On some new forms of lattice integrable equations

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Outline

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- Also, starting from the bilinear form of semidiscrete sine-Gordon equation we find the recently proposed lattice Tzitzeica equation.

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- Also, starting from the bilinear form of semidiscrete sine-Gordon equation we find the recently proposed lattice Tzitzeica equation.
- In the last section we are making the travelling wave reduction and we show that all of them can be integrated to classical QRT mappings.

New integrable lattice KdV equation

We start from a different form of differential-difference KdV equation namely:

$$\dot{w}_n = \frac{w_n}{w_{n-1}}(w_{n+1} - w_{n-1}) \tag{1}$$

The travelling wave reduction of this equation is exactly the autonomous limit of three point delay-Painleve I equation.
There is also a one way Miura transformation from (1) to the well known differential-difference KdV equation of Hirota; if w_n is a solution of (1), then $u_n = w_n/w_{n-1}$ is a solution of

$$\dot{u}_n - u_n^2(u_{n+1} - u_{n-1}) = 0. \tag{2}$$

Indeed if we put $u_n = w_n/w_{n-1}$ we obtain (after rearanging terms):

$$w_{n-2}(\dot{w}_n w_{n-1} - w_n w_{n+1}) - w_n(\dot{w}_{n-1} w_{n-2} - w_n w_{n-1}) = 0.$$

This relation is true since both brackets can be computed from (1) and its downshift along n.

New integrable lattice KdV equation

We apply the Hirota fomalism in order to prove integrability of (1).
Using $w_n = G_n(t)/F_n(t)$ we build its Hirota bilinear form:

$$D_t G_n \cdot F_n = G_{n+1}F_{n-1} - G_{n-1}F_{n+1} \tag{3}$$

$$G_n F_n = G_{n-1}F_{n+1}, \tag{4}$$

where we have introduced the Hirota bilinear operator

$$D_t^k a \cdot b = (\partial_\tau)^k a(t+\tau)b(t-\tau)|_{\tau=0}$$

For $G_n = F_{n+1}$, equation (4) becomes an identity, whilst equation (3) becomes:

$$D_t F_{n+1} \cdot F_n = F_{n+2}F_{n-1} - F_n F_{n+1} \tag{5}$$

which is exactly the bilinear form of the classical integrable differential-difference KdV equation (2). This fact shows that the equation (1) is an integrable system

The Hirota bilinear method for building integrable discretisations

- The differential or differential-difference integrable system has to be correctly bilinearised
- In **the first step** we replace differential Hirota operators with discrete ones preserving gauge invariance
- In **the second step** the multisoliton solution has to be found
- In **the third step** the nonlinear form of the system has to be recovered

The first step

In order to discretize the system (3)-(4) we have to discretize the Hirota operator in (3) by replacing time derivatives with finite differences with the step δ (and $t \to \delta m$):

$$D_t G_n \cdot F_n \to \left(\frac{1}{\delta}[G(n,(m+1)\delta) - G(n,\delta m)] \right) F(n,\delta m) -$$

$$- \left(\frac{1}{\delta}[F(n,(m+1)\delta) - F(n,\delta m)] \right) G(n,\delta m)$$

After that, we impose the invariance of the resulting bilinear equations with respect to multiplication with $\exp(\mu n + \nu m)$ for any μ, ν (the bilinear gauge invariance). Finally, the gauge invariant discrete forms of (3)-(4) are:

$$G_n^{m+1} F_n^m - G_n^m F_n^{m+1} = \delta (G_{n+1}^{m+1} F_{n-1}^m - G_{n-1}^m F_{n+1}^{m+1}) \qquad (6)$$

$$G_n^m F_n^m = G_{n-1}^m F_{n+1}^m \qquad (7)$$

The second step

In order to prove the integrability of the above system we have to compute the 3-soliton solution:

$$F_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} a_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right) \prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \tag{8}$$

$$G_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} b_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right) \prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \tag{9}$$

where a_i is arbitrary and

$$b_i = p_i a_i \quad , \qquad i = \overline{1,3} \tag{10}$$

$$A_{ij} = \left(\frac{p_i - p_j}{p_i p_j - 1}\right)^2 \quad , \qquad i < j = \overline{1,3} \tag{11}$$

$$q_i = \left(\frac{1-\delta p_i^{-1}}{1-\delta p_i}\right)^{1/\delta} , \qquad i = \overline{1,3} \tag{12}$$

The third step

Now we recover the nonlinear form. Dividing (6) by $F_n^m F_n^{m+1}$, we obtain:

$$\frac{G_n^{m+1}}{F_n^{m+1}} - \frac{G_n^m}{F_n^m} = \delta \frac{F_{n-1}^m F_{n+1}^{m+1}}{F_n^m F_n^{m+1}} \left(\frac{G_{n+1}^{m+1}}{F_{n+1}^{m+1}} - \frac{G_{n-1}^m}{F_{n-1}^m} \right)$$

Using the following notations $\omega_n^m = \frac{G_n^m}{F_n^m}$ and $\Gamma_n^m = \frac{F_{n-1}^m F_{n+1}^{m+1}}{F_n^m F_n^{m+1}}$ we obtain:

$$\omega_n^{m+1} - \omega_n^m = \delta \Gamma_n^m (\omega_{n+1}^{m+1} - \omega_{n-1}^m)$$

From (7) we find:

$$\frac{(F_n^m)^2}{F_{n+1}^m F_{n-1}^m} = \frac{\omega_{n-1}^m}{\omega_n^m}$$

But one can see immediately:

$$\frac{\Gamma_{n+1}^m}{\Gamma_n^m} = \frac{F_{n+2}^{m+1} F_n^{m+1}}{(F_{n+1}^{m+1})^2} \frac{(F_n^m)^2}{F_{n+1}^m F_{n-1}^m} = \frac{\omega_{n+1}^{m+1}}{\omega_n^{m+1}} \frac{\omega_{n-1}^m}{\omega_n^m}$$

The third step

Finally, the nonlinear form of our sistem is:

$$\begin{aligned} \omega_n^{m+1} - \omega_n^m &= \delta\Gamma_n^m(\omega_{n+1}^{m+1} - \omega_{n-1}^m) \\ \Gamma_{n+1}^m &= \frac{\omega_{n-1}^m \omega_{n+1}^{m+1}}{\omega_n^m \omega_n^{m+1}} \Gamma_n^m \end{aligned} \tag{13}$$

We can eliminate Γ_n^m and Γ_{n+1}^m and we get the following new higher order lattice KdV equation:

$$\frac{\omega_n^{m+1} - \omega_n^m}{\omega_{n+1}^{m+1} - \omega_{n+1}^m} = \frac{\omega_{n+1}^{m+1} - \omega_{n-1}^m}{\omega_{n+2}^{m+1} - \omega_n^m} \frac{\omega_n^m \omega_n^{m+1}}{\omega_{n-1}^m \omega_{n+1}^{m+1}} \tag{14}$$

Now we go back to equation (5):

$$D_t F_{n+1} \cdot F_n = F_{n+2} F_{n-1} - F_n F_{n+1}$$

which is the bilinear form of the classical integrable differential-difference KdV equation.
Using the first step of the above Hirota method we obtain the classical reduced Hirota-Miwa equation:

$$F_{n+1}^{m+1} F_n^m - F_{n+1}^m F_n^{m+1} = \delta[F_{n+2}^{m+1} F_{n-1}^m - F_{n+1}^m F_n^{m+1}]$$

Considering $W_n^m = F_{n+1}^m / F_n^m$ and dividing the bilinear equation with $F_n^{m+1} F_n^m$, we obtain the following equation:

$$W_n^{m+1} = (1-\delta) W_n^m + \delta \frac{W_{n+1}^{m+1} W_n^{m+1}}{W_{n-1}^m} \qquad (15)$$

which is a quad-lattice equation and can also be obtained integrating (14).

There is a simple Miura transformation from equation (15) to classical lattice KdV equation of Hirota [6]. More precisely, if W_n^m is a solution of (15) then:

$$u_n^m = W_n^{m+1}/W_{n-1}^m \tag{16}$$

obeys the lattice KdV of Hirota:

$$u_n^{m+1} - u_n^m = \frac{\delta}{1-\delta} u_n^{m+1} u_n^m (u_{n+1}^{m+1} - u_{n-1}^m) \tag{17}$$

Indeed, if $u_n^m = W_n^{m+1}/W_{n-1}^m$ then (17) becomes:

$$W_n^{m+2}\left[(1-\delta)W_{n-2}^m W_{n-1}^m + \delta W_{n-1}^{m+1} W_n^{m+1}\right] -$$

$$W_{n-2}^m\left[(1-\delta)W_n^{m+1} W_{n-1}^{m+1} + \delta W_n^{m+2} W_{n+1}^{m+2}\right] = 0$$

which is true since the first square bracket can be computed from one downshift along the n-direction of (15) and the second square bracket from the one upshift along the m-direction.

New integrable lattice mKdV equation

In this section we are going to study a different form of differential-difference mKdV equation applying the Hirota formalism.

The equation under consideration is:

$$\dot{v}_n = 2v_n \frac{v_{n+1} - v_{n-1}}{v_{n+1} + v_{n-1}} \qquad (18)$$

which goes again in the travelling wave reduction to three-point autonomous delay-Painleve II.

There is a Miura from (18) to the classical semidiscrete mKdV equation (self-dual nonlinear network):

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}) \qquad (19)$$

namely, $u_n = \frac{i}{2}\frac{d}{dt}\log v_n$.

New integrable lattice mKdV equation

We build the Hirota bilinear form using the substitution $v_n = G_n(t)/F_n(t)$. Introducing in (18) and decoupling in the bilinear dispersion relation and the soliton-phase constraint we obtain the following system:

$$D_t G_n \cdot F_n = G_{n+1}F_{n-1} - G_{n-1}F_{n+1} \quad (20)$$

$$2G_n F_n = G_{n+1}F_{n-1} + G_{n-1}F_{n+1} \quad (21)$$

The above system is an integrable one since it admits 3-soliton solution of the following form (k_i is the wave number, ω_i is the angular frequency):

$$F_n = \sum_{\mu_1,\mu_2,\mu_3 \in \{0,1\}} \left(\prod_{i=1}^{3} (a_i e^{\eta_i})^{\mu_i} \right) \prod_{i<j}^{3} A_{ij}^{\mu_i \mu_j} \quad (22)$$

$$G_n = \sum_{\mu_1,\mu_2,\mu_3 \in \{0,1\}} \left(\prod_{i=1}^{3} (b_i e^{\eta_i})^{\mu_i} \right) \prod_{i<j}^{3} A_{ij}^{\mu_i \mu_j} \quad (23)$$

where $\eta_i = k_i n + \omega_i t, \quad i = \overline{1,3}$.

New integrable lattice mKdV equation

The dispersion relation has the form:

$$\omega_i = e^{k_i} - e^{-k_i} = 2\sinh k_i \tag{24}$$

Phase factors and defined by:

$$b_i = -a_i = 1 \tag{25}$$

The interaction terms have the following form:

$$A_{ij} = \frac{\cosh(k_i - k_j) - 1}{\cosh(k_i + k_j) - 1} \quad , \qquad i < j = \overline{1,3} \tag{26}$$

The Hirota bilinear method for building integrable discretisations

We discretize the bilinear system (20)-(21) in the same way. Replacing time derivatives in (20) with finite differences and imposing the bilinear gauge invariance we obtain:

$$G_n^{m+1}F_n^m - G_n^m F_n^{m+1} = \delta(G_{n+1}^{m+1}F_{n-1}^m - G_{n-1}^m F_{n+1}^{m+1}) \quad (27)$$

$$2G_n^m F_n^m = G_{n+1}^m F_{n-1}^m + G_{n-1}^m F_{n+1}^m \quad (28)$$

The above system admits the following 3-soliton solution:

$$F_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} a_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right)\prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \quad (29)$$

$$G_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} b_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right)\prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \quad (30)$$

The Hirota bilinear method for building integrable discretisations

The 3-soliton solution has the same phase factors and interaction terms as in the differential-difference case:

$$b_i = -a_i = 1 \quad , \qquad i = \overline{1,3}$$

$$A_{ij} = \frac{\cosh(k_i - k_j) - 1}{\cosh(k_i + k_j) - 1} \quad , \qquad i < j = \overline{1,3}$$

but different dispersion relation:

$$q_i = \left(\frac{1 - \delta p_i^{-1}}{1 - \delta p_i} \right)^{1/\delta} , \quad i = \overline{1,3} \tag{31}$$

where $p_i = e^{k_i}$, $q_i = e^{\omega_i}$, $i = 1, 2, 3$ (k_i is the wave number and ω_i is the angular frequency).

The Hirota bilinear method for building integrable discretisations

Now we can recover the nonlinear form. Dividing (27) by $F_n^m F_n^{m+1}$ and using the following notations $\omega_n^m = \frac{G_n^m}{F_n^m}$, $\Gamma_n^m = \frac{F_{n-1}^m F_{n+1}^{m+1}}{F_n^m F_n^{m+1}}$, we obtain:

$$\omega_n^{m+1} - \omega_n^m = \delta \Gamma_n^m (\omega_{n+1}^{m+1} - \omega_{n-1}^m)$$

From equation (28) we find:

$$\frac{(F_n^m)^2}{F_{n+1}^m F_{n-1}^m} = \frac{\omega_{n-1}^m + \omega_{n+1}^m}{2\omega_n^m}$$

But one can see immediately:

$$\frac{\Gamma_{n+1}^m}{\Gamma_n^m} = \frac{F_{n+2}^{m+1} F_n^{m+1}}{(F_{n+1}^{m+1})^2} \frac{(F_n^m)^2}{F_{n+1}^m F_{n-1}^m} = \frac{2\omega_{n+1}^{m+1}}{\omega_{n+2}^{m+1} + \omega_n^{m+1}} \frac{\omega_{n+1}^m + \omega_{n-1}^m}{2\omega_n^m}$$

The Hirota bilinear method for building integrable discretisations

Finally the nonlinear form of our sistem is:

$$\begin{aligned} \omega_n^{m+1} - \omega_n^m &= \delta \Gamma_n^m (\omega_{n+1}^{m+1} - b\omega_{n-1}^m) \\ \Gamma_{n+1}^m &= \frac{\omega_n^m + \omega_{n-1}^m}{\omega_{n+2}^m + \omega_n^{m+1}} \frac{\omega_{n+1}^{m+1}}{\omega_n^m} \Gamma_n^m \end{aligned} \tag{32}$$

One can eliminate Γ_n^m and Γ_{n+1}^m and we get the following new higher order nonlinear lattice mKdV equation:

$$\frac{\omega_n^{m+1} - \omega_n^m}{\omega_{n+1}^{m+1} - \omega_{n+1}^m} = \frac{\omega_{n+1}^{m+1} - \omega_{n-1}^m}{\omega_{n+2}^{m+1} - \omega_n^m} \frac{\omega_{n+2}^{m+1} + \omega_n^{m+1}}{\omega_n^m + \omega_{n+1}^m} \frac{\omega_n^m}{\omega_{n+1}^{m+1}} \tag{33}$$

Lattice system related to intermediate Sine-Gordon equation

In this section we are going to study the following differential-difference equation:

$$\frac{d}{dt}(u_n u_{n+1}) = \gamma u_n^2 + \kappa u_{n+1}{}^2 \qquad (34)$$

where γ and κ are constants.

For $\gamma = -\kappa = 1$ the above equation is equivalent with the famous intermediate sine-Gordon equation (through the substitution $y(x,t) = i \log(u(x+i\sigma,t)/u(x-i\sigma,t)), n = (x-i\sigma)/2i\sigma$):

$$\partial_t Ty(x,t) + 2\sin y(x,t) = 0$$

where T is a singular integral operator defined through:

$$(Tf)(x) = \frac{1}{2\sigma} P \int_{-\infty}^{\infty} \coth \frac{\pi(z-x)}{2\sigma} f(z) dz$$

Hirota biliniarization

Taking $u_n = G_n/F_n$ in (34) we obtain:

$$G_{n+1}F_n(D_tG_n \cdot F_{n+1} - \kappa G_{n+1}F_n) + G_nF_{n+1}(D_tG_{n+1} \cdot F_n - \gamma G_nF_{n+1}) = 0$$

The equation above can be splitted in the following way:

$$D_tG_n \cdot F_{n+1} - \kappa G_{n+1}F_n = AG_nF_{n+1} \quad (35)$$

$$D_tG_{n+1} \cdot F_n - \gamma G_nF_{n+1} = -AG_{n+1}F_n \quad (36)$$

where A is a gauge constant.

We have obtained the Hirota bilinear form of the differential-difference equation under consideration (34).

The Hirota bilinear method for building integrable discretisations

Replacing time derivatives with finite differences and t with δm in (35) and (36), and imposing the bilinear gauge invariance we contruct the fully discrete gauge invariant bilinear equations:

$$G_n^{m+1}F_{n+1}^m - G_n^m F_{n+1}^{m+1} = \delta(kG_{n+1}^{m+1}F_n^m + AG_n^m F_{n+1}^{m+1}) \quad (37)$$

$$G_{n+1}^{m+1}F_n^m - G_{n+1}^m F_n^{m+1} = \delta(\gamma G_n^m F_{n+1}^{m+1} - AG_{n+1}^{m+1}F_n^m) \quad (38)$$

We take A=1, $\gamma = -\kappa = 1$ and the above system admits the following 3-soliton solution:

$$F_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} a_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right)\prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \quad (39)$$

$$G_n^m = \sum_{\mu_1,\mu_2,\mu_3\in\{0,1\}} \left(\prod_{i=1}^{3} b_i^{\mu_i}(p_i^n q_i^{m\delta})^{\mu_i}\right)\prod_{i<j}^{3} A_{ij}^{\mu_i\mu_j} \quad (40)$$

The Hirota bilinear method for building integrable discretisations

The dispersion relation is:

$$q_i = \frac{1+2\delta+p_i}{1+(1+2\delta)p_i}$$

The phase factors

$$a_i = -b_i = 1, \quad i = \overline{1,3}$$

and the interaction terms

$$A_{ij} = \left(\frac{p_i - p_j}{1 - p_i p_j}\right)^2$$

In order to see the nonlinear form, we take $X_n^m = G_n^m / F_n^m$ and from bilinear equation we obtain (after eliminating the term $F_n^{m+1} F_{n+1}^m / F_n^m F_{n+1}^{m+1}$):

$$X_{n+1}^m = X_n^{m+1} \frac{X_{n+1}^{m+1}(1+\delta) - \delta X_n^m}{X_n^m(1+\delta) - \delta X_{n+1}^{m+1}}$$

which is nothing but classical lattice mKdV equation.

The Hirota bilinear method for building integrable discretisations

The dispersion relation is:

$$q_i = \frac{1 + 2\delta + p_i}{1 + (1 + 2\delta)p_i}$$

The phase factors

$$a_i = -b_i = 1, \quad i = \overline{1,3}$$

and the interaction terms

$$A_{ij} = \left(\frac{p_i - p_j}{1 - p_i p_j} \right)^2$$

In order to see the nonlinear form, we take $X_n^m = G_n^m / F_n^m$ and from bilinear equation we obtain (after eliminating the term $F_n^{m+1} F_{n+1}^m / F_n^m F_{n+1}^{m+1}$):

$$X_{n+1}^m = X_n^{m+1} \frac{X_{n+1}^{m+1}(1 + \delta) - \delta X_n^m}{X_n^m(1 + \delta) - \delta X_{n+1}^{m+1}}$$

which is nothing but classical lattice mKdV equation.

Remark on discrete Tzitzeica equation

Starting from the general differential-difference equation:

$$\frac{\dot{u}_{n+1}}{u_{n+1}} - \frac{\dot{u}_n}{u_n} = \alpha u_n u_{n+1} + \frac{\mu}{u_n u_{n+1}} + \beta_0 u_n + \beta_1 u_{n+1} \tag{41}$$

we get by means of $u_n = G_n(t)/F_n(t)$:

$$D_t G_{n+1} \cdot G_n - \mu F_n F_{n+1} = A G_{n+1} G_n \tag{42}$$

$$D_t F_{n+1} \cdot F_n + \alpha G_{n+1} G_n + \beta_0 G_n F_{n+1} + \beta_1 G_{n+1} F_n = A F_n F_{n+1} \tag{43}$$

where A is a constant.

Remark on discrete Tzitzeica equation

We discretize in a gauge invariant way the above bilinear form and we get the following general bilinear system:

$$G_{n+1}^{m+1}G_n^m - G_{n+1}^m G_n^{m+1} - \delta\mu F_n^{m+1}F_{n+1}^m = \delta A G_{n+1}^m G_n^{m+1} \tag{44}$$

$$\begin{aligned} &F_{n+1}^{m+1}F_n^m - F_{n+1}^m F_n^{m+1} + \delta\alpha G_{n+1}^m G_n^{m+1} + \delta\beta_{00} G_n^{m+1}F_{n+1}^m + \\ &+\delta\beta_{10} G_{n+1}^m F_n^{m+1} + \delta\beta_{01} G_n^m F_{n+1}^{m+1} + \delta\beta_{11} G_{n+1}^{m+1}F_n^m = \delta A F_{n+1}^m F_n^{m+1} \end{aligned} \tag{45}$$

where $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}$ are arbitrary coefficients, which have to be determined according to integrability requirements. Now we consider
$\delta A = -1, \delta\mu = 1, \delta\alpha = -1/c^2, \beta_{00} = -\beta_{01} = 1/\delta c, \beta_{10} = -\beta_{11} = 1/\delta c$.
Then the above bilinear system will have the form:

$$G_{n+1}^{m+1}G_n^m - F_n^{m+1}F_{n+1}^m = 0 \tag{46}$$

$$cF_{n+1}^{m+1}F_n^m - \frac{1}{c}G_{n+1}^m G_n^{m+1} + G_n^{m+1}F_{n+1}^m + G_{n+1}^m F_n^{m+1} - G_n^m F_{n+1}^{m+1} - G_{n+1}^{m+1}F_n^m = 0 \tag{47}$$

Remark on discrete Tzitzeica equation

Calling $W_n^m = G_n^m / F_n^m$ and eliminating the term $F_n^m F_{n+1}^{m+1} / F_{n+1}^m F_n^{m+1}$, we will get exactly the form of lattice Tzitzeica equation found by Adler [9]:

$$\frac{W_n^m W_{n+1}^{m+1}}{c - W_n^m - W_{n+1}^{m+1}} = \frac{1}{c^{-1} W_{n+1}^m W_n^{m+1} - W_n^{m+1} - W_{n+1}^m}$$

We claim that the bilinear system (46), (47) is the Hirota bilinear form of the lattice Tzitzeica equation. Indeed, if we put $G_n^m = \tau_{n+1}^m \tau_n^{m+1}, F_n^m = \tau_n^m \tau_{n+1}^{m+1}$ then the first bilinear equaton is identically verified and the second bilinear equation is exactly the trilinear form found by Adler:

$$\det \begin{pmatrix} \tau_n^{m+2} & \tau_{n+1}^{m+2} & \tau_{n+2}^{m+2} \\ \tau_n^{m+1} & c^{-1} \tau_{n+1}^{m+1} & \tau_{n+2}^{m+1} \\ \tau_n^m & \tau_{n+1}^m & \tau_{n+2}^m \end{pmatrix} - (c - c^{-1}) \tau_n^m \tau_{n+1}^{m+1} \tau_{n+2}^{m+2} = 0.$$

Reduction of the higher order lattice KdV

The general form of a symmetric QRT mapping is the following:

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1} f_2(x_m)}{f_2(x_m) - x_{m-1} f_3(x_m)}$$

where f_1, f_2, f_3 are general quartic polynomials in x_m. Any QRT mapping possesses an invariant which is biquadratic in x_m and x_{m-1}. The integrability comes from the fact that this biquadratic correspondence can be integrated in terms of elliptic functions.

Let us start with the higher lattice KdV equation (14):

$$\frac{\omega_n^{m+1} - \omega_n^m}{\omega_{n+1}^{m+1} - \omega_{n+1}^m} = \frac{\omega_{n+1}^{m+1} - \omega_{n-1}^m}{\omega_{n+2}^{m+1} - \omega_n^m} \frac{\omega_n^m \omega_n^{m+1}}{\omega_{n-1}^m \omega_{n+1}^{m+1}}$$

We consider that $\omega(n,m) = x(n+m) \equiv x_\nu$ with $\nu = n+m$. In this reduction our equation becomes the following four-order mapping:

$$\frac{x_{\nu+1} - x_\nu}{x_{\nu+2} - x_{\nu+1}} - \frac{x_{\nu+2} - x_{\nu-1}}{x_{\nu+3} - x_\nu} \frac{x_\nu x_{\nu+1}}{x_{\nu-1} x_{\nu+2}} = 0$$

Reduction of the higher order lattice KdV

Then we have:

$$\frac{x_{\nu+1}-x_\nu}{x_{\nu+2}-x_{\nu+1}}-\frac{x_{\nu+2}-x_{\nu-1}}{x_{\nu+3}-x_\nu}\frac{x_\nu x_{\nu+1}}{x_{\nu-1}x_{\nu+2}}=$$

$$=\frac{x_{\nu+1}/x_\nu-1}{x_{\nu+2}/x_{\nu+1}-1}\frac{x_\nu}{x_{\nu+1}}-\frac{x_{\nu+2}/x_{\nu-1}-1}{x_{\nu+3}/x_\nu-1}\frac{x_{\nu+1}}{x_{\nu+2}}=$$

$$=\frac{x_{\nu+1}/x_\nu-1}{x_{\nu+2}/x_{\nu+1}-1}\frac{x_\nu}{x_{\nu+1}}-\frac{(x_{\nu+2}/x_{\nu+1})(x_{\nu+1}/x_\nu)(x_\nu/x_{\nu-1})-1}{(x_{\nu+3}/x_{\nu+2})(x_{\nu+2}/x_{\nu+1})(x_{\nu+1}/x_\nu)-1}\frac{x_{\nu+1}}{x_{\nu+2}}\underbrace{=}_{w_\nu=x_{\nu+1}/x_\nu}$$

$$=\frac{w_\nu-1}{w_{\nu+1}-1}\frac{1}{w_\nu}-\frac{w_{\nu+1}w_\nu w_{\nu-1}-1}{w_{\nu+2}w_{\nu+1}w_\nu-1}\frac{1}{w_{\nu+1}}\iff$$

$$\iff\frac{w_{\nu+1}(w_{\nu+2}w_{\nu+1}w_\nu-1)}{w_{\nu+1}-1}=\frac{w_\nu(w_{\nu+1}w_\nu w_{\nu-1}-1)}{w_\nu-1}$$

Reduction of the higher order lattice KdV

In the last equality the left term is the upshift of the right one so:

$$\frac{w_\nu(w_{\nu+1}w_\nu w_{\nu-1}-1)}{w_\nu-1}=\alpha \Leftrightarrow w_{\nu+1}w_{\nu-1}=\frac{\alpha+1}{w_\nu}-\frac{\alpha}{w_\nu^2}$$

where α is an arbitrary constant. The last relation is a QRT mapping which is the autonomous limit of a q-Painleve equation realizing an automorphism of a rational surface of type $A_7^{(1)}$.

Quite surprinsingly, if we do the same travelling wave reduction on the quadrilateral KdV (15):

$$W_n^{m+1}=(1-\delta)W_n^m+\delta\frac{W_{n+1}^{m+1}W_n^{m+1}}{W_{n-1}^m}$$

we will obtain the same QRT mapping as above with $\alpha=-1+1/\delta$. Indeed if $W_n^m=x(n+m)\equiv x_\nu$ then (15) is turned into:

$$x_{\nu+1}x_{\nu-1}-(1-\delta)x_\nu x_{\nu-1}-\delta x_{\nu+2}x_{\nu+1}=0$$

Reduction of the higher order lattice KdV

Dividing the above equation by $x_{\nu+1}x_\nu$ we get:

$$\frac{x_{\nu-1}}{x_\nu} - (1-\delta)\frac{x_{\nu-1}}{x_{\nu+1}} - \delta\frac{x_{\nu+2}}{x_\nu} \underbrace{=}_{w_\nu = x_{\nu+1}/x_\nu}$$

$$= 1/w_{\nu-1} - (1-\delta)/(w_\nu w_{\nu-1}) - \delta w_{\nu+1} w_\nu \underbrace{\Leftrightarrow}_{\alpha=-1+1/\delta}$$

$$w_{\nu+1} w_{\nu-1} = \frac{\alpha+1}{w_\nu} - \frac{\alpha}{w_\nu^2}$$

Reduction of the higher order lattice KdV

Remark: The travelling wave reduction of classical lattice KdV of Hirota gives a QRT mapping which is the autonomous limit of the *d*-Painleve equation (aditive type of rational surface $E_6^{(1)}$). Indeed, if
$u_n^m = u(n+m) \equiv u_\nu, \delta' = \delta/(1-\delta)$ we get from (17):

$$u_{\nu+1} - u_\nu - \delta' u_{\nu+1} u_\nu (u_{\nu+2} - u_{\nu-1}) = 0 \Rightarrow \frac{1}{\delta'}\left(\frac{1}{u_\nu} - \frac{1}{u_{\nu+1}}\right) - (u_{\nu+2} - u_{\nu-1}) = 0 \Rightarrow$$

$$\frac{1}{\delta'}\left(\frac{1}{u_\nu} - \frac{1}{u_{\nu+1}}\right) - (u_{\nu+2} + u_{\nu+1} + u_\nu - u_{\nu+1} - u_\nu - u_{\nu-1}) = 0 \Longleftrightarrow$$

$$\frac{1}{\delta'}\frac{1}{u_\nu} + u_{\nu+1} + u_\nu + u_{\nu-1} = \frac{1}{\delta'}\frac{1}{u_{\nu+1}} + u_{\nu+2} + u_{\nu+1} + u_\nu$$

Because the left hand side is the downshift of the right hand side, every member will be a constant γ, so:

$$u_{\nu+1} + u_\nu + u_{\nu-1} = \gamma - \frac{1}{\delta'}\frac{1}{u_\nu}$$

which is exactly the autonoums form of *d*-Painleve I equation. So, we have integrable discretisations which give both multiplicative and additive mappings by the same reduction.

Reduction of the higher order lattice mKdV

The higher lattice mKdV is (34)

$$\frac{\omega_n^{m+1}-\omega_n^m}{\omega_{n+1}^{m+1}-\omega_{n+1}^m}=\frac{\omega_{n+1}^{m+1}-\omega_{n-1}^m}{\omega_{n+2}^{m+1}-\omega_n^m}\,\frac{\omega_{n+2}^{m+1}+\omega_n^{m+1}}{\omega_{n-1}^m+\omega_{n+1}^m}\,\frac{\omega_n^m}{\omega_{n+1}^{m+1}}$$

By the same procedure and keeping the same notations we arrive at:

$$\frac{w_\nu-1}{w_{\nu+1}-1}-\frac{w_{\nu+1}w_\nu w_{\nu-1}-1}{w_{\nu+2}w_{\nu+1}w_\nu-1}\,\frac{w_{\nu+2}w_{\nu+1}+1}{w_\nu w_{\nu-1}+1}\,\frac{w_\nu}{w_{\nu+1}}=0$$

Now we force $(w_\nu w_{\nu+1}+1)$ as a denominator in the second term. We shall get

$$\frac{w_\nu-1}{w_{\nu+1}-1}-\frac{w_{\nu+1}w_\nu w_{\nu-1}-1}{w_{\nu+2}w_{\nu+1}w_\nu-1}\,\frac{w_{\nu+2}w_{\nu+1}+1}{w_{\nu+1}w_\nu+1}\,\frac{w_{\nu+1}w_\nu+1}{w_\nu w_{\nu-1}+1}\,\frac{w_\nu}{w_{\nu+1}}=0\iff$$

$$\iff\frac{(w_\nu-1)(w_{\nu+1}w_\nu+1)(w_\nu w_{\nu-1}+1)}{(w_{\nu+1}w_\nu w_{\nu-1}-1)w_\nu}=\frac{(w_{\nu+1}-1)(w_{\nu+2}w_{\nu+1}+1)(w_{\nu+1}w_\nu+1)}{(w_{\nu+2}w_{\nu+1}w_\nu-1)w_{\nu+1}}$$

Reduction of the higher order lattice mKdV

Again the left hand side is the downshift of the right hand side, so both members are equal to a constant σ. Accordingly, the integrated reduction of the lattice mKdV is the following more complicated QRT-form:

$$w_{\nu+1} = \frac{1 - w_\nu(\sigma+1) + w_{\nu-1}(w_\nu - w_\nu^2)}{-(w_\nu - w_\nu^2) - w_{\nu-1}(w_\nu^2(1+\sigma) - w_\nu^3)}$$

Remark: Travelling wave reduction of Titeica equation gives trivially the QRT mapping:

$$w_{\nu+1} w_{\nu-1} = \frac{c - w_{\nu+1} - w_{\nu-1}}{c^{-1} w_\nu^2 - 2w_\nu}$$

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