Singularities and integrability of birational dynamical systems on the projective plane

Adrian-Stefan Carstea, Tomoyuki Takenawa

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Research Group on Geometry and Physics
(http://events.theory.nipne.ro/gap/)
NIPNE, Bucharest, Romania
and
Tokyo University of Marine Science and Technology, Tokyo, Japan

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From nonlinear discrete equations(mappings) to surface theory

The main motivation! Finding an INTEGRABILITY DETECTORS for discrete equations
Example:

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Integrability$\equiv$ internal symmetry, existence of invariants, computing general solution etc.

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Singularity pattern $(f, 0, \infty, \infty, 0, -f)$. So after a **finite** number of steps the singularities are confined and **initial information is recovered**- *singularity confinement*

Our example can be written as:

$$\phi : \begin{cases} x_{n+1} & = y_n \\ y_{n+1} & = \quad -x_n - y_n + \frac{a}{y_n} \end{cases} . \tag{1}$$

seen as a chain of birational mappings ... $\rightarrow (\underline{x}, \underline{y}) \rightarrow (x, y) \rightarrow (\bar{x}, \bar{y}) \rightarrow$ ... where $\underline{x} = x_{n-1}, x = x_n, \bar{x} = x_{n+1}$ and so on.
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Singularity confinement:

$$\underbrace{(f, 0)}_{(x_0, y_0)} \rightarrow \underbrace{(0, \infty)}_{(x_1, y_1)} \rightarrow \underbrace{(\infty, \infty)}_{(x_2, y_2)} \rightarrow \underbrace{(\infty, 0)}_{(x_3, y_3)} \rightarrow \underbrace{(0, f)}_{(x_4, y_4)}$$

and the secret is the follwing:
If $(x_0, y_0) = (f, \epsilon)$ then the foolowing products are *finite*

$$x_1 y_1 = a + O(\epsilon), \quad \frac{x_2}{y_2} = -1 + O(\epsilon), \quad x_3 y_3 = -a + O(\epsilon)$$

So lets construct a surface by glueing

$$\mathbb{C}^2 \cup \mathbb{C}^2 = \left(x_1, \frac{1}{x_1 y_1}\right) \cup \left(x_1 y_1, \frac{1}{y_1}\right)$$

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But this is nothing but blow up of the affine space $\mathrm{Spec}\mathbb{C}[x, Y]$ with the center $(x, Y) = (0, 0)$ which gives the surface $(Y = 1/y)$:

$$X_1 = \{(x, Y, [z_0 : z_1]) \in \mathrm{Spec}\mathbb{C}[x, Y] \times \mathbb{P}^1 | x z_0 = Y z_1\} =$$

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So by blowing up $\mathbb{C}^2$ in the points $(x_1, y_1) = (0, \infty), (x_2, y_2) = (\infty, \infty), (x_3, y_3) = (\infty, 0)$ the equation then make sense on this new surface.
Accordingly we do analize any discrete order two nonlinear equation by identifying the singularities and blow them up.
From now on we shall replace $\mathbb{C}^2$ with $\mathbb{P}^1 \times \mathbb{P}^1$ and any nonlinear equation will be a birational mapping $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ After blowing up the singular points we get a surface X and our mapping is lifted to a regular mapping:

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Algorithm for analysing mappings

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity

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- linearisability = infinite number of blow ups, analytical stability, ruled surface S

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- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
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If the fibers contain exceptional curves of first kind the elliptic surface is called relatively non-minimal. To make it minimal one has to blow down that curves.

Analytical stability and blowing-down structure

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Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational automorphism with iterates growing quadratically with n.
For any such automorphism we can blow up $\mathbb{P}^1 \times \mathbb{P}^1$ and construct a rational surface X such that: $\tilde{\phi} : X \to X$ and $\tilde{\phi}$ is analytically stable which means:
$(\tilde{\phi^*})^n = (\tilde{\phi^n})^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$
Analitical stability is equivalent with the following: There is no divisor D such that exist $k > 0$ and $\tilde{\phi}(D) =$point, $\tilde{\phi}^k(D) =$ indeterminate

$$D \to \bullet \to \bullet \to \ldots \bullet \to D'$$

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\phi}} & X \\ \mu \downarrow & & \downarrow \mu \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

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- Blow down the (-1) curves in the following way: Let C be the (-1) divisor class and F_1, F_2 two divisor classes such that

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- the genus formula is helping here $g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X)$ which must be zero
- then we have a new coordinate system where X is minimal given by the following transformation:

$$\mathbb{C}^2 \ni (x, y) \longrightarrow \left(\frac{y'}{x'}, \frac{y''}{x''}\right) \in \mathbb{P}^1 \times \mathbb{P}^1$$

Singularities and surfaces

Basic example

$$x_{n+1} = -x_{n-1} \frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \tag{2}$$

$$\begin{aligned} \overline{x} &= y \\ \overline{y} &= -x \frac{(y - a)(y - 1/a)}{(y + a)(y + 1/a)} \end{aligned} \tag{3}$$

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Indeterminate points for ϕ and ϕ^{-1}:

$$\begin{aligned} P_1 : (x,y) &= (0,-a), & P_2 : (x,y) &= (0,-1/a), \\ P_3 : (X,y) &= (0,a), & P_4 : (X,y) &= (0,1/a), \\ P_5 : (x,y) &= (a,0), & P_6 : (x,y) &= (1/a,0), \\ P_7 : (x,Y) &= (-a,0), & P_8 : (x,Y) &= (-1/a,0). \end{aligned}$$

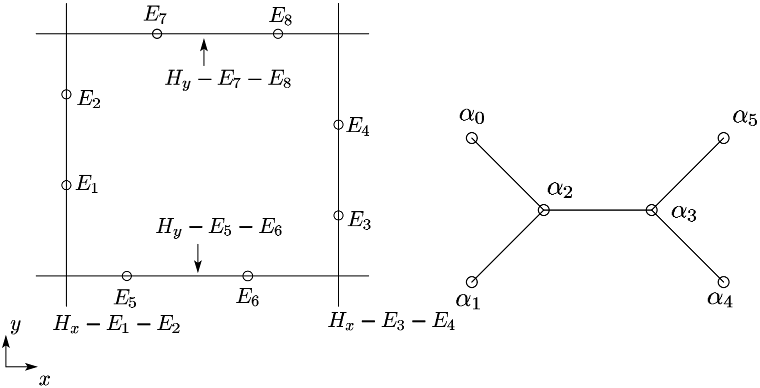

Figure: Space of initial conditions and orthogonal complement

The Picard group of X is a **Z**-module

$$\mathrm{Pic}(X) = \mathbb{Z}H_x \oplus \mathbb{Z}H_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i,$$

H_x, H_y are the total transforms of the lines $x = \mathrm{const.}$, $y = \mathrm{const.}$
E_i are the total transforms of the eight blowing up points.
The intersection form:

$$H_z \cdot H_w = 1 - \delta_{zw}, \quad E_i \cdot E_j = -\delta_{ij}, \quad H_z \cdot E_k = 0$$

for $z, w = x, y$. Anti-canonical divisor of X:

$$-K_X = 2H_x + 2H_y - \sum_{i=1}^{8} E_i.$$

If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice ($h_i, e_j \in \mathbf{Z}$) the induced bundle mapping is acting on it as

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$$
\begin{aligned}
&\phi_*(h_0, h_1, e_1, \ldots, e_8)\\
=&(h_0, h_1, e_1, \ldots, e_8)\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
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It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$:

$$D_0 = H_x - E_1 - E_2,\ D_1 = H_y - E_5 - E_6$$
$$D_2 = H_x - E_3 - E_4,\ D_3 = H_y - E_7 - E_8$$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

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F \equiv & \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0 \\
\Leftrightarrow \; & kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.
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So the conservation law will be:

$$I = \left(\frac{(x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)}{xy} \right)^2$$

Symmetries

Related to orthogonal complement of the space of initial condition $A_3^{(1)}$

$$\mathrm{rank}\mathrm{Pic}(X) = \mathrm{rank} < H_0, H_1, E_1, ...E_8 >_{\mathbb{Z}} = 10$$

Define:

$$< D > = \sum_{i=0}^{3} \mathbb{Z} D_i$$

$$< D >^{\perp} = \{\alpha \in \mathrm{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3\}$$

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Elementary reflections:

$$w_i : \mathrm{Pic}(x) \to \mathrm{Pic}(X), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

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Permutation of roots:

$$\sigma_{10}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$\sigma_{tot}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$$

Hence the group generated by reflections and permutations becomes an extended Weyl group

This extended Weyl group becomes the group of Cremona isometries for the space of initial conditions X since:

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Accordingly our mapping *lives* in a Weyl group and has the following decomposition in elementary reflections:

$$\phi_* = \sigma_{tot} \circ w_3 \circ w_5 \circ w_4 \circ w_3$$

All elements $\omega \in \widetilde{W}(D_5^{(1)})$ which commutes with ϕ_*, namely $(\omega \circ \phi_* = \phi_* \circ \omega)$ form the **symmetries** of the mapping.
The equation is related to the translations in this affine Weyl group. In general for an affine Weyl group with null vector δ the traslation of an element D with respect to the root α_i is given by

$$t_{\alpha_i} : D \to D - (D, \delta)\alpha_i + (D, \alpha_i + \delta)\delta$$

and our mapping is "the fourth root" of a translation:

$$\phi_*^4 \equiv t_{\alpha_3} \circ t_{\alpha_3} \circ t_{\alpha_4} \circ t_{\alpha_5} = t_{2\alpha_3+\alpha_4+\alpha_5}$$

Differential Nahm equations are nonlinear ODE order two describing symmetric monopoles associted to some rotational symmetry groups. The solutions are expressed through rational expressions of Weierstrass elliptic functions and their derivatives (Hitchin, Manton, Murray -'95)

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Hirota-Kimura discretisation

It applies to some class of ODE (quadratic) and has close relation with Hirota bilinear method. More precisely start with:

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij} x_j^2 + \sum_{j<k} b_{ijk} x_j x_k + c_i$$

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In order to find the time discretisation first we bilinearize it by using projective substitution $x_i = G_i/F$ and we get:

$$D_t G_i \cdot F = \sum_{j=1}^{N} a_{ij} G_j^2 + \sum_{j<k} b_{ijk} G_j G_k + c_i F^2$$

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$$\bar{G}_i F - G_i \bar{F} = \epsilon(\sum_{j=1}^{N} a_{ij} G_j \bar{G}_j + \sum_{j<k} b_{ijk}(\alpha \bar{G}_j G_k + (1-\alpha) G_j \bar{G}_k) + c_i F \bar{F})$$

or in the nonlinear form (Kahan '93, Hirota-Kimura, '00)

$$\bar{x}_i - x_i = \epsilon(\sum_{j=1}^{N} a_{ij} x_j \bar{x}_j + \sum_{j<k} b_{ijk}(\alpha \bar{x}_j x_k + (1-\alpha) x_j \bar{x}_k) + c_i)$$

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$$K(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)}$$

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$$K(\epsilon) = \frac{y(3x-y)^2(4x+y)^3}{1+\epsilon^2 c_2 + \epsilon^4 c_4 + \epsilon^6 c_6}$$

with

$$c_2 = -35x^2 + 7y^2$$

$$c_4 = 7(37x^4 + 22x^2y^2 - 2xy^3 + 2y^4)$$

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The tetrahedral symmetry (simple can be brought to QRT):

$$\bar{x} - x = \epsilon(x\bar{x} - y\bar{y})$$

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use the substitution $u = (1-\epsilon x)/y$, $v = (1+\epsilon x)/y$ and we get QRT-mapping ($\bar{u} = v$) and

$$3\bar{u}\underline{u} - u(\bar{u} + \underline{u}) - u^2 + 4\epsilon^2 = 0$$

with the invariant

$$K = \frac{-3(u-\bar{u})^2 + 4\epsilon^2}{2\epsilon^2(u+\bar{u})(u\bar{u}-\epsilon^2)} \equiv \frac{3x^2y - y^3}{1-\epsilon^2(x^2+y^2)}$$

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The cases of octahedral and icosahedral symmetry cannot be transformed to QRT forms by these type of substitutions.
So we need to analyse carefully the singularity structure. What is seen is that we have more singularities and apparently some of them are useless making the corresponding rational elliptic surface to be more complicated.

The case of octahedral symmetry:

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$$\left\{ \begin{array}{ll} \bar{u} & = v \\ \bar{v} & = \dfrac{(u + 2v - 20\epsilon)(v + 10\epsilon)}{4u - v + 10\epsilon} \end{array} \right. . \qquad (4)$$

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The space of initial conditions is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following nine points:

$$E_1 : (u, v) = (-10\epsilon, 0),\ E_2(0, 10\epsilon),\ E_3(10\epsilon, 5\epsilon),$$
$$E_4(5\epsilon, 0),\ E_5(0, -5\epsilon),\ E_6(-5\epsilon, -10\epsilon)$$
$$E_7(\infty, \infty),\ E_8 : (1/u, u/v) = (0, -1/2),\ E_9 : (1/u, u/v) = (0, -2).$$

The action on the Picard group:

$$\bar{H}_u = 2H_u + H_v - E_1 - E_3 - E_7 - E_8,\ \bar{H}_v = H_u$$
$$\bar{E}_1 = E_2,\ \bar{E}_2 = H_u - E_3,\ \bar{E}_3 = E_4,\ \bar{E}_4 = E_5,\ \bar{E}_5 = E_6,$$
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Three invariant divisor classes:

$$\alpha_0 = H_u + H_v - E_1 - E_2 - E_7,\ \alpha_1 = H_u + H_v - E_1 - E_2 - E_8 - E_9,$$
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The curve corresponding to α_0 is a (-1) curve which must be blown down. $E_1 \to H_a = H_u + H_v - E_2 - E_7$ and $E_2 \to H_b = H_u + H_v - E_1 - E_7$, 0-curves intersecting each other: The corresponding curves are given by:

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So if we set $a = (v - 10\epsilon)/u \quad b = (u + 10\epsilon)/v$ our dynamical system becomes

$$\left\{ \begin{array}{rcl} \bar{a} & = & \dfrac{3ab - 2a + 2}{a - 4} \\ \bar{b} & = & \dfrac{4 - a}{2a + 1} \end{array} \right. . \qquad (5)$$

This system has the following space of initial conditions which define a minimal rational elliptic surface:

$$F_1 : (a, b) = (0, \infty),\ \ F_2 : (a, b) = (\infty, 0),$$
$$F_3 : (a, b) = (-1/2, 4),\ \ F_4 : (a, b) = (-2, \infty)$$
$$F_5 : (a, b) = (\infty, -2),\ \ F_6 : (a, b) = (4, -1/2),$$
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The invariant is nothing but the proper transform of the anti-canonical divisor:

$$K_X = 2H_a + 2H_b - \oplus_{i=1}^{8} F_i$$

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which is the same as the one given at the beginning [Suris et al. 2012]

$$K(\epsilon) = \frac{y(2x+3y)(x-y)^2}{1-10\epsilon^2(x^2+4y^2)+\epsilon^4(9x^4+272x^3y-352xy^3+696y^4)}$$

The case of icosahedral symmetry:

$$\bar{x} - x = \epsilon(2x\bar{x} - y\bar{y})$$

$$\bar{y} - y = -\epsilon(5y\bar{x} + 5x\bar{y} - y\bar{y})$$

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The space of initial condition is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following 12 points:

$$\begin{aligned}
&E_1 : (x, y) = (\infty, \infty),\ E_2(-1/7\epsilon, -3/7\epsilon),\ E_3(-1/7\epsilon, 4/7\epsilon),\\
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&E_7(1/3\epsilon, 0),\ E_8(1/\epsilon, 0),\ E_9(-1/\epsilon, 0),\\
&E_{10}(-1/3\epsilon, 0),\ E_{11}(-1/5\epsilon, 0). E_{12} : (1/x, x/y) = (0, 1/3)
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Singularity confinement gives the following pattern:

$$H_y - E_1\ (y = \infty) \to \text{point} \to \cdots (4\ \text{points}) \cdots \to \text{point} \to H_y - E_1$$
$$\cdots \to \text{point} \to \text{point} \to H_x - E_1\ (x = \infty) \to \text{point} \to \text{point} \to \cdots .$$

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The curve $4x + y = 0 : H_x + H_y - E_1 - E_3 - E_5$ is invariant and we blow it down

So $E_3 \to H_v = H_x + H_y - E_1 - E_5$ and $E_5 \to H_u = H_x + H_y - E_1 - E_3$ with

$$H_u \cdot H_u = H_v \cdot H_v = 0, H_u \cdot H_v = 1$$

where the linear systems of H_u and H_v are given by

$$\begin{aligned} |H_u| : & u_0(1 + 7\epsilon x) + u_1(4x + y) \\ |H_v| : & v_0(1 - 7\epsilon x) + v_1(4x + y). \end{aligned}$$

So $E_3 \to H_v = H_x + H_y - E_1 - E_5$ and $E_5 \to H_u = H_x + H_y - E_1 - E_3$ with

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If we take the new variables u and v as

$$u = \frac{2(1 + 7\epsilon x)}{\epsilon(4x + y)}, \; v = \frac{2(1 - 7\epsilon x)}{\epsilon(4x + y)},$$

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then we have a new space for initial conditions given by nine blow up points:

$$\begin{aligned} &F_1 : (u, v) = (2, -2), F_2 : (0, -4), F_3 : (4, 0), F_4 : (6, -1), F_5 : (5, -2), \\ &F_6 : (4, -3), F_7 : (3, -4), F_8 : (2, -5), F_9 : (1, -6). \end{aligned}$$

The dynamical system becomes an automorphism having the following topological singularity patterns

$$H_v - F_9 \to F_2 \to F_1 \to F_3 \to H_u - F_4$$
$$H_v - F_3 \to F_4 \to F_5 \to F_6 \to F_7 \to F_8 \to F_9 \to H_u - F_2$$

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The invariant (-1) curve $H_u + H_v - F_1 - F_2 - F_3$, which should be blown down.

$$F_3 \to H_s = H_u + H_v - F_1 - F_2, \quad F_2 \to H_t = H_u + H_v - F_1 - F_3$$

where the linear systems of H_s and H_t are given by

$$|H_s| : s_0 u(v+2) + s_1(u - v - 4)$$
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and hence we take the new variables s and t as

$$s = -\frac{3u(v+2)}{2(u-v-4)}, t = -\frac{3v(u-2)}{2(u-v-4)}$$

$$\begin{cases} \bar{s} & = & \dfrac{2st - 3s - 3t + 9}{s + t - 3} \\ \bar{t} & = & \dfrac{2(s - 3)(t + 3)}{3s - t - 9} \end{cases} .$$

with the blow-up points

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$$F_1' : (s,t) = (3,0),\ \ F_2'(0,3),\ \ F_3'(-3,2),\ \ F_4' : (\frac{s}{t-3}, t-3) = (5,0),$$

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The invariants can be computed by using the the anticanonical divisor:

$$K = \frac{(s-t)^2 + 4(s+t) - 21}{(s-2)(t-2)(2st-5s-5t+15)} = \frac{-56\epsilon^6 y(-3x+y)^2(4x+y)^3}{d_1 d_2 d_3} \tag{6}$$

where

$$d_1 = -3 - 12\epsilon x + 15\epsilon^2 x^2 - 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2$$
$$d_2 = -3 + 12\epsilon x + 15\epsilon^2 x^2 + 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2$$
$$d_3 = -3 + 27\epsilon^2 x^2 + 10\epsilon^2 xy + 10\epsilon^2 y^2.$$

Tropical dynamical systems

Motivation: How simple can a nonlinearity be? How do behave the discrete equations with the simplest nonlinearity?

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Mathematically the tropicalisation has been introduced as follows: Calling $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ we introduce the semiring $\{\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e\}$ through the following definitions:

- $a \oplus b := \max(a, b), \quad a \otimes b := a + b$
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A nonlinear discrete equation (ordinary or partial) with *positive definite dependent variable* x_n can be ultradiscretised or tropicalised using the following substitution and formula:

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Example:

$$x_{n+1}x_{n-1} = a\frac{1 + x_n}{x_n^2}, \qquad I_n = \frac{a(1 + x_n + x_{n+1}) + x_n^2 x_{n+1}^2}{x_n x_{n+1}}$$

If $x_n = \exp(X_n/\epsilon)$, $a = \exp A/\epsilon$ then we get the tropical equation and the invariant:

$$X_{n+1}+X_{n-1} = A+\max(0, X_n)-2X_n, \quad \mathcal{I} = \max(2X_n+2X_{n+1}, A, A+X_n, A+X_{n+1})-X_{n+1}-X_n$$

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YES! We shall thus examine the behaviour of a singularity appearing at, say, $n = 1$ where $X_1 = \epsilon$, while X_0 is regular and look at the *propagation of this singularity both forwards and backwards*. Introducing the notation $\mu \equiv \max(\epsilon, 0)$, the presence of μ indicates that the value of X **is singular.** We get (for $X_0 > A$):

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$\vdots$
$X_{-3} = A - \epsilon$
$X_{-2} = X_0 - A + 2\epsilon$
$X_{-1} = -X_0 + A - \epsilon$
$X_0 = X_0$
$X_1 = \epsilon$
$X_2 = A - X_0 - 2\epsilon + \mu$
$X_3 = 2X_0 - A + 3\epsilon - 2\mu$
$X_4 = A - X_0 - \epsilon + \mu$
$X_5 = -\epsilon$
$X_6 = X_0 + 2\epsilon$
$\vdots$

Conclusions

- Singularities are essential in analysing discrete dynamical systems.
- The singularity structure may give a non-minimal elliptic surface. In order to make it minimal one has to blow down some -1 divisor classes (one has to prove the existence of the blow-down structure)
- after minimization the mapping can be "solved"
- we expect to find analogies in the case of tropical dynamical systems using tropical algebraic geometry.

All the results are published here:

1.A. S. Carstea, T. Takenawa, Journal of Nonlinear Mathematical Physics, vol. 20, Supplement 1 (Special Issue on Geometry of the Painlevé equations) 17-33, (2013). Also on arXiv:1211.5393

2. A. S. Carstea, T. Takenawa, Journal of Phyiscs A: Math. Theor. 45, 15, 155206, (2012)

3. A. S. Carstea, *On the geometry of Q_4 mapping*, Contemporary Mathematics, vol. 593, 231-239, (2013)

4. B. Grammaticos, A. Ramani, K. M. Tamizhmani, T. Tamizhmani, A.S. Carstea, J. Phys. A: Math. Theor. 40, F725, (2007)