Dissipative vector fields and applications to stability theory

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Motivation

The purpose of this talk is to present from a dynamical-geometrical perspective, the class of smooth vector fields (defined on a finite dimensional smooth Riemannian manifold) that admit a given set of first integrals, and dissipate an a-priori given set of scalar quantities, with prescribed dissipation rates.
As application, we provide a method to construct dissipative perturbations of completely integrable systems in order to control the stability of periodic orbits.

A coordinate-free formulation of hyperplanes intersection

Next result provides a coordinate free formulation for the intersection of k linear hyperplanes with prescribed normal directions.

Proposition

Let $(E, \langle\cdot,\cdot\rangle)$ be an n-dimensional inner product space over a field $\mathbb{K}$ of characteristic zero, and let $\{v_1, \dots, v_k\} \subset E$ be a set of linearly independent vectors ($k \in \mathbb{N}$, $0 < k < n-1$). Then the solutions $u \in E$ of the system

$$\langle u, v_1\rangle = \cdots = \langle u, v_k\rangle = 0,$$

are the elements of the $(n-k)$-dimensional vector subspace

$$E[v_1, \dots, v_k] := \operatorname{span}_{\mathbb{K}} \left\{ \star\left(\bigwedge_{i=1, i\neq a}^{n-k} \omega_i \wedge \bigwedge_{l=1}^{k} v_l \right) : a \in \{1, \dots, n-k\} \right\},$$

A coordinate-free formulation of hyperplanes intersection

where

$$\{\omega_1, \ldots, \omega_{n-k}\} \subset E$$

is a set of linearly independent vectors such that $\{v_1, \ldots, v_k, \omega_1, \ldots, \omega_{n-k}\}$ forms a basis of E.

Remark

- For $k = n$, the Proposition (3.1) becomes trivial since the only solution of the system $\langle u, v_1 \rangle = \cdots = \langle u, v_n \rangle = 0$ is $u = 0$.
- For $k = n - 1$, the conclusion of Proposition (3.1) becomes as follows:
  The solutions $u \in E$ of the system

  $$\langle u, v_1 \rangle = \cdots = \langle u, v_{n-1} \rangle = 0,$$

  are the elements of the 1-dimensional vector subspace

  $$E[v_1, \dots, v_{n-1}] := \operatorname{span}_{\mathbb{K}} \left\{ \star \left(\bigwedge_{l=1}^{n-1} v_l \right) \right\}.$$

A coordinate-free formulation of hyperplanes intersection

Let us give now the main result of this section, which provides a coordinate free formulation of the linear variety described by the intersection of k linear hyperplanes and respectively p affine hyperplanes of an n - dimensional inner product space $(E, \langle \cdot, \cdot \rangle)$.

A coordinate-free formulation of hyperplanes intersection

Theorem

Let $(E, \langle\cdot,\cdot\rangle)$ be an n-dimensional inner product space over a field $\mathbb{K}$ of characteristic zero. Let $k, p \in \mathbb{N}$, $k > 0$, $p > 1$, $k + p < n - 1$, $\lambda_1, \ldots, \lambda_p \in \mathbb{K} \setminus \{0\}$ be given, and let $\{v_1, \ldots, v_k, w_1, \ldots, w_p\} \subset E$ be a set of linearly independent vectors.
Then the solutions $u \in E$ of the system

$$\begin{cases} \langle u, v_1\rangle = \cdots = \langle u, v_k\rangle = 0, \\ \langle u, w_1\rangle = \lambda_1, \ldots, \langle u, w_p\rangle = \lambda_p, \end{cases} \tag{1}$$

are given by $u = u_0 + u_\perp$, where

$$u_0 = \left\| \bigwedge_{i=1}^{p} w_i \wedge \bigwedge_{j=1}^{k} v_j \right\|_{k+p}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} \lambda_i \Theta_i,$$

with

$$\Theta_i = \star\left[\bigwedge_{j=1, j\neq i}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l \wedge \star\left(\bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l\right)\right],$$

and

$$u_\perp \in \operatorname{span}_{\mathbb{K}}\left\{\star\left(\bigwedge_{i=1, i\neq a}^{n-(k+p)} \omega_i \wedge \bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l\right) : a \in \{1, \ldots, n-(k+p)\}\right\},$$

where

$$\{\omega_1, \ldots, \omega_{n-(k+p)}\} \subset E$$

is a set of linearly independent vectors such that $\{v_1, \ldots, v_k, w_1, \ldots, w_p, \omega_1, \ldots, \omega_{n-(k+p)}\}$ forms a basis of E.

A coordinate-free formulation of hyperplanes intersection

Remark

If one adopt the notation $\mathscr{H}_{(v_i;0)} := \{u \in E \mid \langle u, v_i \rangle = 0\}$, $i \in \{1, \ldots, k\}$, and respectively $\mathscr{H}_{(w_j;\lambda_j)} := \{u \in E \mid \langle u, w_j \rangle = \lambda_j\}$, $j \in \{1, \ldots, p\}$, then the intersection of the above defined linear and respectively affine hyperplanes, is the linear variety

$$\bigcap_{i=1}^{k} \mathscr{H}_{(v_i;0)} \cap \bigcap_{j=1}^{p} \mathscr{H}_{(w_j;\lambda_j)} = u_0 + E[w_1, \ldots, w_p, v_1, \ldots, v_k], \tag{2}$$

where

$$u_0 = \left\| \bigwedge_{i=1}^{p} w_i \wedge \bigwedge_{j=1}^{k} v_j \right\|_{k+p}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} \lambda_i \Theta_i,$$

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with

$$\Theta_i = \star\left[\bigwedge_{j=1, j\neq i}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l \wedge \star\left(\bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l\right)\right],$$

and

$$E[w_1, \ldots, w_p, v_1, \ldots, v_k] = \bigcap_{i=1}^{k} \mathscr{H}_{(v_i;0)} \cap \bigcap_{j=1}^{p} \mathscr{H}_{(w_j;0)}$$

$$= \mathrm{span}_{\mathbb{K}}\left\{\star\left(\bigwedge_{i=1, i\neq a}^{n-(k+p)} \omega_i \wedge \bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l\right) : a \in \{1, \ldots, n-(k+p)\}\right\},$$

where

$$\{\omega_1, \ldots, \omega_{n-(k+p)}\} \subset E$$

is a set of linearly independent vectors such that $\{v_1, \ldots, v_k, w_1, \ldots, w_p, \omega_1, \ldots, \omega_{n-(k+p)}\}$ forms a basis of E.

A coordinate-free formulation of hyperplanes intersection

Let us point out that the Theorem (1) remains valid also for the limit cases $p \in \{0,1\}$, $k = 0$, $k + p \in \{n-1, n\}$, the only difference from the general case being the inconsistency of the notations, which in these limit cases may lead to confusions. Hence, for these limit cases we prefer to state separately the conclusion of the Theorem (1).

Remark

- For $p = 0$, the Theorem (1) reduces to Proposition (3.1).
- For $p = 1$, the conclusion of the Theorem (1) becomes as follows:
  The solutions $u \in E$ of the system

$$\begin{cases} \langle u, v_1 \rangle = \cdots = \langle u, v_k \rangle = 0, \\ \langle u, w_1 \rangle = \lambda_1, \end{cases}$$

  are given by $u = u_0 + u_\perp$, where

$$u_0 = \left\| w_1 \wedge \bigwedge_{j=1}^{k} v_j \right\|_{k+1}^{-2} \cdot (-1)^{n-1} \lambda_1 \Theta_1,$$

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with

$$\Theta_1 = \star \left[\bigwedge_{l=1}^{k} v_l \wedge \star \left(w_1 \wedge \bigwedge_{l=1}^{k} v_l \right) \right],$$

and

$$u_\perp \in \operatorname{span}_{\mathbb{K}} \left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-(k+1)} \omega_i \wedge w_1 \wedge \bigwedge_{l=1}^{k} v_l \right) : a \in \{1, \ldots, n-(k+1)\} \right\},$$

where

$$\{\omega_1, \ldots, \omega_{n-(k+1)}\} \subset E$$

is a set of linearly independent vectors such that $\{v_1, \ldots, v_k, w_1, \omega_1, \ldots, \omega_{n-(k+1)}\}$ forms a basis of E.

Remark

For $k = 0$, the conclusion of the Theorem (1) becomes as follows: The solutions $u \in E$ of the system

$$\langle u, w_1 \rangle = \lambda_1, \ldots, \langle u, w_p \rangle = \lambda_p,$$

are given by $u = u_0 + u_\perp$, where

$$u_0 = \left\| \bigwedge_{i=1}^{p} w_i \right\|_p^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} \lambda_i \Theta_i,$$

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with

$$\Theta_i = \star\left[\bigwedge_{j=1, j\neq i}^{p} w_j \wedge \star\left(\bigwedge_{j=1}^{p} w_j\right)\right],$$

and

$$u_\perp \in \mathrm{span}_{\mathbb{K}}\left\{\star\left(\bigwedge_{i=1, i\neq a}^{n-p} \omega_i \wedge \bigwedge_{j=1}^{p} w_j\right) : a \in \{1, \ldots, n-p\}\right\},$$

where

$$\{\omega_1, \ldots, \omega_{n-p}\} \subset E$$

is a set of linearly independent vectors such that $\{w_1, \ldots, w_p, \omega_1, \ldots, \omega_{n-p}\}$ forms a basis of E.

Remark

In the case when $k+p=n-1$, the conclusions of the Theorem (1) still hold true, the only difference being the fact that the direction of the linear variety is one-dimensional and can be expressed only in terms of the vectors $v_1, \ldots, v_k, w_1, \ldots, w_p$:

$$u_{\perp} \in \operatorname{span}_{\mathbb{K}} \left\{ \star \left(\bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l \right) \right\}$$

A coordinate-free formulation of hyperplanes intersection

Remark

For $k+p=n$, the conclusion of the Theorem (1) becomes as follows:
The system

$$\begin{cases} \langle u, v_1 \rangle = \cdots = \langle u, v_k \rangle = 0, \\ \langle u, w_1 \rangle = \lambda_1, \ldots, \langle u, w_p \rangle = \lambda_p, \end{cases}$$

has a unique solution which is given by $u = u_0$, where

$$u_0 = \left\| \bigwedge_{i=1}^{p} w_i \wedge \bigwedge_{j=1}^{k} v_j \right\|_n^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} \lambda_i \Theta_i,$$

with

$$\Theta_i = \star \left[\bigwedge_{j=1, j \neq i}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l \wedge \star \left(\bigwedge_{j=1}^{p} w_j \wedge \bigwedge_{l=1}^{k} v_l \right) \right].$$

Local generators of affine distributions on Riemannian manifolds

The purpose of this section is to translate on smooth Riemannian manifolds the results given in the previous section. This approach follows naturally, and has direct applications to dynamical systems. As we will see in the next section, the results presented here will provide an explicit characterization of conservative and also dissipative dynamical systems.

Local generators of affine distributions on Riemannian manifolds

Theorem

Let (M,g) be an n-dimensional smooth Riemannian manifold, and fix $k,p \in \mathbb{N}$ two natural numbers such that $k>0$, $p>1$, $k+p<n-1$. Let $h_1,\dots,h_p \in \mathscr{C}^\infty(U,\mathbb{R})$ be a given set of non-zero smooth functions defined on an open subset $U \subseteq M$, and respectively let $\{X_1,\dots,X_k,Y_1,\dots,Y_p\} \subset \mathfrak{X}(U)$ be a set of linearly independent vector fields on U.
Then the solutions $X \in \mathfrak{X}(U)$ of the system

$$\begin{cases} g(X,X_1)=\cdots=g(X,X_k)=0, \\ g(X,Y_1)=h_1,\dots,g(X,Y_p)=h_p, \end{cases} \tag{3}$$

are given by $X = X_0 + X_\perp$,

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where

$$X_0 = \left\| \bigwedge_{i=1}^{p} Y_i \wedge \bigwedge_{j=1}^{k} X_j \right\|_{k+p}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i \Theta_i,$$

with

$$\Theta_i = \star \left[\bigwedge_{j=1, j\neq i}^{p} Y_j \wedge \bigwedge_{l=1}^{k} X_l \wedge \star \left(\bigwedge_{j=1}^{p} Y_j \wedge \bigwedge_{l=1}^{k} X_l \right) \right],$$

and $X_\perp(x) \in \mathrm{span}_{\mathbb{R}}^{\perp_{g(x)}} \{X_1(x), \ldots, X_k(x), \ldots, Y_1(x), \ldots, Y_p(x)\}$, for any $x \in U$.

Local generators of affine distributions on Riemannian manifolds

Moreover, for each $x \in U$, there exists an open neighborhood $U_x \subseteq U$, such that for any $x' \in U_x$

$$X_\perp(x') \in \operatorname{span}_{\mathbb{R}}\{\star_{x'}\left(\bigwedge_{i=1, i\neq a}^{n-(k+p)} Z_i(x') \wedge \bigwedge_{j=1}^{p} Y_j(x') \wedge \bigwedge_{l=1}^{k} X_l(x')\right) :$$

$$a \in \{1, \ldots, n-(k+p)\}\},$$

where

$$\{Z_1, \ldots, Z_{n-(k+p)}\} \subset \mathfrak{X}(U_x)$$

is an **arbitrary** set of linearly independent vector fields on U_x, such that the vector fields

$$\{X_1, \ldots, X_k, Y_1, \ldots, Y_p, Z_1, \ldots, Z_{n-(k+p)}\}$$

are linearly independent on the open subset $U_x \subseteq U$, i.e., they form a moving frame on U_x.

Local generators of affine distributions on Riemannian manifolds

Remark

The set of vector fields

$$\mathfrak{X}[X_1, \ldots, X_k, Y_1, \ldots, Y_p] = \{X \in \mathfrak{X}(U) \mid g(X, X_i) = g(X, Y_j) = 0; \\ 1 \leq i \leq k; 1 \leq j \leq p\}$$

forms an $[n-(k+p)]$-dimensional smooth distribution, locally generated around each point $x \in U$, in some open neighborhood $U_x \subseteq U$, by the set of vector fields

$$\left\{ \star \left(\bigwedge_{i=1, i \neq a}^{n-(k+p)} Z_i \wedge \bigwedge_{j=1}^{p} Y_j \wedge \bigwedge_{l=1}^{k} X_l \right) : a \in \{1, \ldots, n-(k+p)\} \right\} \subset \mathfrak{X}(U_x).$$

Local generators of affine distributions on Riemannian manifolds

Recall that in contrast with the vector fields $X_1, \ldots, X_k, Y_1, \ldots, Y_p$, which are globally defined on U, the vector fields $Z_1, \ldots, Z_{n-(k+p)}$ are only locally defined since their existence depend on x, and is guaranteed in general only in some open neighborhood U_x around x. Moreover, they are arbitrary chosen in order to be linearly independent and to complete locally the set of vector fields $\{X_1, \ldots, X_k, Y_1, \ldots, Y_p\}$ up to a moving frame in U_x.

Local generators of affine distributions on Riemannian manifolds

By a similar argument as in the proof of Proposition (3.1), the above defined set of local generators does not depend on the set of locally defined linearly independent vector fields $Z_1, \ldots, Z_{n-(k+p)}$, as long as

$$\{X_1, \ldots, X_k, Y_1, \ldots, Y_p, Z_1, \ldots, Z_{n-(k+p)}\}$$

forms a moving frame.

Local generators of affine distributions on Riemannian manifolds

Let us fix some general notations. Let $\mathscr{A} \subset \mathfrak{X}(U)$ be a smooth r-dimensional affine distribution on the open subset U of a smooth $n-$dimensional manifold M. This means that for each $x \in U$, there exists an open neighborhood $U_x \subseteq U$, a smooth vector field $X_0 \in \mathfrak{X}(U_x)$, and r linearly independent smooth vector fields $\{X_1, \ldots, X_r\} \subset \mathfrak{X}(U_x)$ such that

$$\mathscr{A}_{x'} = X_0(x') + \operatorname{span}_{\mathbb{R}}\{X_1(x'), \ldots, X_r(x')\},$$

for each $x' \in U_x$.

Local generators of affine distributions on Riemannian manifolds

A set of locally defined vector fields

$$\{X_0\} \biguplus \{X_1, \ldots, X_r\},$$

fulfilling the above requirements, is called *a set of local generators* of the smooth affine distribution $\mathscr{A}$.
Recall that the $r-$dimensional smooth distribution that assigns to each $x \in U$ the direction of the affine space $\mathscr{A}_x$, is denoted by $L(\mathscr{A})$, and is called *the linear part* of the affine distribution $\mathscr{A}$.
Consequently, $L(\mathscr{A})$ can be generated locally around x, as

$$L(\mathscr{A})_{x'} = \mathrm{span}_{\mathbb{R}}\{X_1(x'), \ldots, X_r(x')\},$$

for every $x' \in U_x$.

Local generators of affine distributions on Riemannian manifolds

Using the above notation for a set of local generators of a smooth affine distribution, the conclusion of the Theorem (2) can be reformulated as follows.

Theorem

In the hypothesis of Theorem (2)*, the solutions* $X \in \mathfrak{X}(U)$ *of the system* (3) *form the* $[n-(k+p)]$*-dimensional smooth affine distribution*

$$\mathfrak{A}[X_0; X_1, \ldots, X_k, Y_1, \ldots, Y_p] := X_0 + \mathfrak{X}[X_1, \ldots, X_k, Y_1, \ldots, Y_p],$$

locally generated by the following set of $[n-(k+p)]+1$ *vector fields*

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-(k+p)} Z_i \wedge \bigwedge_{j=1}^{p} Y_j \wedge \bigwedge_{l=1}^{k} X_l \right) : a \in \{1, \ldots, n-(k+p)\} \right\}.$$

Local generators of affine distributions on Riemannian manifolds

As in the case of Theorem (1), let us now discuss some special cases of Theorems (2), (3), namely the Riemannian analogous of Remarks (3.3), (3.4), (3.5), (3.6).

Local generators of affine distributions on Riemannian manifolds

Remark

- For $p = 0$, the conclusion of Theorem (2) becomes as follows: The distribution

$$\mathfrak{X}[X_1, \ldots, X_k] = \{X \in \mathfrak{X}(U) \mid g(X, X_1) = \cdots = g(X, X_k) = 0\},$$

is locally generated by the set of vector fields

$$\left\{ \star \left(\bigwedge_{i=1, i \neq a}^{n-k} Z_i \wedge \bigwedge_{l=1}^{k} X_l \right) : a \in \{1, \ldots, n-k\} \right\},$$

where the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-k}, X_1, \ldots, X_k\}$$

forms a moving frame.

Local generators of affine distributions on Riemannian manifolds

- For $p=1$, the conclusion of Theorem (2) becomes as follows: The affine distribution

$$\mathfrak{A}[X_0; X_1, \ldots, X_k, Y_1] = \{X \in \mathfrak{X}(U) \mid g(X, X_1) = \ldots$$
$$= g(X, X_k) = 0, g(X, Y_1) = h_1\} = X_0 + \mathfrak{X}[X_1, \ldots, X_k, Y_1],$$

is locally generated by the set of vector fields

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-(k+1)} Z_i \wedge Y_1 \wedge \bigwedge_{l=1}^{k} X_l \right) : a \in \{1, \ldots, n-(k+1)\} \right\},$$

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where

$$X_0 = \left\| Y_1 \wedge \bigwedge_{j=1}^{k} X_j \right\|_{k+1}^{-2} \cdot (-1)^{n-1} h_1 \cdot \left(\star \left[\bigwedge_{l=1}^{k} X_l \wedge \star \left(Y_1 \wedge \bigwedge_{l=1}^{k} X_l \right) \right] \right),$$

and respectively the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-(k+1)}, X_1, \ldots, X_k, Y_1\}$$

forms a moving frame.

Local generators of affine distributions on Riemannian manifolds

Remark

For $k=0$, the conclusion of Theorem (2) becomes as follows:
The affine distribution

$$\begin{aligned}\mathfrak{A}[X_0; Y_1, \ldots, Y_p] &= \{X \in \mathfrak{X}(U) \mid g(X, Y_j) = h_j, \quad 1 \leq j \leq p\} \\ &= X_0 + \mathfrak{X}[Y_1, \ldots, Y_p],\end{aligned}$$

is locally generated by the set of vector fields

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-p} Z_i \wedge \bigwedge_{j=1}^{p} Y_j \right) : a \in \{1, \ldots, n-p\} \right\},$$

where

$$X_0 = \left\| \bigwedge_{i=1}^{p} Y_i \right\|_p^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i \cdot \left(\star \left[\bigwedge_{j=1, j \neq i}^{p} Y_j \wedge \star \left(\bigwedge_{j=1}^{p} Y_j \right) \right] \right),$$

and respectively the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-p}, Y_1, \ldots, Y_p\}$$

forms a moving frame.

Local generators of affine distributions on Riemannian manifolds

Remark

For $k+p=n-1$, the conclusion of Theorem (2) becomes as follows:
The affine distribution $\mathfrak{A}[X_0; X_1, \ldots, X_k, Y_1, \ldots, Y_p]$ is locally generated by the set vector fields

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{j=1}^{p} Y_j \wedge \bigwedge_{l=1}^{k} X_l \right) \right\}.$$

Remark

For $k+p=n$, the conclusion of Theorem (2) reduces to:

$$\begin{aligned}&\mathfrak{A}[X_0;X_1,\dots,X_k,Y_1,\dots,Y_p]\\&\quad=\{X\in\mathfrak{X}(U)\mid g(X,X_i)=0,g(X,Y_j)=h_j,1\le i\le k,1\le j\le p\}\\&\quad=\{X_0\},\end{aligned}$$

where

$$X_0=\left\|\bigwedge_{i=1}^{p}Y_i\wedge\bigwedge_{l=1}^{k}X_l\right\|_n^{-2}\cdot$$
$$\sum_{i=1}^{p}(-1)^{n-i}h_i\cdot\left(\star\left[\bigwedge_{j=1,j\neq i}^{p}Y_j\wedge\bigwedge_{l=1}^{k}X_l\wedge\star\left(\bigwedge_{j=1}^{p}Y_j\wedge\bigwedge_{l=1}^{k}X_l\right)\right]\right).$$

Applications to dynamical systems

The aim of this section is to apply the main results from the previous section in the case of linear/affine distributions associated to conservative/dissipative dynamical systems defined eventually on an open subset U of a Riemannian manifold (M, g).

Applications to dynamical systems

Before stating the main results, let us recall that a smooth function $F \in \mathscr{C}^\infty(U,\mathbb{R})$ is said to be a first integral (or conservation law) of the vector field $X \in \mathfrak{X}(U)$ if $\mathscr{L}_X F = 0$, where $\mathscr{L}_X$ stands for the Lie derivative along the vector field X, or equivalently one say that X conserves F.
Similarly, a vector field $X \in \mathfrak{X}(U)$ is said to dissipate the smooth function $H \in \mathscr{C}^\infty(U,\mathbb{R})$ with dissipation rate $h \in \mathscr{C}^\infty(U,\mathbb{R})$, if $\mathscr{L}_X H = h$.
In the Riemannian setting, these conditions are obviously equivalent to $g(X,\nabla_g F) = 0$, and respectively $g(X,\nabla_g H) = h$, where ∇_g stands for the gradient operator with respect to the Riemannian metric g.

In what follows, a vector field $X \in \mathfrak{X}(U)$ will be called **dissipative** if there exist $k, p \in \mathbb{N}$ with $k + p > 0$, and a set of smooth functions $\{I_1, \ldots, I_k, D_1, \ldots, D_p, h_1, \ldots, h_p\} \subset \mathscr{C}^\infty(U, \mathbb{R})$ such that the vector field X conserves $I_1, \ldots, I_k$ and dissipates $D_1, \ldots, D_p$ with (corresponding) dissipation rates $h_1, \ldots, h_p$. If $p = 0$, the vector field X will be called **conservative**.

Hence, one can apply the Theorem (2) in the case of linear/affine distributions associated to conservative/dissipative vector fields defined eventually on an open subset U of a Riemannian manifold (M,g).

Theorem

Let (M,g) be an n-dimensional smooth Riemannian manifold, and fix $k,p \in \mathbb{N}$ two natural numbers such that $k > 0$, $p > 1$, $k + p < n - 1$. Let $h_1, \ldots, h_p \in \mathscr{C}^\infty(U,\mathbb{R})$ be a given set of non-zero smooth functions defined on an open subset $U \subseteq M$, and respectively let $I_1, \ldots, I_k, D_1, \ldots, D_p \in \mathscr{C}^\infty(U,\mathbb{R})$ be given, such that

$$\{\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p\} \subset \mathfrak{X}(U)$$

form a set of linearly independent vector fields on U.

Then the solutions $X \in \mathfrak{X}(U)$ of the system

$$\begin{cases} \mathscr{L}_X I_1 = \cdots = \mathscr{L}_X I_k = 0, \\ \mathscr{L}_X D_1 = h_1, \ldots, \mathscr{L}_X D_p = h_p, \end{cases}$$

form the affine distribution (consisting of dissipative vector fields)

$$\mathfrak{A}[X_0; \nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p] = \\ X_0 + \mathfrak{X}[\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p],$$

locally generated by the set of vector fields

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-(k+p)} Z_i \wedge \bigwedge_{j=1}^{p} \nabla_g D_j \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \right) : a \in \{1, \ldots, n-(k+p)\} \right\}$$

where

$$X_0 = \left\| \bigwedge_{i=1}^{p} \nabla_g D_i \wedge \bigwedge_{j=1}^{k} \nabla_g I_j \right\|_{k+p}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i \Theta_i,$$

$$\Theta_i = \star \left[\bigwedge_{j=1, j\neq i}^{p} \nabla_g D_j \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \wedge \star \left(\bigwedge_{j=1}^{p} \nabla_g D_j \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \right) \right],$$

and respectively the set of locally defined vector fields

$$\{\nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1, \ldots, \nabla_g D_p, Z_1, \ldots, Z_{n-(k+p)}\}$$

forms a moving frame.

Applications to dynamical systems

A dynamical version of Theorem (4) can be formulated as follows.

Theorem

Let $\dot{x} = X(x)$ be the dynamical system generated by a vector field $X \in \mathfrak{X}(U)$ which conserves the smooth (functionally independent) functions

$$I_1, \ldots, I_k, D_1, \ldots, D_p \in \mathscr{C}^\infty(U, \mathbb{R}).$$

Then the perturbed dynamical system

$$\dot{x} = X(x) + X_0(x),$$

with X_0 given in Theorem (4), is a dissipative dynamical system, generated by the dissipative vector field $X + X_0$ which conserves $I_1, \ldots, I_k$, and dissipates $D_1, \ldots, D_p$ with (corresponding) dissipation rates $h_1, \ldots, h_p$.

Remark

- For $p = 0$, the conclusion of Theorem (4) becomes as follows: The distribution

$$\mathfrak{X}[\nabla_g I_1, \ldots, \nabla_g I_k] = \{X \in \mathfrak{X}(U) \mid \mathscr{L}_X I_1 = \cdots = \mathscr{L}_X I_k = 0\},$$

is locally generated by the set of vector fields

$$\left\{ \star \left(\bigwedge_{i=1, i\neq a}^{n-k} Z_i \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \right) : a \in \{1, \ldots, n-k\} \right\},$$

where the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-k}, \nabla_g I_l, \ldots, \nabla_g I_k\}$$

forms a moving frame.

- For $p=1$, the conclusion of Theorem (4) becomes as follows: The affine distribution

$$\begin{aligned}
&\mathfrak{A}[X_0;\nabla_g I_1,\dots,\nabla_g I_k,\nabla_g D_1]\\
&\qquad=\{X\in\mathfrak{X}(U)\,|\,\mathscr{L}_X I_1=\cdots=\mathscr{L}_X I_k=0,\mathscr{L}_X D_1=h_1\}\\
&\qquad=X_0+\mathfrak{X}[\nabla_g I_1,\dots,\nabla_g I_k,\nabla_g D_1],
\end{aligned}$$

is locally generated by the set of vector fields

$$\{X_0\}\biguplus\left\{\star\left(\bigwedge_{i=1,i\neq a}^{n-(k+1)} Z_i\wedge\nabla_g D_1\wedge\bigwedge_{l=1}^{k}\nabla_g I_l\right):a\in\{1,\dots,n-(k+1)\}\right\}$$

where

$$X_0 = \left\| \nabla_g D_1 \wedge \bigwedge_{j=1}^{k} \nabla_g I_j \right\|_{k+1}^{-2}$$

$$\cdot (-1)^{n-1} h_1 \cdot \left(\star \left[\bigwedge_{l=1}^{k} \nabla_g I_l \wedge \star \left(\nabla_g D_1 \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \right) \right] \right),$$

and respectively the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-(k+1)}, \nabla_g I_1, \ldots, \nabla_g I_k, \nabla_g D_1\}$$

forms a moving frame.

The first part of Remark (5.1) (namely for $p = 0$) provides a set of local generators for the distribution given by the conservative vector fields $X \in \mathfrak{X}(U)$ admitting the set of (functionally independent) first integrals $I_1, \ldots, I_k \in \mathscr{C}^\infty(U, \mathbb{R})$.
Moreover, if $p = 0$ and $k = n-1$, then the conclusion of Remark (5.1) becomes as follows:
The vector field $\star\left(\bigwedge_{l=1}^{k} \nabla_g I_l\right)$ generates locally the distribution of **completely integrable vector fields**

$$\mathfrak{X}[\nabla_g I_1, \ldots, \nabla_g I_{n-1}] = \{X \in \mathfrak{X}(U) \mid \mathscr{L}_X I_1 = \cdots = \mathscr{L}_X I_{n-1} = 0\}.$$

Remark

For $k = 0$, the conclusion of Theorem (4) becomes as follows:
The affine distribution

$$\begin{aligned}\mathfrak{A}[X_0; \nabla_g D_1, \dots, \nabla_g D_p] &= \{X \in \mathfrak{X}(U) \mid \mathscr{L}_X D_j = h_j, \quad 1 \leq j \leq p\} \\ &= X_0 + \mathfrak{X}[\nabla_g D_1, \dots, \nabla_g D_p],\end{aligned}$$

is locally generated by the set of vector fields

$$\{X_0\} \biguplus \left\{ \star \left(\bigwedge_{i=1, i \neq a}^{n-p} Z_i \wedge \bigwedge_{j=1}^{p} \nabla_g D_j \right) : a \in \{1, \dots, n-p\} \right\},$$

where

$$X_0 = \left\| \bigwedge_{i=1}^{p} \nabla_g D_i \right\|_p^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i \cdot \left(\star \left[\bigwedge_{j=1, j \neq i}^{p} \nabla_g D_j \wedge \star \left(\bigwedge_{j=1}^{p} \nabla_g D_j \right) \right] \right),$$

and respectively the set of locally defined vector fields

$$\{Z_1, \ldots, Z_{n-p}, \nabla_g D_1, \ldots, \nabla_g D_p\}$$

forms a moving frame.

Remark

For $k+p=n-1$, the conclusion of Theorem (4) becomes as follows:
The affine distribution

$$\mathfrak{A}[X_0;\nabla_g I_1,\ldots,\nabla_g I_k,\nabla_g D_1,\ldots,\nabla_g D_p]$$

is locally generated by the set of vector fields

$$\{X_0\} \biguplus \left\{ \star\left(\bigwedge_{j=1}^{p} \nabla_g D_j \wedge \bigwedge_{l=1}^{k} \nabla_g I_l \right) \right\}.$$

Remark

For $k+p=n$, the conclusion of Theorem (4) reduces to:

$$\begin{aligned}
&\mathfrak{A}[X_0;\nabla_g I_1,\dots,\nabla_g I_k,\nabla_g D_1,\dots,\nabla_g D_p] \\
&\qquad =\{X\in\mathfrak{X}(U)\mid \mathscr{L}_X I_i=0,\mathscr{L}_X D_j=h_j, 1\le i\le k, 1\le j\le p\} \\
&\qquad =\{X_0\},
\end{aligned}$$

where

$$X_0=\left\|\bigwedge_{i=1}^{p}\nabla_g D_i\wedge\bigwedge_{l=1}^{k}\nabla_g I_l\right\|_n^{-2}\cdot\sum_{i=1}^{p}(-1)^{n-i}h_i\Theta_i,$$

$$\Theta_i=\star\left[\bigwedge_{j=1,j\neq i}^{p}\nabla_g D_j\wedge\bigwedge_{l=1}^{k}\nabla_g I_l\wedge\star\left(\bigwedge_{j=1}^{p}\nabla_g D_j\wedge\bigwedge_{l=1}^{k}\nabla_g I_l\right)\right].$$

The first result of this section is an explicit formula for the characteristic multipliers of a given periodic orbit of a general codimension-one dissipative dynamical system. Because of the local nature of the main results, one can suppose that we work on an open subset $U \subseteq \mathbb{R}^n$.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Let $\dot{x} = X(x)$, $X \in \mathfrak{X}(U)$, be a given codimension-one dissipative dynamical system, i.e., there exists $k, p \in \mathbb{N}$ such that $k + p = n - 1$, and some smooth functions $I_1, \dots, I_k, D_1, \dots, D_p$, h_1, $\dots$, $h_p \in \mathscr{C}^\infty(U, \mathbb{R})$ such that the vector field X conserves $I_1, \dots, I_k$, and dissipates $D_1, \dots, D_p$ with associated dissipation rates $h_1 D_1, \dots, h_p D_p$.

Suppose that $\Gamma := \{\gamma(t) \subset U : 0 \leq t \leq T\}$ is a $T-$periodic orbit of $\dot{x} = X(x)$ such that $\Gamma \subset ID^{-1}(\{0\})$, and moreover, $0 \in \mathbb{R}^{n-1}$ is a regular value of the map $ID := (I_1, \dots, I_k, D_1, \dots, D_p) : U \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1}$.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Let us recall first that for a general dynamical system $\dot{x} = X(x)$, generated by a smooth vector field $X \in \mathfrak{X}(U)$, defined on an open subset $U \subseteq \mathbb{R}^n$, and respectively a given $T-$periodic orbit $\Gamma := \{\gamma(t) \subset U : 0 \leq t \leq T\}$, the characteristic multipliers of Γ are the eigenvalues of the fundamental matrix $u(T)$, where u is the solution of the variational equation

$$\frac{du}{dt} = DX(\gamma(t))u(t), \ u(0) = I_{n,n},$$

and $I_{n,n}$ stands for the identity matrix of dimensions $n \times n$.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Taking into account the complexity of the variational equation, the computation of characteristic multipliers in general is almost impossible, since there exist no general methods to solve explicitly the variational equation.
One of the main results of this section is to complete this task for the class of codimension-one dissipative dynamical systems.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Theorem

Let $\dot{x} = X(x)$ be a codimension-one dissipative dynamical system generated by a smooth vector field $X \in \mathfrak{X}(U)$ defined eventually on an open subset $U \subseteq \mathbb{R}^n$, such that there exist $k, p \in \mathbb{N}$, $k + p = n - 1$, and respectively $I_1, \ldots, I_k, D_1, \ldots, D_p$, h_1, $\ldots$, $h_p \in \mathscr{C}^\infty(U, \mathbb{R})$ such that $\mathscr{L}_X I_1 = \cdots = \mathscr{L}_X I_k = 0$, and $\mathscr{L}_X D_1 = h_1 D_1$, $\ldots$, $\mathscr{L}_X D_p = h_p D_p$. Suppose that $\Gamma = \{\gamma(t) \subset U : 0 \leq t \leq T\}$ is a $T-$periodic orbit of $\dot{x} = X(x)$, such that the following conditions hold true:

Stability of periodic orbits of codimension-one dissipative dynamical systems

- $\Gamma \subset ID^{-1}(\{0\})$, and $0 \in \mathbb{R}^{n-1}$ is a regular value of the map
$$ID = (I_1, \ldots, I_k, D_1, \ldots, D_p) : U \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1},$$
- $\nabla I_1(\gamma(t)), \ldots, \nabla I_k(\gamma(t)), \nabla D_1(\gamma(t)), \ldots, \nabla D_p(\gamma(t)), X(\gamma(t))$ are linearly independent for each $0 \leq t \leq T$.

Then, the characteristic multipliers of the periodic orbit Γ are

$$\underbrace{1, \ldots, 1}_{k+1 \text{ times}}, \exp\left(\int_0^T h_1(\gamma(s))ds\right), \ldots, \exp\left(\int_0^T h_p(\gamma(s))ds\right).$$

Stability of periodic orbits of codimension-one dissipative dynamical systems

Next section has two main purposes, namely, the first purpose is to provide sufficient conditions to guarantee the partial orbital phase asymptotic stability of periodic orbits of a codimension-one dissipative dynamical system, whereas the second purpose is to give sufficient conditions to guarantee the instability of periodic orbits of a codimension-one dissipative dynamical system.

Let us start by recalling some definitions concerning the stability of the periodic orbits of a general dynamical system. In order to do that, let $\dot{x} = X(x)$ be a dynamical system generated by a smooth vector field $X \in \mathfrak{X}(U)$, defined eventually on an open subset $U \subseteq \mathbb{R}^n$. Suppose $\Gamma = \{\gamma(t) \subset U : 0 \leq t \leq T\}$ is a $T-$periodic orbit of $\dot{x} = X(x)$.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Definition

- *The periodic orbit* Γ *is called* **orbitally stable** *if, given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\operatorname{dist}(x(t,x_0),\Gamma) < \varepsilon$ *for all* $t > 0$ *and for all* $x_0 \in U$ *such that* $\operatorname{dist}(x_0,\Gamma) < \delta$.
- *The periodic orbit* Γ *is called* **unstable** *if it is not orbitally stable.*
- *The periodic orbit* Γ *is called* **orbitally asymptotically stable** *if it is orbitally stable and (by choosing* δ *smaller if necessary),* $\operatorname{dist}(x(t,x_0),\Gamma) \to 0$ *as* $t \to \infty$.
- *The periodic orbit* Γ *is called* **orbitally phase asymptotically stable**, *if it is asymptotically orbitally stable and there is a* $\delta > 0$ *such that for each* $x_0 \in U$ *with* $\operatorname{dist}(x_0,\Gamma) < \delta$, *there exists* $\theta_0 = \theta_0(x_0)$ *such that*

$$\lim_{t\to\infty} \|x(t,x_0) - \gamma(t+\theta_0)\| = 0.$$

Stability of periodic orbits of codimension-one dissipative dynamical systems

Theorem

Let $\dot{x} = X(x)$ be a codimension-one dissipative dynamical system generated by a smooth vector field $X \in \mathfrak{X}(U)$ defined eventually on an open subset $U \subseteq \mathbb{R}^n$, such that there exists $k, p \in \mathbb{N}$ with $p > 0$, $k + p = n - 1$, and respectively $I_1, \ldots, I_k, D_1, \ldots, D_p$, h_1, $\ldots$, $h_p \in \mathscr{C}^\infty(U, \mathbb{R})$ such that $\mathscr{L}_X I_1 = \cdots = \mathscr{L}_X I_k = 0$, and $\mathscr{L}_X D_1 = h_1 D_1$, $\ldots$, $\mathscr{L}_X D_p = h_p D_p$. Suppose $\Gamma = \{\gamma(t) \subset U : 0 \leq t \leq T\}$ is a $T-$periodic orbit of $\dot{x} = X(x)$, such that the following conditions hold true:

Stability of periodic orbits of codimension-one dissipative dynamical systems

- $\Gamma \subset ID^{-1}(\{0\})$, and $0 \in \mathbb{R}^{n-1}$ is a regular value of the map

$$ID = (I_1, \ldots, I_k, D_1, \ldots, D_p) : U \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1},$$

- $\nabla I_1(\gamma(t)), \ldots, \nabla I_k(\gamma(t)), \nabla D_1(\gamma(t)), \ldots, \nabla D_p(\gamma(t)), X(\gamma(t))$ are linearly independent for each $0 \leq t \leq T$.

Stability of periodic orbits of codimension-one dissipative dynamical systems

Then, if moreover $0 \in \mathbb{R}^k$ is a regular value of the map $I = (I_1, \ldots, I_k) : U \subseteq \mathbb{R}^n \to \mathbb{R}^k$, and if

$$\int_0^T h_1(\gamma(s))ds < 0, \ldots, \int_0^T h_p(\gamma(s))ds < 0,$$

then the periodic orbit Γ is orbitally phase asymptotically stable, with respect to perturbations along the invariant manifold $I^{-1}(\{0\})$.

On the other hand, if there exists $i_0 \in \{1, \ldots, p\}$ such that $\int_0^T h_{i_0}(\gamma(s))ds > 0$, then the periodic orbit Γ is unstable.

The purpose of next section is to apply the results from the previous section in order to partially orbitally asymptotically stabilize, a given periodic orbit of a completely integrable dynamical system.

In order to do that, let us consider a completely integrable dynamical system $\dot{x} = X(x)$, $X \in \mathfrak{X}(U)$, defined eventually on an open subset $U \subseteq \mathbb{R}^n$ (i.e., it admits a set of $n-1$ first integrals, $I_1, \ldots, I_k, D_1, \ldots, D_p \in \mathscr{C}^\infty(U, \mathbb{R})$, independent at least on an open subset $V \subseteq U$). Suppose that $\Gamma = \{\gamma(t) \subset V : 0 \leq t \leq T\}$ is a $T-$periodic orbit of the system $\dot{x} = X(x)$.

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

The idea for the stabilization procedure is to perturb the completely integrable system $\dot{x} = X(x)$, in such a way that the perturbed dynamical system becomes a dissipative dynamical system on V, which admits also Γ as a periodic orbit, and moreover verifies the hypothesis of Theorem (7). Note that using classical perturbation methods, the persistence of periodic orbits after perturbations, follows as a consequence of the implicit function theorem. The method introduced in this section, provide for the class of completely integrable dynamical systems, an explicit perturbation which preserve (under reasonable conditions) an a-priori given periodic orbit.

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

Theorem

Let $\dot{x} = X(x)$ be a completely integrable dynamical system generated by a smooth vector field $X \in \mathfrak{X}(U)$ defined eventually on an open subset $U \subseteq \mathbb{R}^n$, and let $k, p \in \mathbb{N}$ be two natural numbers, with $k + p = n - 1$, such that there exist $n - 1$ first integrals $I_1, \ldots, I_k, D_1, \ldots, D_p \in \mathscr{C}^\infty(U, \mathbb{R})$, independent on an open subset $V \subseteq U$. Suppose the system $\dot{x} = X(x)$ admits a $T-$periodic orbit $\Gamma = \{\gamma(t) \subset V : 0 \leq t \leq T\}$ such that:

- *$\Gamma \subset ID^{-1}(\{0\})$, and $0 \in \mathbb{R}^{n-1}$ is a regular value of the map*

$$ID = (I_1, \ldots, I_k, D_1, \ldots, D_p) : V \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1},$$

- *$\nabla I_1(\gamma(t)), \ldots, \nabla I_k(\gamma(t)), \nabla D_1(\gamma(t)), \ldots, \nabla D_p(\gamma(t)), X(\gamma(t))$ are linearly independent for each $0 \leq t \leq T$.*

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

If moreover, $0 \in \mathbb{R}^k$ is a regular value of the map $I = (I_1, \ldots, I_k) : V \subseteq \mathbb{R}^n \to \mathbb{R}^k$, then for any choice of smooth functions $h_1, \ldots, h_p \in \mathscr{C}^\infty(V, \mathbb{R})$ such that

$$\int_0^T h_1(\gamma(s))ds < 0, \ldots, \int_0^T h_p(\gamma(s))ds < 0,$$

Γ, as a periodic orbit of the dissipative dynamical system $\dot{x} = X(x) + X_0(x)$, $x \in V$,

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

$$X_0 = \left\| \bigwedge_{i=1}^{p} \nabla D_i \wedge \bigwedge_{j=1}^{k} \nabla I_j \right\|_{n-1}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i D_i \Theta_i,$$

$$\Theta_i = \star \left[\bigwedge_{j=1, j\neq i}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \wedge \star \left(\bigwedge_{j=1}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \right) \right],$$

is orbitally phase asymptotically stable, with respect to perturbations along the invariant manifold $I^{-1}(\{0\})$.

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

On the other hand, for any choice of smooth functions $k_1,\dots,k_p \in \mathscr{C}^\infty(V,\mathbb{R})$, such that there exists $i_0 \in \{1,\dots,p\}$ for which

$$\int_0^T k_{i_0}(\gamma(s))ds > 0,$$

Γ, as a periodic orbit of the dissipative dynamical system $\dot{x} = X(x) + X_0(x)$, $x \in V$,

$$X_0 = \left\| \bigwedge_{i=1}^{p} \nabla D_i \wedge \bigwedge_{j=1}^{k} \nabla I_j \right\|_{n-1}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} k_i D_i \Theta_i,$$

$$\Theta_i = \star \left[\bigwedge_{j=1, j\neq i}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \wedge \star \left(\bigwedge_{j=1}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \right) \right],$$

is an unstable periodic orbit.

Remark

In the hypothesis of the Theorem (8), note that:

- the condition $\Gamma \subset ID^{-1}(\{0\})$ implies that for any choice of smooth functions $h_1, \ldots, h_p \in \mathscr{C}^\infty(V, \mathbb{R})$, the control vector field $X_0 \in \mathfrak{X}(V)$, given by

$$X_0 = \left\| \bigwedge_{i=1}^{p} \nabla D_i \wedge \bigwedge_{j=1}^{k} \nabla I_j \right\|_{n-1}^{-2} \cdot \sum_{i=1}^{p} (-1)^{n-i} h_i D_i \Theta_i,$$

$$\Theta_i = \star \left[\bigwedge_{j=1, j\neq i}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \wedge \star \left(\bigwedge_{j=1}^{p} \nabla D_j \wedge \bigwedge_{l=1}^{k} \nabla I_l \right) \right],$$

verifies that $X_0(\gamma(t)) = 0$, for every $t \in [0, T]$;

Orbitally asymptotically stabilizing the periodic orbits of completely integrable dynamical systems

- each of the smooth functions $h_1, \ldots, h_p \in \mathscr{C}^\infty(V, \mathbb{R})$ might be chosen of the type e.g., $h(x) = -(\psi^2(x) + c)$, $x \in V$, with $\psi \in \mathscr{C}^\infty(V, \mathbb{R})$ and $c \in (0, \infty)$, since

$$\int_0^T h(\gamma(s))ds = -\int_0^T \psi^2(\gamma(s))ds - Tc \leq -Tc < 0;$$

- the smooth function $k_{i_0} \in \mathscr{C}^\infty(V, \mathbb{R})$ might be chosen of type e.g., $k(x) = \varphi^2(x) + c$, $x \in V$, with $\varphi \in \mathscr{C}^\infty(V, \mathbb{R})$ and $c \in (0, \infty)$, since

$$\int_0^T k(\gamma(s))ds = \int_0^T \varphi^2(\gamma(s))ds + Tc \geq Tc > 0.$$

Example

Example

Let us consider the family of harmonic oscillators, described by the three dimensional vector field

$$X(x,y,z) = y\partial_x - x\partial_y \in \mathfrak{X}(\mathbb{R}^3).$$

The induced dynamical system,

$$\dot{\mathbf{x}} = X(\mathbf{x}),\ \mathbf{x} = (x,y,z) \in \mathbb{R}^3, \tag{4}$$

admits a $2\pi-$*periodic orbit given by*
$\Gamma = \{\gamma(t) = (\sin t, \cos t, 0) : 0 \leq t \leq 2\pi\}$.
Moreover, the system (4) *is completely integrable, since it has two independent first integrals, namely*

$$I(x,y,z) = x^2 + y^2 - 1,\ D(x,y,z) = z.$$

Example

By straightforward computations we obtain that the vector field X_0 from Theorem (8), in this case has the expression

$$X_0(x,y,z) = zu(x,y,z)\partial_z, \ (x,y,z) \in \mathbb{R}^3,$$

and consequently it verifies the condition $X_0 \circ \gamma = 0$, for any smooth real function $u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$.
Consequently, the perturbed system

$$\dot{\mathbf{x}} = X(\mathbf{x}) + X_0(\mathbf{x}), \ \mathbf{x} = (x,y,z) \in \mathbb{R}^3, \tag{5}$$

is a codimension-one dissipative dynamical system associated to $I, D, u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$, i.e., $\mathscr{L}_{X+X_0} I = 0$, and respectively $\mathscr{L}_{X+X_0} D = uD$.
Since $X_0 \circ \gamma = 0$, for any smooth real function $u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$, we obtain that Γ is a periodic orbit of the dissipative system $\dot{\mathbf{x}} = X(\mathbf{x}) + X_0(\mathbf{x})$, for any smooth real function $u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$.

Example

Hence, by Theorem (8), we obtain the following conclusions:

- for any smooth function $u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\int_0^{2\pi} u(\sin t, \cos t, 0)dt < 0$, the periodic orbit Γ of the associated perturbed system (5) is orbitally phase asymptotically stable, with respect to perturbations along the cylinder $I^{-1}(\{0\})$;
- for any smooth function $u \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\int_0^{2\pi} u(\sin t, \cos t, 0)dt > 0$, the periodic orbit Γ of the associated perturbed system (5) is unstable.

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