2-COCYCLES
and
GEODESIC EQUATIONS

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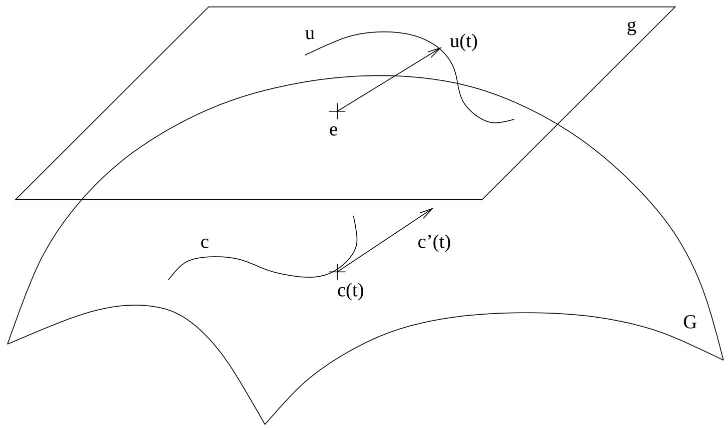
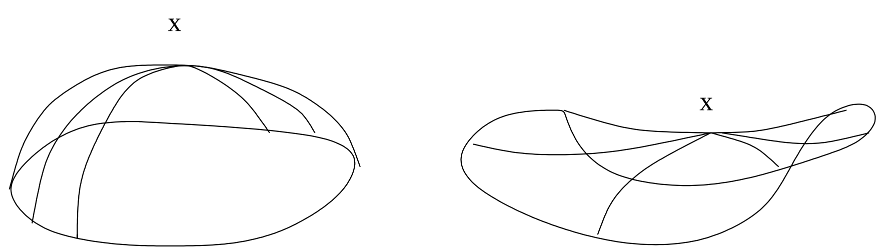
GEODESICS ON LIE GROUPS

Geodesic on a Lie group with right invariant Riemannian metric



Curvature and stability of geodesics



C(x,r) C(x,r)

(a) K>0 (b) K<0

GEODESIC EQUATIONS

Let G be a Lie group with right invariant metric. The *energy functional* applied to a smooth curve $g : [a, b] \to G$ is

$$E(g) = \frac{1}{2}\int_a^b \|g'(t)\|^2 dt = \frac{1}{2}\int_a^b \langle \delta^r g(t), \delta^r g(t)\rangle dt,$$

where $\delta^r g = g' g^{-1}$ denotes the right logarithmic derivative of g.

The curve $g : [a, b] \to G$ is a geodesic for the right invariant metric on G if and only if its right logarithmic derivative $u = \delta^r g : [a, b] \to \mathfrak{g}$ satisfies the *Euler equation*:

$$\frac{d}{dt}u + \mathrm{ad}_u^\top u = 0,$$

where $\mathrm{ad}_u^\top : \mathfrak{g} \to \mathfrak{g}$ is the adjoint of ad_u for the inner product on the Lie algebra $\mathfrak{g}$:

$$\langle \mathrm{ad}_u^\top v, w\rangle = \langle v, [u, w]\rangle.$$

DIFFEOMORPHISM GROUPS

For a compact manifold M, the diffeomorphism group $\mathsf{Diff}(M)$ is a Lie group with Lie algebra $\mathfrak{X}(M)$ endowed with the opposite bracket [Kriegl-Michor, Omori, Milnor, Ebin-Marsden]

With the help of the adjoint action $\mathsf{Ad}(\varphi)X = (\varphi^{-1})^*X$ one gets the Lie algebra bracket

$$\mathsf{ad}(X)Y = \frac{d}{dt}\Big|_0 (\mathsf{Fl}^X_{-t})^*Y = -[X,Y],$$

where $\mathsf{Fl}^X_t \in \mathsf{Diff}(M)$ is the flow of the vector field X at time t:

$$\frac{d}{dt}\mathsf{Fl}^X_t = X \circ \mathsf{Fl}^X_t \, .$$

The time 1 flow is the exponential map

$$\exp : \mathfrak{X}(M) \to \mathsf{Diff}(M), \quad \exp(X) = \mathsf{Fl}^X_1$$

SUBGROUPS OF DIFFEOMORPHISMS

The group of symplectic diffeomorphisms of (M,ω)

$$\mathrm{Diff}_{\mathrm{symp}}(M) = \{\varphi \in \mathrm{Diff}(M) \ : \ \varphi^*\omega = \omega\}$$
$$\mathfrak{X}_{\mathrm{symp}}(M) = \{X \in \mathfrak{X}(M) \ : \ L_X\omega = 0\}$$

and its subgroup $\mathrm{Diff}_{\mathrm{ham}}(M)$ of *Hamiltonian* diffeomorphisms (the kernel of Calabi's flux homomorphism) with Lie algebra

$$\mathfrak{X}_{\mathrm{ham}}(M) = \{X_h \in \mathfrak{X}(M) \ : \ i_{X_h}\omega = dh, \ h \in C^\infty(M)\}$$

The group of volume preserving diffeomorphisms of (M,μ)

$$\mathrm{Diff}_{\mathrm{vol}}(M) = \{\varphi \in \mathrm{Diff}(M) \ : \ \varphi^*\mu = \mu\}$$
$$\mathfrak{X}_{\mathrm{vol}}(M) = \{X \in \mathfrak{X}(M) \ : \ L_X\mu = 0\}$$

and its subgroup $\mathrm{Diff}_{\mathrm{ex}}(M)$ of *exact volume preserving* diffeomorphisms (the kernel of Thurston's flux homomorphism) with

$$\mathfrak{X}_{\mathrm{ex}}(M) = \{X_\alpha \in \mathfrak{X}(M) \ : \ i_{X_\alpha}\mu = d\alpha, \ \alpha \in \Omega^{m-2}(M)\}$$

IDEAL FLUID FLOW

”A fluid moves to get out of its own way as efficiently as possible.”
Joe Monaghan

[Arnold ’66] The ideal fluid flow with velocity field u in $\mathfrak{g} = \mathfrak{X}_{\mathrm{vol}}(M)$ and pressure function p

$$\partial_t u = -\nabla_u u - \operatorname{grad} p$$

is the geodesic equation on $G = \mathrm{Diff}_{\mathrm{vol}}(M)$ with right invariant L^2-metric, i.e. $\langle u, v\rangle = \int_M g(u,v)\mu$.

Vorticity equation. The vorticity 2-form $\omega = du^\flat$ is transported by the flow:

$$\partial_t \omega + L_u \omega = 0.$$

On a simply connected surface it becomes

$$\Delta \partial_t f + \{\Delta f, f\} = 0$$

where f denotes the stream function of u.

TOTALLY GEODESIC

In [Haller, Teichmann, V. '02] we determine those Riemannian manifolds M on which the subgroup $\mathsf{Diff}_{\mathrm{ex}}(M)$ is totally geodesic in the group $\mathsf{Diff}_{\mathrm{vol}}(M)$ for the right invariant L^2–metric.

Let $\mathbb{T}^k = \mathbb{R}^k/\Lambda$ be a flat torus, equipped with the metric induced from the Euclidean metric on $\mathbb{R}^k$, such that Λ acts on an oriented Riemannian manifold F by orientation preserving isometries. The total space of the associated fiber bundle $\mathbb{R}^k \times_\Lambda F \to \mathbb{T}^k$ is an oriented Riemannian manifold in a natural way, called a *twisted product*. Locally over $\mathbb{T}^k$ the metric is the product metric.

Theorem. The only Riemannian manifolds M with the required property are twisted products $M = \mathbb{R}^k \times_\Lambda F$ of a flat torus with a closed connected oriented Riemannian manifold F with vanishing first Betti number.

PREQUANTIZATION EXTENSION

(M, ω) prequantizable symplectic manifold: there exists a principal circle bundle $P \to M$ and a principal connection $\theta \in \Omega^1(P)$ with curvature ω.

The Lie algebra extension

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathfrak{X}_{\mathrm{ham}}(M) \to 0,$$

with Poisson bracket $\{f, g\} = \omega(X_g, X_f)$ on $C^\infty(M)$, integrates to the prequantization central group extension

$$1 \to S^1 \to \mathrm{Quant}(P) \to \mathrm{Diff}_{\mathrm{ham}}(M) \to 1,$$

with $\mathrm{Quant}(P) = \{\psi \in \mathrm{Aut}(P) : \psi^*\theta = \theta\}$ *the quantomorphism group* [Kostant, Souriau ’70].

The prequantization Lie algebra extension splits for compact M.

QUASI-GEOSTROPHIC EQUATION

Let g be a Riemannian metric on M with induced volume form $\mu = \omega^n$.

The geodesic equation on the quantomorphism group $\mathtt{Quant}(P)$ for right invariant H^1–metric

$$\langle f_1, f_2 \rangle = \int_M (f_1 f_2 + g(\nabla f_1, \nabla f_2))\mu$$

on its Lie algebra $C^\infty(M)$ is

$$\Delta \partial_t f - \partial_t f + \{\Delta f, f\} = 0.$$

Here $f \in C^\infty(M)$ denotes the stream function of the velocity field u and the equation describes quasi-geostrophic motion in f-plane approximation.

2-COCYCLES

A Lie algebra 2–cocycle $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfies the cyclic identity

$$\sigma([X_1, X_2], X_3) + \sigma([X_2, X_3], X_1) + \sigma([X_3, X_1], X_2) = 0.$$

It defines a Lie bracket on $\mathbb{R} \times \mathfrak{g}$

$$[(a, X), (b, Y)] = (\sigma(X, Y), [X, Y]),$$

hence an exact sequence of Lie algebras $0 \to \mathbb{R} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$ such that the image of $\mathbb{R}$ is central in $\widehat{\mathfrak{g}}$. The continuous Lie algebra cohomology space $H^2(\mathfrak{g})$ parameterizes isomorphism classes of extensions of $\mathfrak{g}$ by $\mathbb{R}$.

A normalized group 2–cocycle on the Lie group G is a locally smooth map $c : G \times G \to \mathbb{R}$ satisfying $c(g, e) = c(e, g) = 0$ and the cocycle condition

$$c(g, g') + c(gg', g'') = c(g', g'') + c(g, g'g'').$$

The cocycle c defines a Lie group structure on $\mathbb{R} \times G$ by

$$(x, g)(x', g') = (x + x' + c(g, g'), gg'),$$

thus obtaining a central Lie group extension $\widehat{G}$ of G.

GEODESIC EQUATION

The geodesic equation on a 1–dimensional central Lie group extension $\widehat{G}$ of G with right invariant metric determined by the scalar product

$$\langle (a, X), (b, Y) \rangle_{\widehat{\mathfrak{g}}} = \langle X, Y \rangle_{\mathfrak{g}} + ab$$

on its Lie algebra $\widehat{\mathfrak{g}}$ is

$$\frac{d}{dt} u = -\,\mathsf{ad}(u)^\top u - a k(u), \quad a \in \mathbb{R},$$

where the skew-symmetric map $k : \mathfrak{g} \to \mathfrak{g}$ is defined by σ via

$$\langle k(X), Y \rangle = \sigma(X, Y), \quad \forall X, Y \in \mathfrak{g}. \tag{1}$$

VIRASORO-BOTT GROUP

The Lie algebra cocycle corresponding to the Bott cocycle on $\mathrm{Diff}_+(S^1)$

$$c(\varphi, \psi) := \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi)' d \log \psi'$$

(with $\varphi' \in C^\infty(S^1)$ defined by $\varphi^* dx = \varphi' dx$) is the Virasoro cocycle on the Lie algebra $\mathfrak{X}(S^1)$

$$\sigma(u, v) = \int_{S^1} (u'v'' - u''v') dx$$

The coadjoint action is

$$\mathrm{Ad}^*(\varphi)(p, a) = (\mathrm{Ad}^*(\varphi)p + cS(\varphi^{-1})dt^2, a), \quad a \in \mathbb{R}$$

where $p \in Q(S^1)$ and $S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwartzian derivative that measures the deviation of f from being a Moebius transformation

KDV EQUATIONS

Burgers eq (dispersionless KdV eq)

$$\partial_t u = -3uu'$$

is the geodesic equation on the group $\mathrm{Diff}(S^1)$ of diffeomorphisms on the circle for the right invariant metric defined by the L^2 scalar product

$$\langle u, v \rangle = \int_{S^1} uv dx$$

on its Lie algebra $\mathfrak{X}(S^1)$ (with $[u, v] = u'v - uv'$).

Korteweg-de Vries eq

$$\partial_t u = -3uu' - 2u''',$$

is the geodesic equation on the Bott-Virasoro group $\widehat{\mathrm{Diff}}(S^1)$ for right invariant L^2 metric.

CAMASSA-HOLM EQUATIONS

Camassa-Holm equation

$$\partial_t(u - u'') = -3uu' + 2u'u'' + uu'''$$

is the geodesic equation on $\mathtt{Diff}(S^1)$ for right invariant H^1 metric defined with the scalar product $\langle v, u\rangle_1 = \int_{S^1}(vu + v'u')dx$ on $\mathfrak{X}(S^1)$.

Extended Camassa-Holm

Considering the right invariant H^1-metric on the Bott-Virasoro group (central extension of $\mathtt{Diff}(S^1)$) an extended Camassa-Holm equation is obtained

$$\partial_t(u - u'') = -3uu' + 2u'u'' + uu''' - 2u'''.$$

INTEGRABILITY

Given a Lie group G with Lie algebra $\mathfrak{g}$, not every abelian Lie algebra extension of $\mathfrak{g}$ can be integrated to an abelian Lie group extension of G.
There are two obstructions for the integration of a Lie algebra cocycle σ on $\mathfrak{g}$. Let σ^ℓ be the closed left invariant 2-form on G determined by σ. The *period group* Π_σ of σ is the image of the period homomorphism

$$\mathrm{per}_\sigma : \pi_2(G) \to \mathbb{R}, \quad \mathrm{per}_\sigma([c]) = \int_{S^2} c^* \sigma^\ell$$

The *flux homomorphism* $F_\sigma : \pi_1(G) \to \mathfrak{g}^*$ assigns to each piecewise smooth loop γ in G the linear map

$$X \in \mathfrak{g} \mapsto -\int_\gamma i_{X^r} \sigma^\ell \in \mathbb{R}.$$

Theorem. [Neeb '04] For a Lie algebra 2–cocycle σ with discrete period group Π_σ and vanishing flux homomorphism F_σ, the abelian Lie algebra extension $0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ defined by σ integrates to

$$1 \to \mathbb{R}/\Pi_\sigma \to \widehat{G} \to G \to 1.$$

LICHNEROWICZ EXTENSION OF $\mathsf{Diff}_{\mathsf{ex}}(M)$

The Lie algebra cohomology $H^2(\mathfrak{X}_{\mathsf{ex}}(M)) = H^2_{dR}(M)$, so the universal central extension [Lichnerowicz'74, Roger'95] is

$$0 \to H^{m-2}_{dR}(M) \to \Omega^{m-2}(M)/d\Omega^{m-3}(M) \to \mathfrak{X}_{\mathsf{ex}}(M) \to 0$$

with Lie bracket $\{[\alpha_1],[\alpha_2]\} = [i_{X_{\alpha_1}} i_{X_{\alpha_2}} \mu]$ and cohomology class

$$(X,Y) \mapsto [r i_Y i_X \mu]$$

where $r : \Omega^{k-2}(S) \to Z^{k-2}(S)$ is any continuous linear projection.

Each $[\eta] \in H^2_{dR}(M)$ determines a Lichnerowicz cocycle

$$\sigma_\eta(X,Y) = \int_M \eta(X,Y)\mu.$$

If $[\eta]$ is integral, there exists a corresponding Lie group extension of $\mathsf{Diff}_{\mathsf{ex}}(M)$ [Ismagilov'96]; it can be obtained as the pull-back of the prequantization central extension in an infinite dimensional setting [Neeb, V. '03].

INFINITE CONDUCTIVITY EQUATION

The infinite conductivity equation models the motion of a high density electronic gas in a magnetic field B with velocity u:

$$\partial_t u = -\nabla_u u - u \times B - \mathrm{grad}\, p.$$

Each closed 2-form η on the compact 3-dimensional manifold M defines a Lichnerowicz Lie algebra 2-cocycle σ_η on the Lie algebra $\mathfrak{X}_{\mathrm{vol}}(M)$ of divergence free vector fields:

$$\sigma_\eta(u, v) = \int_M \eta(u, v)\mu = \int_M g(u \times B, v)\mu,$$

where B is the divergence free vector field B on M defined with $\eta = -i_B\mu$.

The infinite conductivity equation is the geodesic equation for the right invariant L^2 metric on a central extension integrating the Lichnerowicz cocycle [Zeitlin, Roger].

ROGER COCYCLES

Each closed 1-form $\alpha \in \Omega^1(M)$ determines a Lie algebra 2-cocycle on the Lie algebra $C^\infty(M)$ of smooth functions on the compact symplectic manifold (M, ω) with Poisson bracket, namely the Roger cocycle

$$\sigma_\alpha(X_f, X_g) = \int_M f\alpha(X_g)\omega^n.$$

Theorem. The Lie algebra cohomology is $H^2(\mathfrak{X}_{\mathsf{ham}}(M)) = H^1_{dR}(M)$ (conjectured by Roger) [Janssens, V.].

CODIMENSION ONE SUBMANIFOLDS

If the cohomology class $[\alpha] \in H^1(M)$ is Poincaré dual to the homology class $[N]$ of a codimension one submanifold $N \subset M$, then the cocycle

$$\sigma_N(f,g) := \int_N f dg \wedge \omega^{n-1}$$

and the scaled Roger cocycle $\frac{1}{n}\sigma_\alpha$ are cohomologous cocycles on $C^\infty(M)$.

Theorem. Let $\pi : (P,\theta) \to (M,\omega)$ be a prequantum bundle over the symplectic manifold. Every $[N] \in \mathrm{Im}\,\pi_*$ determines a Roger cocycle σ_N that is integrable to a central extension of $\mathtt{Quant}(P)$ [Janssens, V.].

QUASI-GEOSTROPHIC MOTION

On the 2-torus we consider the closed 1–form $\alpha = \beta dy$, $\beta \in \mathbb{R}$.

Theorem. The geodesic equation on the central extension of the quantomorphism group that corresponds to the Roger cocycle σ_α, endowed with H^1–metric, is the equation for quasigeostrophic motion in β–plane approximation [Zeitlin, Pasmanter '94]

$$\Delta\partial_t f - \partial_t f + \{\Delta f, f\} + \beta\partial_x f = 0,$$

with β the gradient of the Coriolis parameter.

It is written for the vorticity function Δf, with f the stream function of u.

CONTACTOMORPHISM GROUP

The prequantum circle bundle $(P^{2n+1}, \theta) \to (M^{2n}, \omega)$ is a contact manifold with contact form the principal connection $\theta \in \Omega^1(P)$ with curvature the symplectic form ω. Let E be the Reeb vector field.

The Lie algebra of the group of contactomorphisms

$$\mathrm{Diff}_\theta(P) = \{\varphi \in \mathrm{Diff}(P) : \varphi^*\theta = e^\Lambda \theta, \Lambda \in C^\infty(P)\}$$

can be identified with $C^\infty(P)$ via $u \mapsto \theta(u)$. It is endowed with contact Poisson bracket $\{f, g\} = X_f(g) - gE(f)$, where X_f is the contact vector field that corresponds to f.

GEODESIC EQUATION

The geodesic equation on the contactomorphism group $\mathrm{Diff}_\theta(P^{2n+1})$ with L^2-metric coming from a Riemannian metric on P compatible with the contact form, i.e.

$$\langle u, E\rangle = \theta(u), \quad \mathrm{div}\, u = (n+1)E(f)$$

is a pde for $f \in C^\infty(P)$ [Ebin-Preston'14]

$$\partial_t m + X_f(m) + (n+2)mE(f), \quad m = f - \Delta f$$

For $n = 0$ it becomes the Camassa-Holm equation. It is well-posed and possesses similar conservation laws $\int_P m\mu$, $\int_P m_\pm^{\frac{n+1}{n+2}}\mu$, $\int_P mf\mu$.

If E is a Killing vector field, then $\mathrm{Quant}(P)$ is totally geodesic submanifold of $\mathrm{Diff}_\theta(P)$. The (well-posed) geodesic equation on $\mathrm{Quant}(P)$ is found again to be the quasi-geostrophic f-plane equation on $C^\infty(M)$.