{k,n} &= e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \qquad \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k, \qquad \Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1. \\
\frac{d\xi_k}{dt} &= 2\mu_k, \\
\frac{d\delta_k}{dt} &= 2\mu_k^2 + 2\nu_k^2, \\
\frac{d\nu_k}{dt} &= 8\nu_0^3 \sum_n e^{-\Delta_{k,n}} i(R_{k,n} - R^*_{k,n}), \\
\frac{d\mu_k}{dt} &= -8\nu_0^3 \sum_n e^{-\Delta_{k,n}} (R_{k,n} + R^*_{k,n}), \\
\frac{d\vec{n}_k}{dt} &= 4\nu_0^2 i \sum_{n=k\pm 1} e^{-\Delta_{k,n}} \left[(e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} \vec{n}_n - R_{k,n} \vec{n}_k) + (e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} \vec{n}_n + R^*_{k,n} \vec{n}_k) \right] \\
&+ C_k \vec{n}_k.
\end{aligned}
$$

The condition $(\vec{n}_k^\dagger, \vec{n}_k) = 1$ holds provided $C_k = -C_k^*$.

Let us now rewrite these equations in terms of the variables Q_k characteristic for the CTC:

$$\begin{aligned} Q_k &= -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \\ \nu_0 &= \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \end{aligned} \tag{4}$$

$$\frac{d(\mu_k + i\nu_k)}{dt} = 4\nu_0\left[(\vec{n}_k^\dagger, \vec{n}_{k-1})e^{Q_k - Q_{k-1}} - (\vec{n}_{k+1}^\dagger, \vec{n}_k)e^{Q_{k+1} - Q_k}\right].$$

Besides, from (4) there follows

$$\frac{dQ_k}{dt} = -4\nu_0(\mu_k + i\nu_k).$$

Therefore we get the generalized CTC for the MNLS:

$$\frac{d^2Q_k}{dt^2} = 16\nu_0^2\left[(\vec{n}_{k+1}^\dagger, \vec{n}_k)e^{Q_{k+1} - Q_k} - (\vec{n}_k^\dagger, \vec{n}_{k-1})e^{Q_k - Q_{k-1}}\right].$$

Perturbed Manakov system

$$\begin{aligned} iu_t + (1/2)u_{xx} + (|u|^2 + |v|^2)u &= iR_1, \\ iv_t + (1/2)v_{xx} + (|u|^2 + |v|^2)v &= iR_2, \end{aligned}$$

where

$$R_1 = (i(\alpha+\delta) - \rho)u + (i\gamma - \kappa)v + i\beta u_{xx} - \sigma|v|^2 u,$$

$$R_2 = (i(\alpha-\delta) + \rho)v + (i\gamma - \kappa)u + i\beta v_{xx} - \sigma|u|^2 v,$$

where
α — isotropic gain coefficient;
δ, γ — gain dichroism;
β – dispersion
ρ — linear birefringence
κ — linear coupling
σ — differential cross-phase coefficient
For birefringent fibers $\sigma = -1/3$.

Hamiltonian form:

$$\begin{aligned} H &= \int_{-\infty}^{\infty} dx \, \left\{ \left(\frac{1}{2} - i\beta \right) \left(|u_x|^2 + |v_x|^2 \right) - \frac{1}{2} \left(|u|^2 + |v|^2 \right)^2 - \sigma |u|^2 |v|^2 \right. \\ &+ \left. i\alpha \left(|u|^2 + |v|^2 \right) + (i\delta - \rho) \left(|u|^2 - |v|^2 \right) + (i\gamma - \kappa)(u^* v + u v^*) \right\} . \end{aligned}$$

Suppose that the N-soliton solution to the Manakov system (5) for the soliton train is determined by the ordered sum of the tail-tail overlapping initial one-soliton pulses

$$\begin{pmatrix} u \\ v \end{pmatrix} (x,0) = \sum_{k=1}^{N} 2\nu_k \begin{pmatrix} \cos\theta_k \exp[i(2\mu_k \xi_k^{(0)} + \delta_k^{(1)})] \\ \sin\theta_k \exp[i(2\mu_k \xi_k^{(0)} + \delta_k^{(2)})] \end{pmatrix} e^{2i\mu_k (x - \xi_k^{(0)})} \operatorname{sech} 2\nu_k (x - \xi_k^{(0)}).$$

$$\epsilon = \left| \int_{-\infty}^{\infty} dx u_k^\dagger(x) u_n(x) \right| = \left| \int_{-\infty}^{\infty} dx \frac{4\nu_k \nu_n e^{i(\phi_n - \phi_k)}}{\cosh z_k \cosh z_n} \right| \simeq 8\nu_0 \Delta_{kn} e^{-\Delta_{kn}} .$$

Here $n = k \pm 1$, ν_0 is the mean value of the soliton amplitudes, $\nu_0 = (1/N) \sum_k \nu_k$ and $\Delta_{k,n} = 2\nu_0 |\xi_k^{(0)} - \xi_n^{(0)}|$.

$$z_k = 2\nu_k(x-\xi_k), \quad R_k^{(+)} = R_k \cos\theta_k + S_k \sin\theta_k, \quad R_k^{(-)} = R_k \sin\theta_k - S_k \cos\theta_k,$$

$$R_k = r_k \exp\left(-i\frac{\mu_k}{\nu_k} z_k - i\chi_k^{(1)}\right), \qquad S_k = s_k \exp\left(-i\frac{\mu_k}{\nu_k} z_k - i\chi_k^{(2)}\right).$$

Insert the N-train anzats into the Hamiltonian

$$H_{\mathrm{pert};k} = -i\beta 16\nu_k^3\left(\mu_k^2 + \frac{\nu_k^3}{3}\right) + 4i\alpha\nu_k + 4(i\delta - \rho)\nu_k \cos(2\theta_k)$$

$$+(i\gamma - \kappa)4\nu_k \sin(2\theta_k)\cos(2\gamma_k) - \frac{8\nu_k^3}{3}\sigma \sin^2(2\theta_k).$$

For

$$V(x) = V_2 x^2 + V_1 x + V_0 + A_1 \cos(\Omega_1 x + \Omega_0).$$

we get perturbed GCTC $\lambda_k = \mu_k + i\nu_k$:

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left[(\vec{n}^\dagger_{k+1}, \vec{n}_k) e^{Q_{k+1}-Q_k} - (\vec{n}^\dagger_k, \vec{n}_{k-1}) e^{Q_k - Q_{k-1}} \right] - V_2 \xi_k - \frac{V_1}{2} + M_k, \\ \frac{dQ_k}{dt} &= -4\nu_0(\mu_k + i\nu_k) - iD_k - \frac{i}{N}\sum_{j=1}^N D_j, \\ M_k &= \frac{\pi A \Omega_1^2}{8\nu_k} \frac{1}{\sinh Z_k} \sin(\Omega_1 \xi_k + \Omega_0), \\ D_k &= V_2 \left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2 \right) - V_1 \xi_k - V_0 - \frac{\pi^2 A \Omega_1^2}{16\nu_k^2} \frac{\cosh Z_k}{\sinh^2 Z_k} \cos(\Omega_1 \xi_k + \Omega_0), \end{aligned}$$

where $Z_k = \pi\Omega_1/(4\nu_k)$.

Effects of the polarization vectors on the soliton interaction

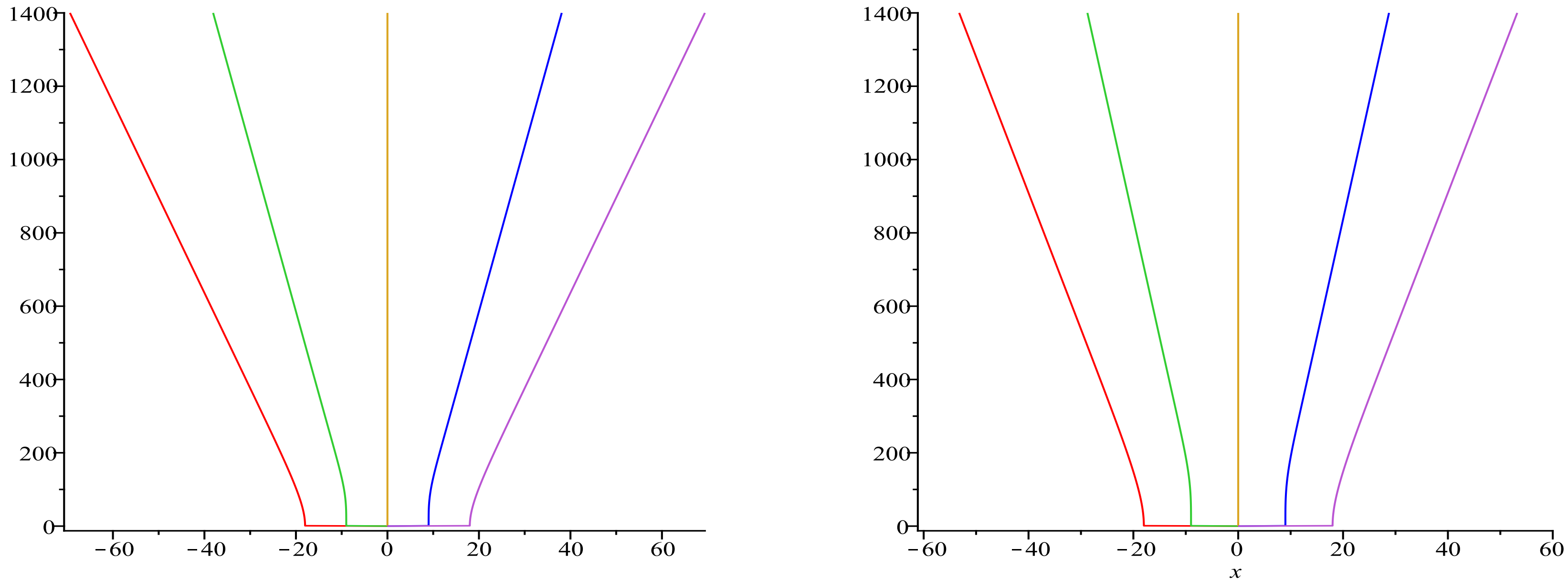


Figure 2: The initial soliton parameters as like in (**??**) with $r_0 = 9$. Left panel: scalar soliton train; Right panel: vector soliton train with $r_0 = 9$ and $m_{0s} = 0.7$.

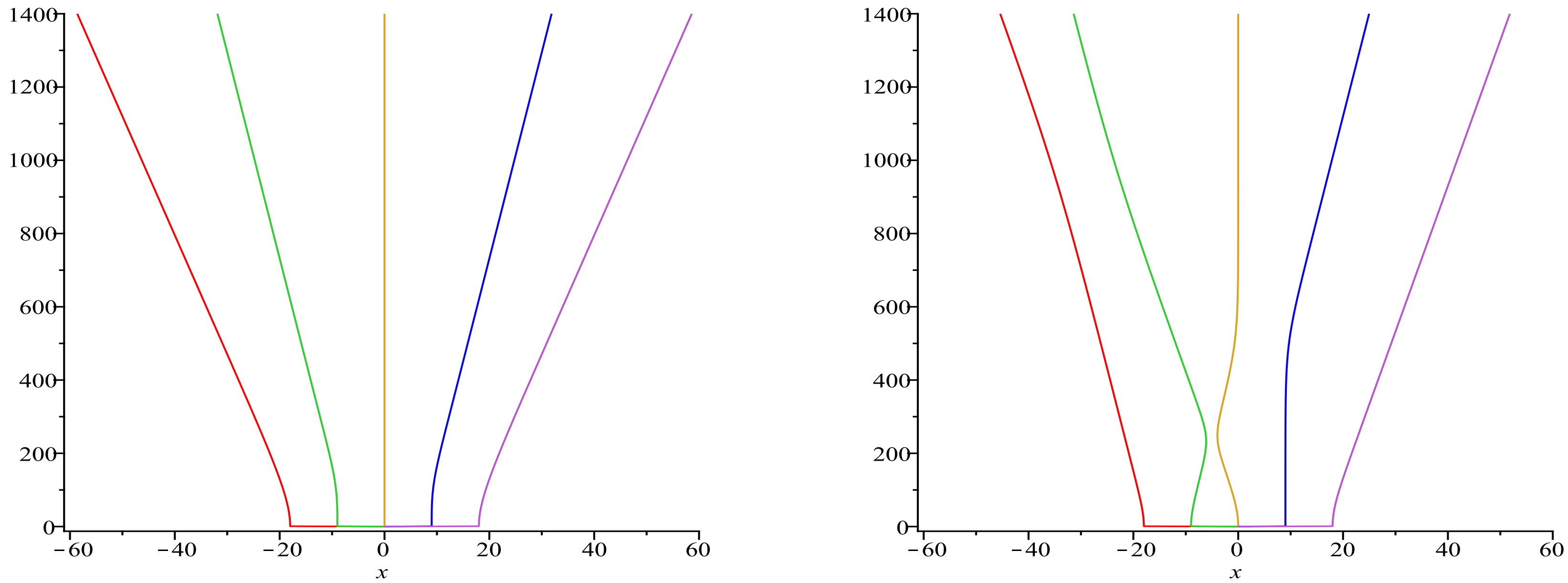


Figure 3: Left panel: vector soliton train with $m_{0s} = 0.8$; Right panel: vector soliton train with $m_{01} = m_{03} = m_{04} = 0.8$ and $m_{02} = 0.031$.

However, if we choose m_{0s} to be different we may encounter additional problems, see figure 3. From this figure it is obvious that taking substantially different values for m_{0s} we may encounter violation of adiabaticity. That means that our approximation is valid only for such configurations of polarization vectors for which m_{0s} are of the same order of magnitude.

Effects of quadratic potentials

Figure 4: $\nu_k = 0.5$, $\mu_k = 0$,$\delta_k = k\pi$, $\theta_k = \frac{2k\pi}{5}$, $V(x) = 0.000025x^2$

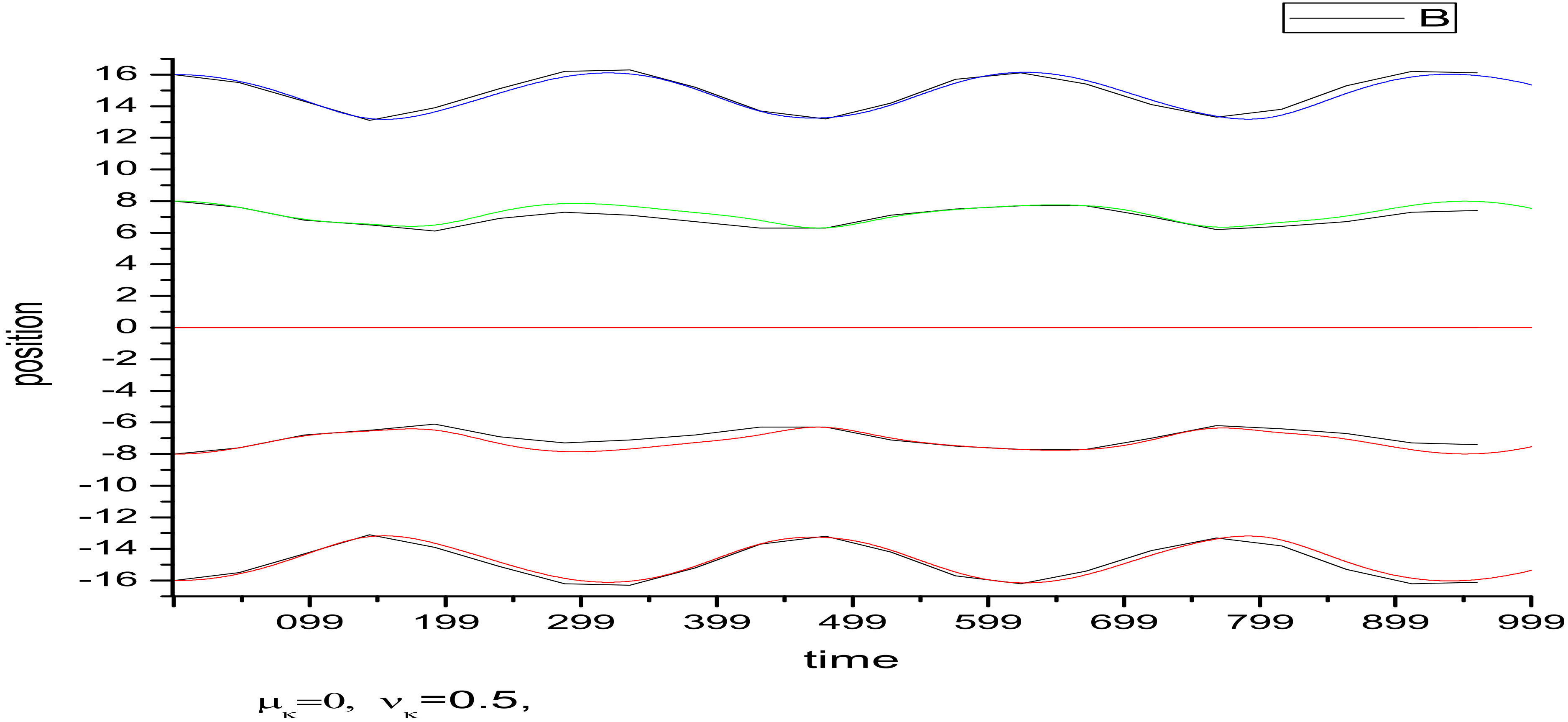


Figure 5: $\nu_k = 0.5$, $\mu_k = 0$,$\delta_k = k\pi$, $\theta_k = \frac{2k\pi}{5}$, $V(x) = 0.000025(x-25)^2$

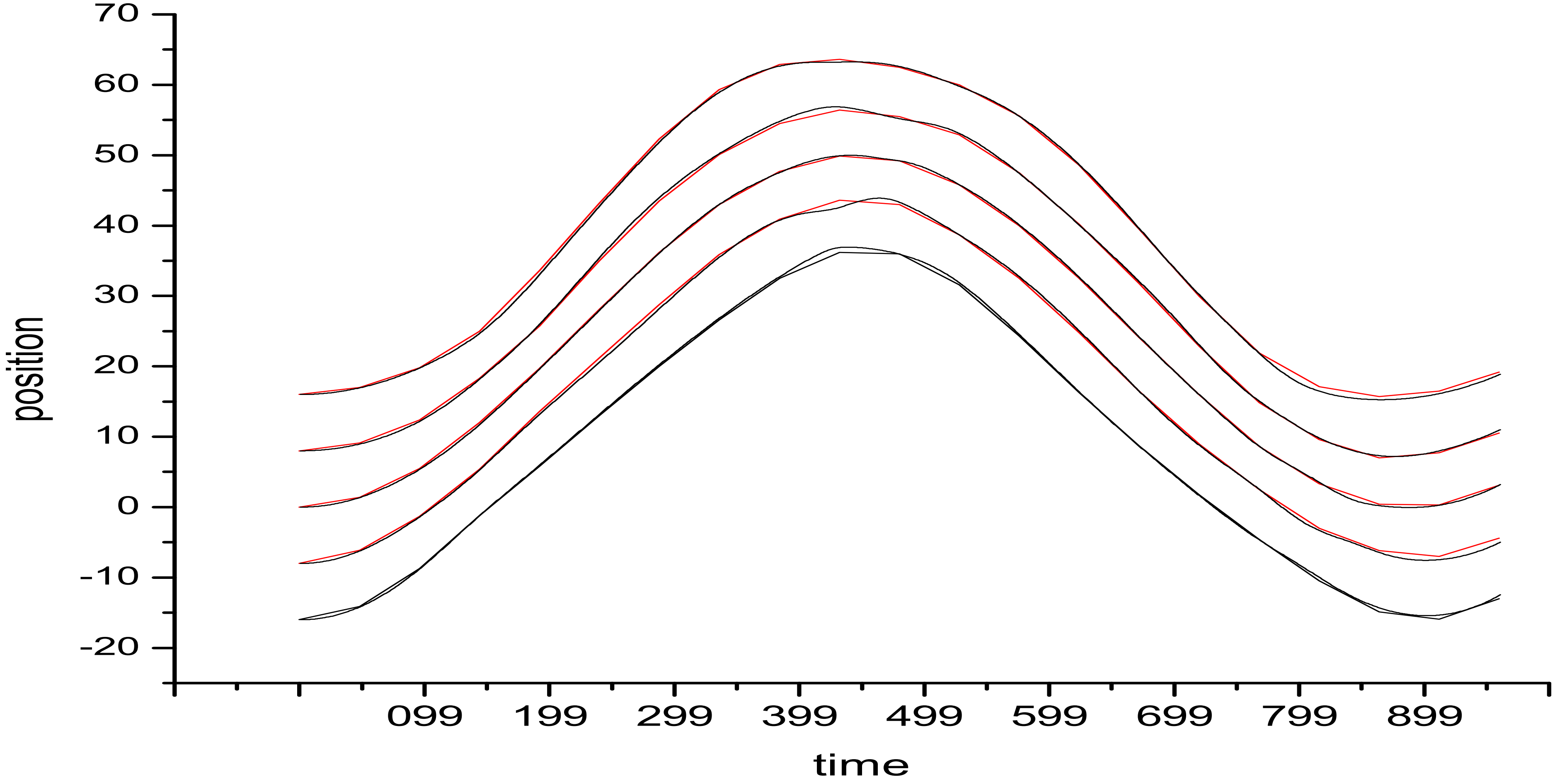


Figure 6: $\nu_k = 0.5$, $\mu_k = 0$,$\delta_k = k\pi$, $\theta_k = \frac{2k\pi}{5}$, $V(x) = 0.00005(x-25)^2$

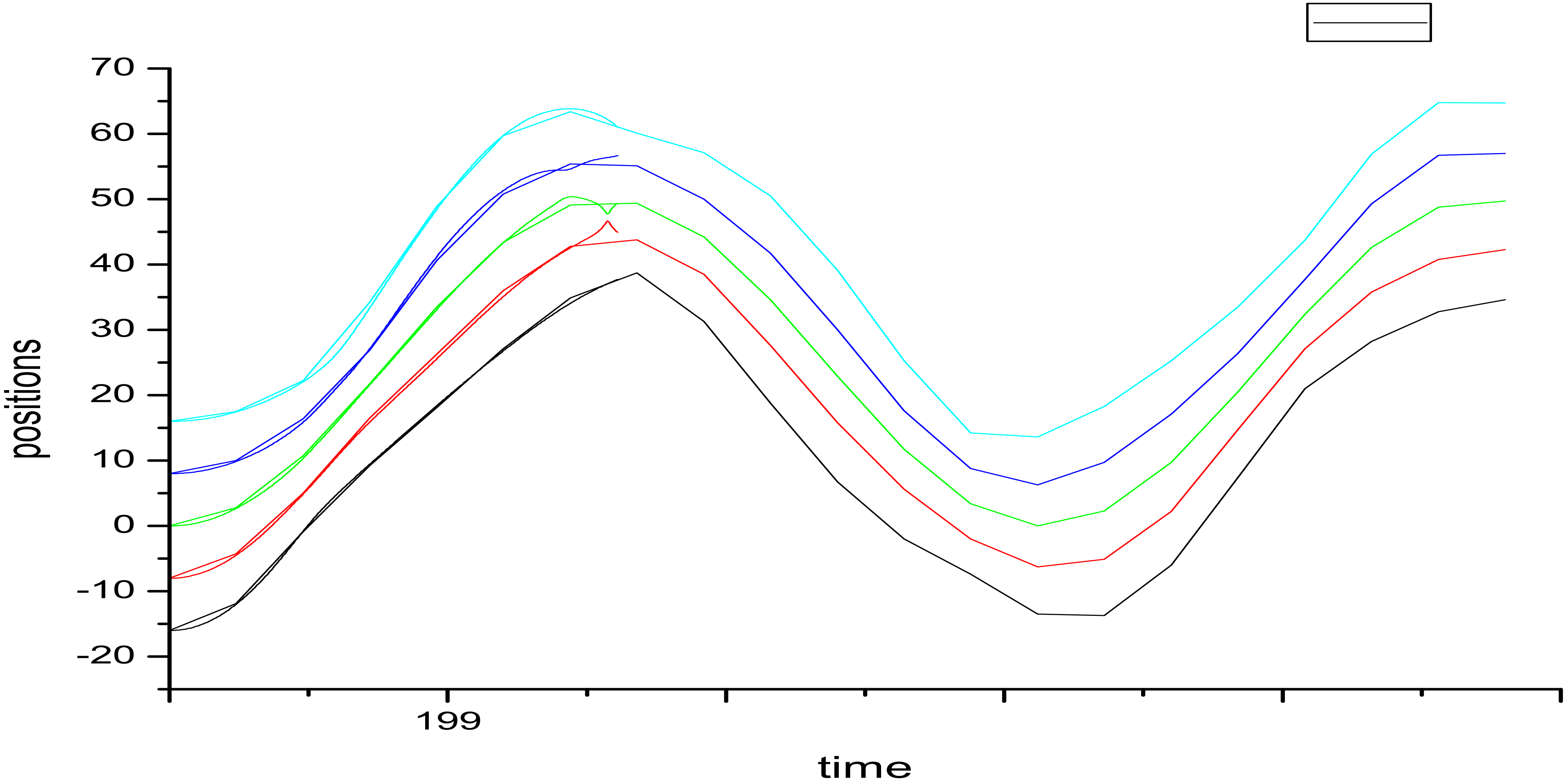


Figure 7: $\nu_k = 0.5$, $\mu_k = 0$,$\delta_k = k\pi$, $\theta_k = \frac{2k\pi}{5}$, $V(x) = 0.000055(x-25)^2$

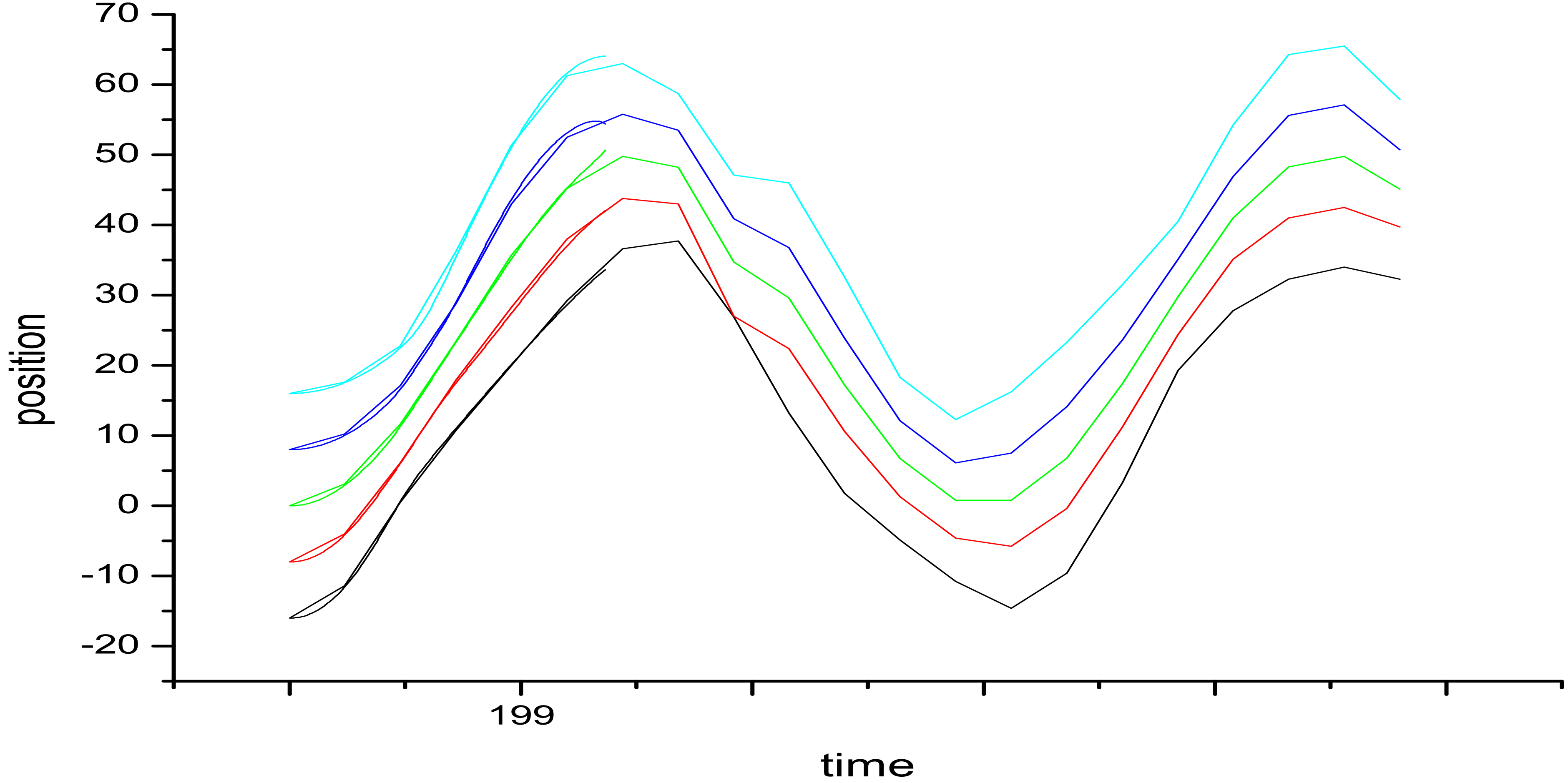
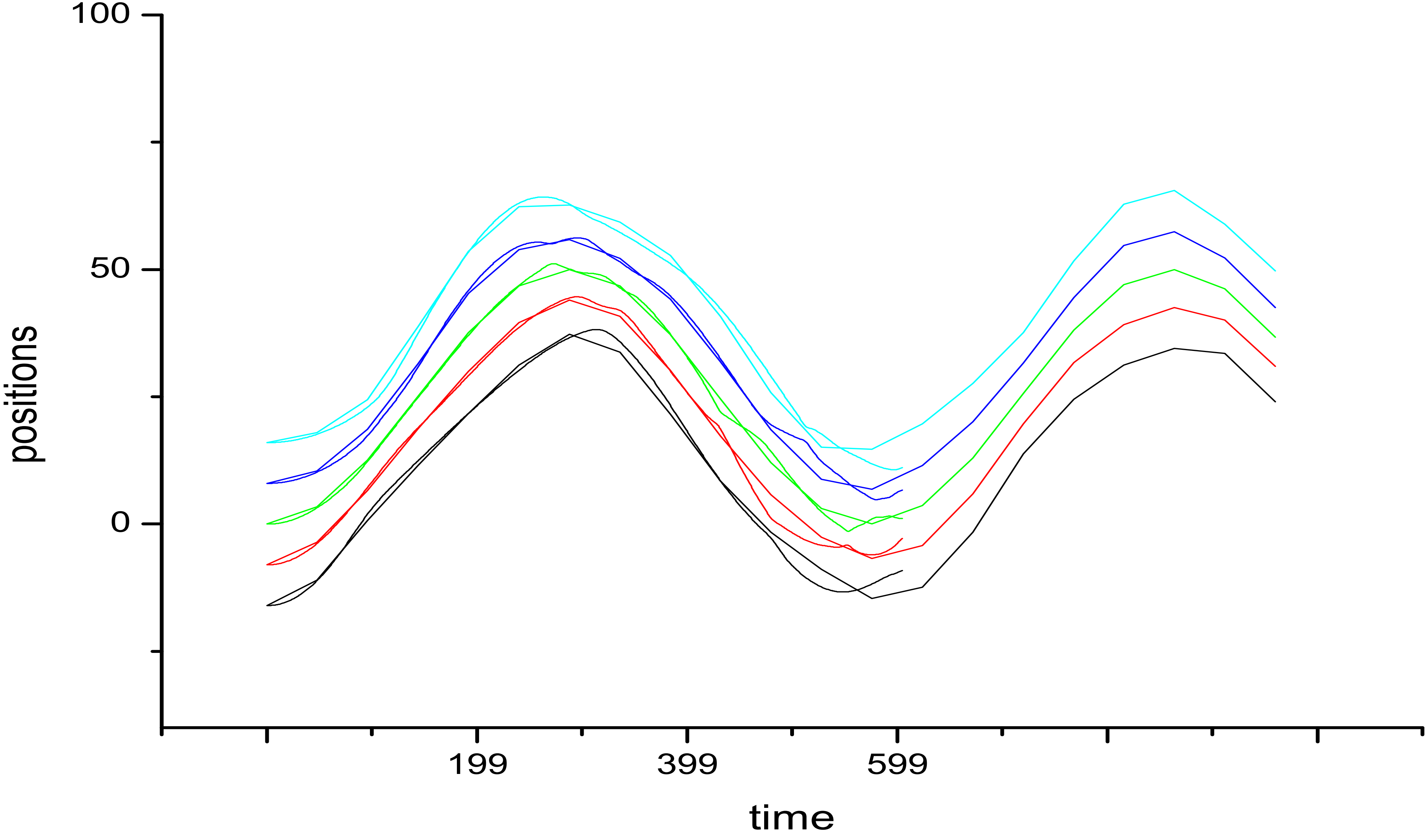


Figure 8: $\nu_k = 0.5$, $\mu_k = 0$,$\delta_k = k\pi$, $\theta_k = \frac{2k\pi}{5}$, $V(x) = 0.00006(x-25)^2$



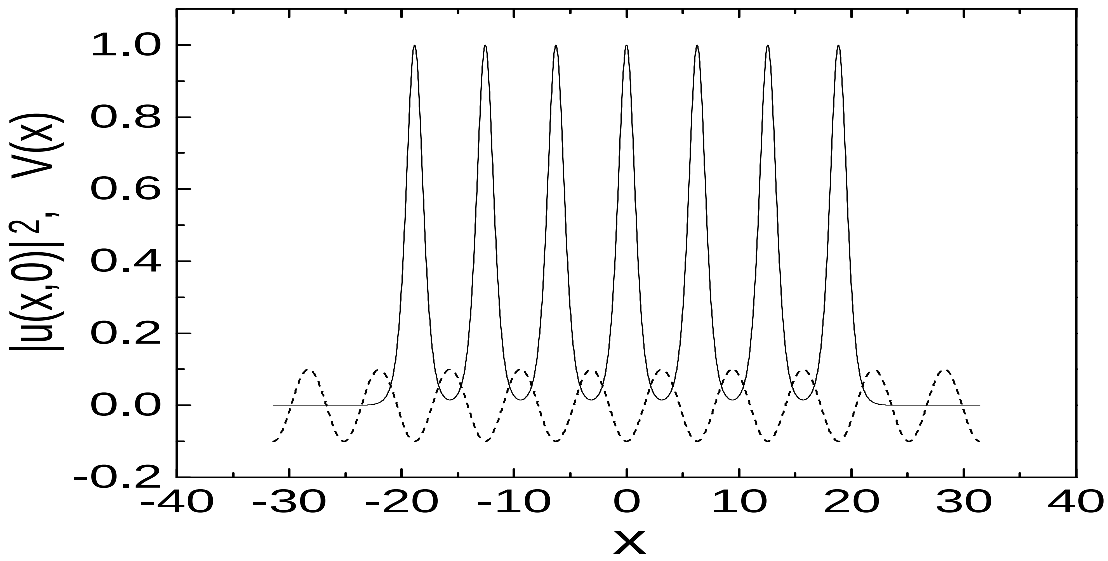


Figure 9: The periodic potential and a 7-soliton trains with $r_0 = 7$.

Effects of periodic potentials

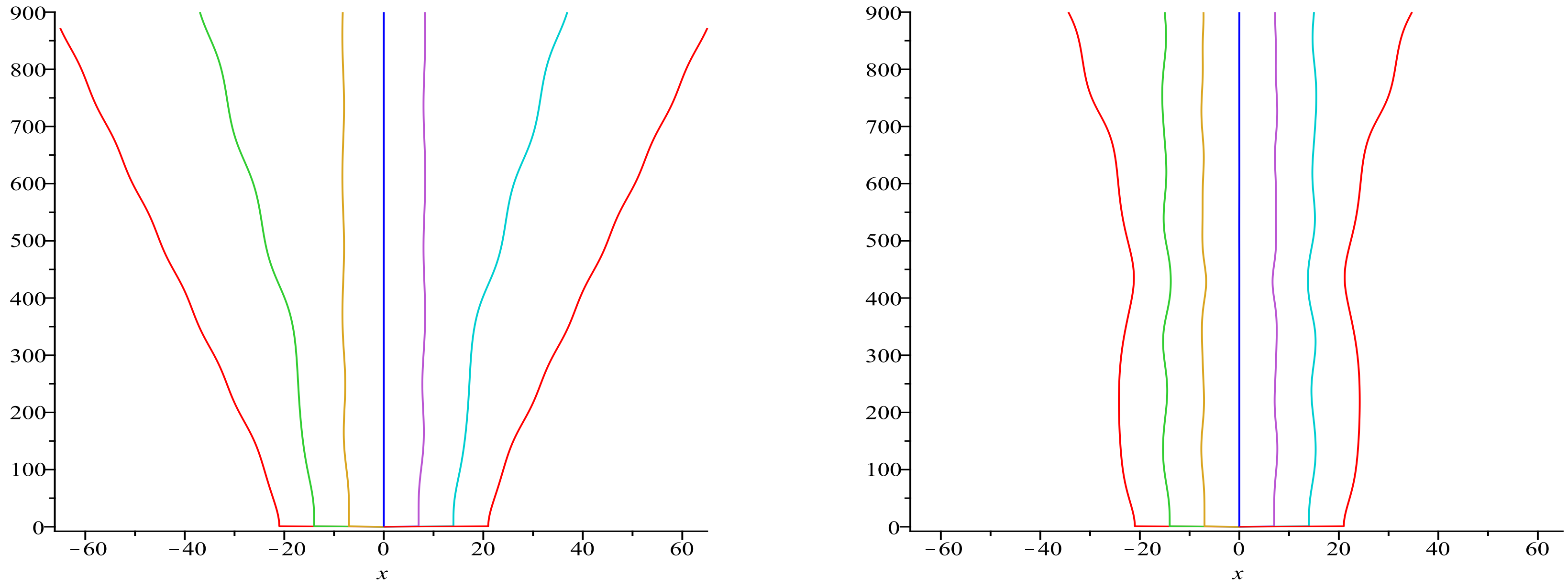


Figure 10: The effect of the periodic potential on 7-soliton trains with $r_0 = 7$ and subcritical intensities. Left panel: $A = 0.00075$; Right panel: $A = 0.0012$

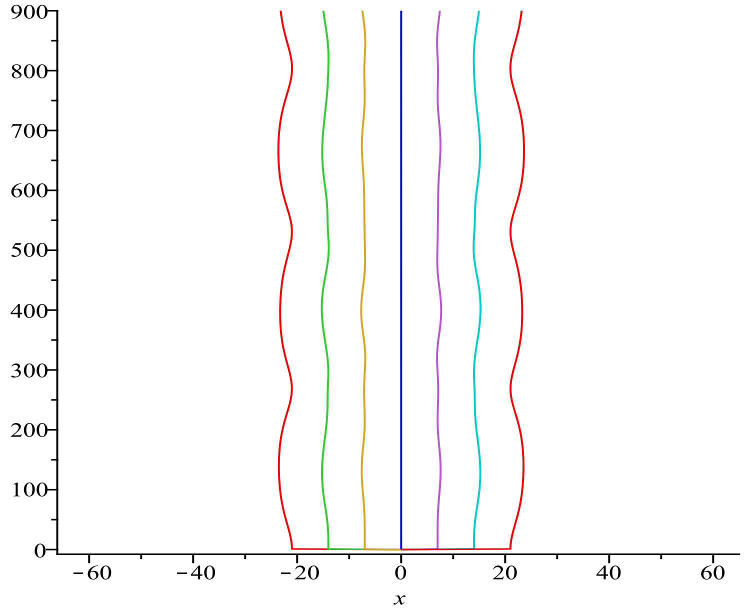


Figure 11: periodic potential on 7-soliton trains with critical intensity: $A = 0.0013$.

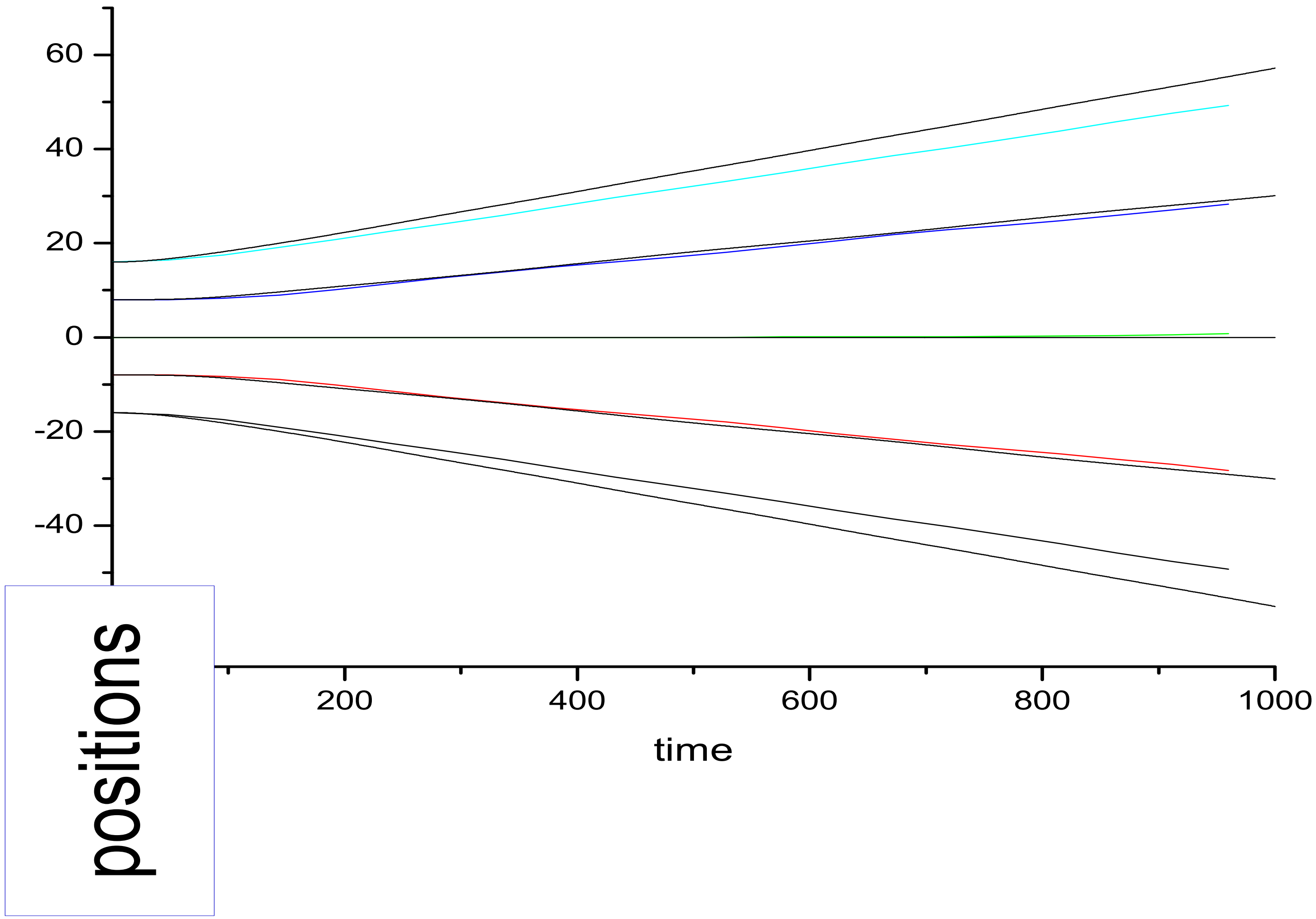


Figure 12: periodic potential on 5-soliton trains with $r_0 = 8$ with sub-critical intensity: $A = 0.001$.

Effects of potentials wells and humps

$$V(x) = \sum_s c_s V_s(x, y_s), \qquad V_s(x, y_s) = \frac{1}{\cosh^2(2\nu_0 x - y_s)}.$$

The potential $-V(x, y_s)$ (resp. $V_s(x, y_s)$) is a well (resp. hump) with width 1.7 at half-height/depth. Adjusting one or more terms in () with different c_s and y_s we can describe wells and/or humps with different widths/depths and positions.

The PCTC which takes into account such types of potentials is given by:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{q_{k+1}-q_k}(\vec{n}^\dagger_{k+1}, \vec{n}_k) - e^{q_k - q_{k-1}}(\vec{n}^\dagger_k, \vec{n}_{k-1}) \right) + M_k + iN_k,$$

$$\frac{dq_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon),$$

where $\lambda_k = \mu_k + i\nu_k$, $X_k = 2\mu_k \Xi_k + D_k$ and

$$q_k = -2\nu_0 \xi_k + k \ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0 \xi_k), \qquad \delta_0 = \frac{1}{N} \sum_{s=1}^{N} \delta_s.$$

The coefficients N_k, M_k, Ξ_k and D_k are given by:

$$N_k = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k} \operatorname{Im} \left(V(y_k) u_k \mathrm{e}^{-i\phi_k} \right), \qquad M_k = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dz_k \, \sinh z_k}{\cosh^2 z_k} \operatorname{Re} \left(V(y_k) u \right.$$

$$\Xi_k = -\frac{1}{4\nu_k^2} \int_{-\infty}^{\infty} \frac{dz_k \, z_k}{\cosh z_k} \operatorname{Im} \left(V(y_k) u_k \mathrm{e}^{-i\phi_k} \right), \quad D_k = \frac{1}{2\nu_k} \int_{-\infty}^{\infty} \frac{dz_k \, (1 - z_k \tanh z_k)}{\cosh z_k} \mathrm{R}$$

where $y_k = z_k/(2\nu_0) + \xi_k$. For our specific choice of $V(x)$ we get:

$$M_k = \sum_s 2c_s \nu_k P(\Delta_{k,s}), \qquad N_k = 0, \qquad \Xi_k = 0, \qquad D_k = \sum_s c_s R(\Delta_{k,s}),$$

where $\Delta_{k,s} = 2\nu_0\xi_k - y_s$ and

$$P(\Delta) = \frac{\Delta + 2\Delta\cosh^2(\Delta) - 3\sinh(\Delta)\cosh(\Delta)}{\sinh^4(\Delta)},$$

$$R(\Delta) = \frac{6\Delta\sinh(\Delta)\cosh(\Delta) - (2\Delta^2 + 3)\sinh^2(\Delta) - 3\Delta^2}{2\sinh^4(\Delta)}.$$

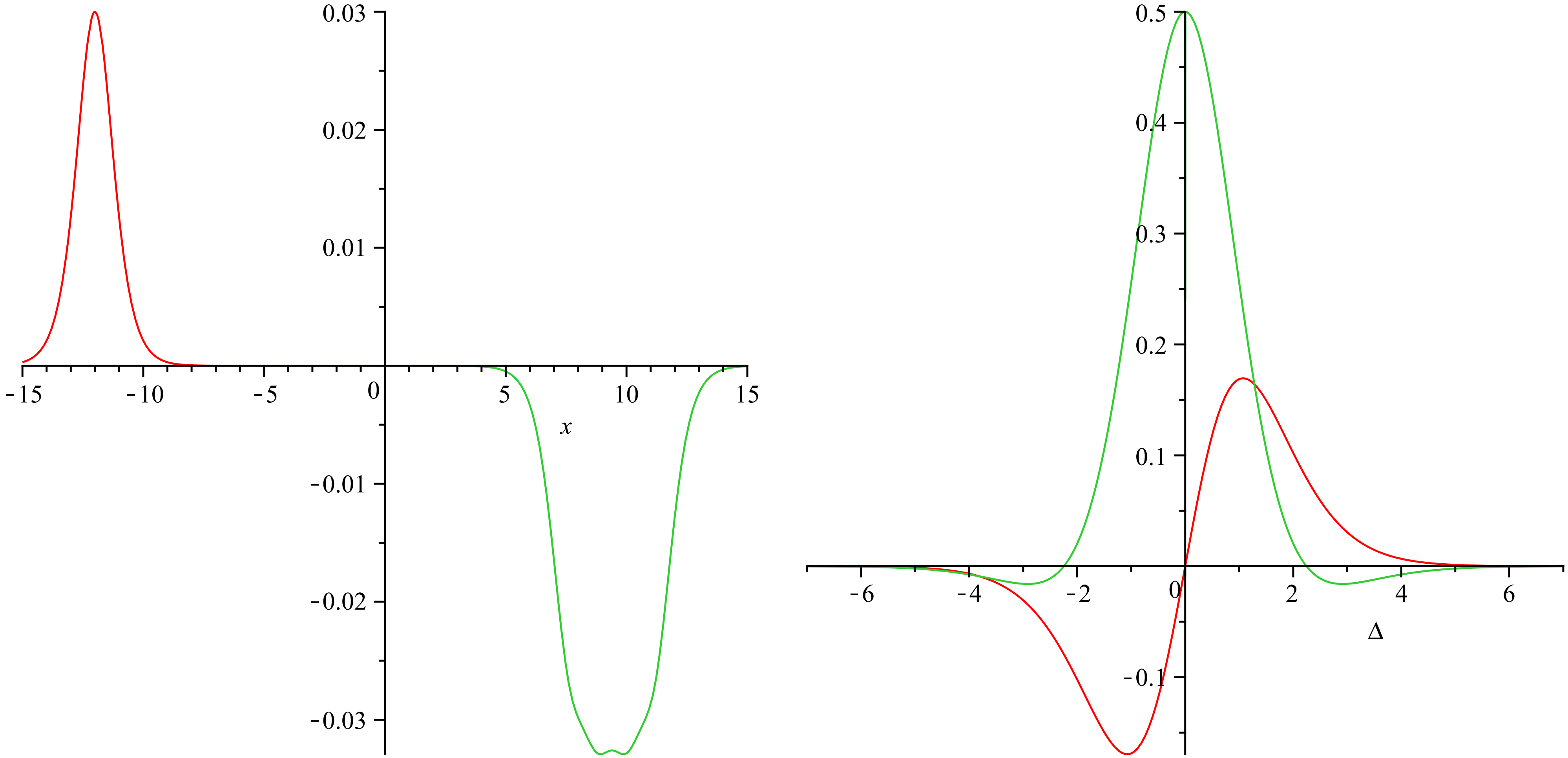


Figure 13: Left panel: Examples of potential hump with $c_0 = 0.03, y_0 = -12$ and a potential well which is a superposition of four terms $V_s(x, y_s)$ with $c_s = 0.02,\ s = 1, 2, 3, 4$ and $y_1 = 7.6, y_2 = 8.8, y_3 = 10.0, y_4 = 11.2$. Right panel: The integrals $R(\Delta)$ (green line) and $P(\Delta)$ (red line).

As it is well known the 3-soliton systems allow for three types of dynamical regimes for large times:

We use two types of initial phases configurations:

$$\text{a)} \quad \delta_{1,0} = 0, \quad \delta_{2,0} = \pi, \quad \delta_{3,0} = 0, \tag{5a}$$

$$\text{b)} \quad \delta_{1,0} = 0, \quad \delta_{2,0} = \frac{\pi}{2} + 2\mu_0 r_0, \quad \delta_{3,0} = -\frac{\pi}{2} + 2\mu_0 r_0. \tag{5b}$$

AFR) asymptotically free regime when all 3 solitons move away with different velocities. This regime takes place if the initial amplitudes and phases are given by

$$\Delta\nu < \nu_{\rm cr} = 2\sqrt{2\cos(\theta_1 - \theta_2)}\nu_0 \exp(-\nu_0 r_0)$$
$$\delta_{1,0} = 0, \quad \delta_{2,0} = \pi, \quad \delta_{3,0} = 0,$$

MAR) mixed asymptotic regime, when two of the solitons form bound state and the third soliton goes away from them with different velocity; Such regime takes place if the amplitudes and the phases are chosen as

$$\begin{aligned} &\Delta\nu < \nu_{\mathrm{cr}} = 2\sqrt{2\cos(\theta_1 - \theta_2)}\nu_0 \exp\left(-\nu_0 r_0\right) \\ &\delta_{1,0} = 0, \quad \delta_{2,0} = \frac{\pi}{2} + 2\mu_0 r_0, \quad \delta_{3,0} = -\frac{\pi}{2} + 2\mu_0 r_0. \end{aligned} \tag{6}$$

BSR) bound state regime when all solitons move asymptotically with the same velocity. Such regime takes place for amplitudes and the phases like

$$\begin{aligned} &\Delta\nu > \nu_{\mathrm{cr}} \\ &\delta_{1,0} = 0, \quad \delta_{2,0} = \pi, \quad \delta_{3,0} = 0. \end{aligned} \tag{7}$$

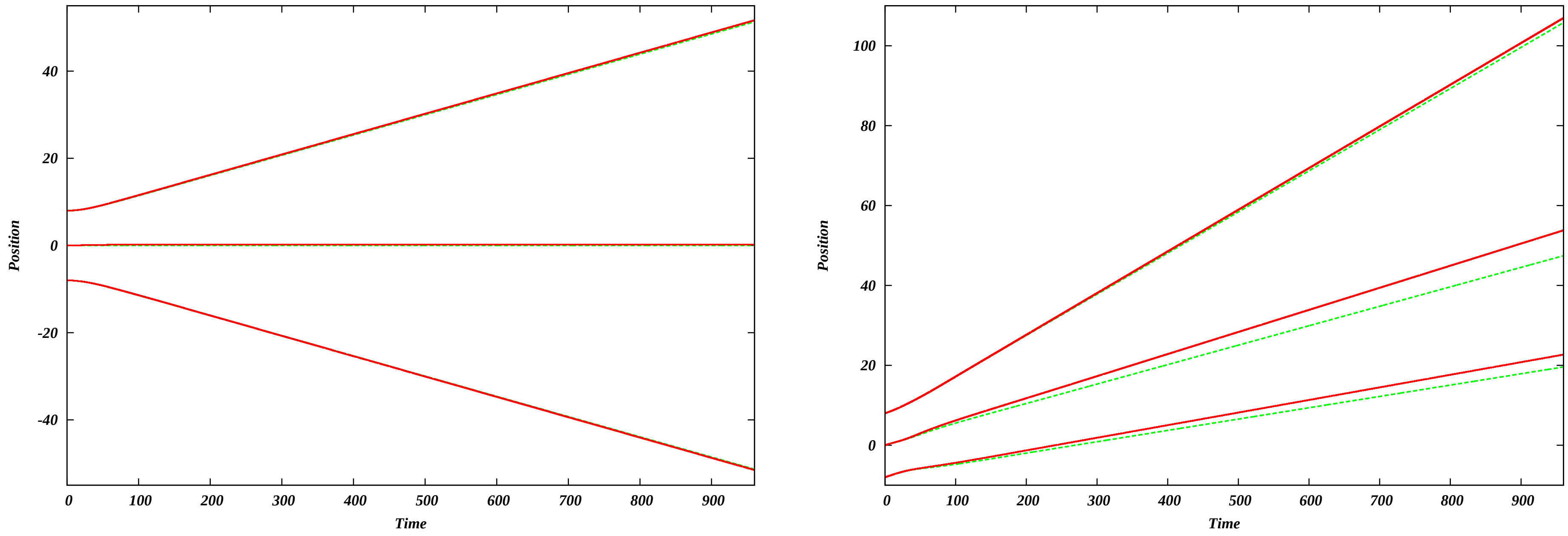


Figure 14: Left panel: Three solitons in AFR, $V(x) = 0$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5aa). Right panel: Three solitons in MAR, $V(x) = 0$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Solid lines correspond to MM, dashed – to PCTC.

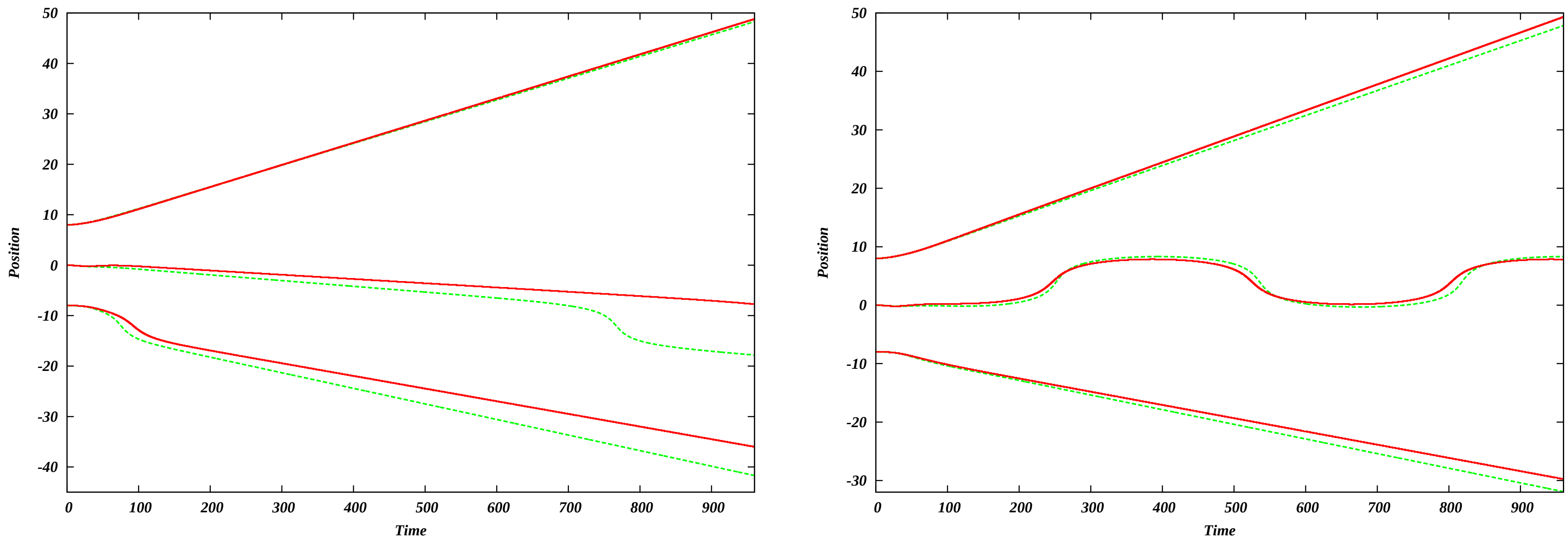


Figure 15: Left panel: Three solitons in MAR, $V(x) = -0.01V_1(x, -12)$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Right panel: Three solitons in MAR, $V(x) = -0.01V_1(x, 4)$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Solid lines correspond to MM, dashed – to PCTC.

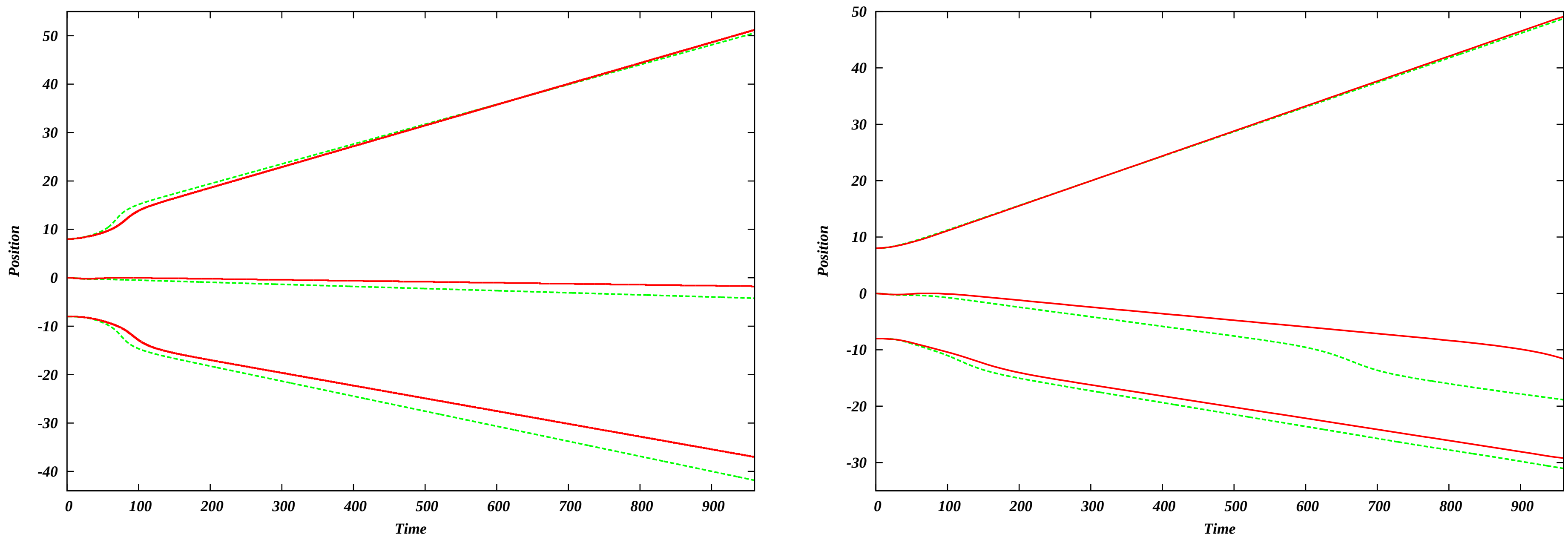


Figure 16: Left panel: Three solitons in MAR, $V(x) = -0.01[V_1(x, 12) + V_1(x, -12)]$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Right panel: Three solitons in MAR, $V(x) = -0.001[V_1(x, -12)+V_1(x, 4)]$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Solid lines correspond to MM, dashed – to PCTC.

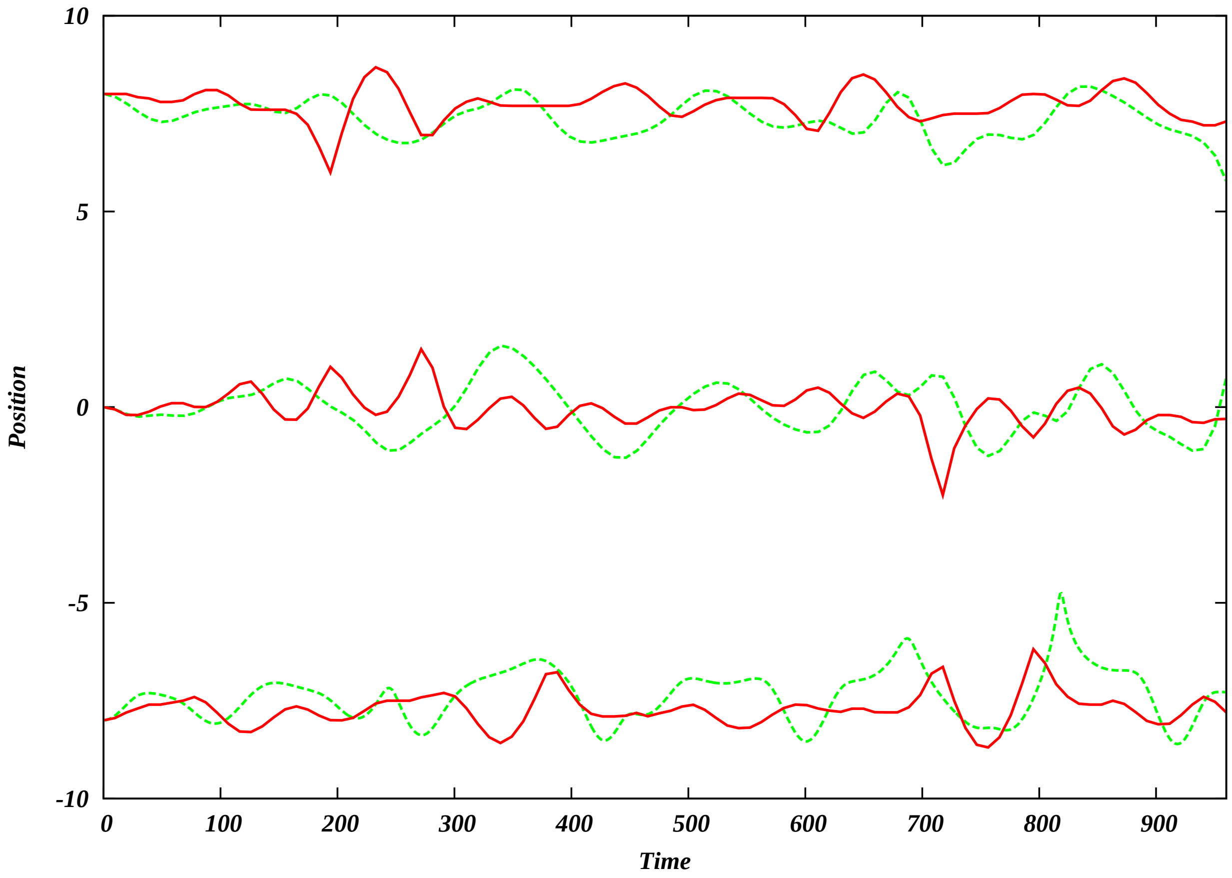


Figure 17: Three solitons in MAR, $V(x) = 0.1[V_1(x,-12) + V_1(x,12)]$, $\Delta\nu = 0.01$, $\mu_0 = 0$ and phases as in (5ab). Solid lines correspond to MM, dashed – to PCTC.

To be continued

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on the next seminar/conference !!!

Thank you for the attention!