Quantum mechanics and geometry on Siegel-Jacobi spaces

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Outline

• **For mathematicians**: *Jacobi group*– $G_n^J = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$
$(2n+1)(n+1)$- dim. Generalized Jacobi groups:Takase, Yang, Lee...

•Jacobi groups - unimodular, **non**reductive, algebraic gr. of Harish-Chandra type (Satake), and *Coherent State* type group (Moskovici & Verona, Lisiecki, Neeb,...)

•The Siegel-Jacobi domains - reductive, **non**symmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding. **not** Einstein manifold.

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• The denomination *Jacobi group* for G_n^J was introduced in Math. by Eichler and Zagier, *Theory of Jacobi forms* (1985).
Eichler and Zagier have introduced the notion of *Jacobi form* on $\mathrm{SL}_2(\mathbb{Z})$ as a holomorphic function on $\mathcal{X}_1^J$ (=$\mathbb{C}\times$ upper half plane), satisfying three properties.
One of this properties, generalized to other groups, was studied by Pyatetskii-Shapiro, who referred to it as the *Fourier-Jacobi* expansion, and to some coefficients as *Jacobi forms*, a name adopted by Eichler and Zagier to denote also the group appearing in this context.

The denomination *Jacobi group* was adopted also in the monograph Berndt & Schmidt *Elements of the Representation Theory of the Jacobi group* (1998) .

R. Berndt (1984), E. Kähler (1983); *Poincaré group* or *The New Poincaré group* investigated by Erich Kähler as the 10-dimensional group G^K (a double cover of the de Sitter group $\mathrm{SO}_0(4,1)$) which invariates the metric $ds^2 = \frac{dx^2+dy^2+dz^2+dt^2}{t^2}$. Quaternionic 2×2 matrices.

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For experts (after Bryant, Chern, Gardner, Goldsmidt, Griffith, 1991)

$$
\begin{array}{ccccc}
 & & \mathrm{SL}_2(\mathbb{R}) & \hookrightarrow & G^J \\
 & & \downarrow^{\cap} & & \downarrow^{\cap} \\
G^K & \hookleftarrow & \mathrm{SL}_2(\mathbb{C}) & & \mathrm{Sp}_2(\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
G^{dS} & \hookleftarrow & \mathrm{SO}_0(1,3) & \hookrightarrow & G^{AdS} \\
 & \searrow & \downarrow^{\cap} & \swarrow & \\
 & & G^P & & \\
 & & \downarrow & & \\
 & & G^G & &
\end{array}
$$

$G^{dS} = \mathrm{SO}_0(4,1)$ (de Sitter); $G^{AdS} = \mathrm{SO}_0(2,3)$ (Anti-de Sitter);
$G^P = \mathrm{SO}_0(1,3) \ltimes \mathbb{R}^4$ (Poincaré); $G^G = (\mathrm{SO}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4$ (Galilei).

• Kirillov:

$\mathfrak{st}(n,\mathbb{R})$ (Kirillov 1974, Section 18.4) or $\mathfrak{tsp}(2n+2,\mathbb{R})$ (Kirillov 2004).
$\mathfrak{st}(n,\mathbb{R}) \approx$ the subalgebra of Weyl algebra A_n of polynomials of degree maximum two in the variables $p_1,\dots,p_n$, $q_1,\dots,q_n$ with the Poisson bracket.
$\mathfrak{h}_n$ = the nilpotent ideal $\approx$ polynomials of degree $\leqslant 1$.
$\mathfrak{sp}(n,\mathbb{R}) \approx$ to the subspace of symmetric homogeneous polynomials of degree 2.

• **In Physics**: Schrödinger (Hagen, or conformal Galilean...) group.
1972, U. Niederer: *the maximal kinematical invariance group (MKI) of the free Schrödinger equation.*
The Schrödinger group — 12-parameter group =

- *the Galilei group* G_3^G +
- *the group of dilations +*
- *1-parameter group of transformations ("expansions"- similar to the special conformal transformations of the conformal group).*

$$\Delta(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) = 0,\ \Delta(t,\boldsymbol{x}) = i\partial_0 + \frac{1}{2m}\Delta_3,\ (t,\boldsymbol{x}) \in \mathbb{R}^4$$

$$\psi(t,\boldsymbol{x}) \rightarrow (T_g\psi)(t,\boldsymbol{x}) = f_g[g^{-1}(t,\boldsymbol{x})]\psi[g^{-1}(t,\boldsymbol{x})],$$
$$g(t,\boldsymbol{x}) = \left(d^2\frac{t+b}{1+\alpha(t+b)}, d\frac{R\boldsymbol{x}+\boldsymbol{v}t+\boldsymbol{a}}{1+\alpha(t+b)}\right) \qquad (1.1)$$

$\alpha, b, d \in \mathbb{R}, R \in \mathrm{SO}(3), \boldsymbol{v}, \boldsymbol{a} \in \mathbb{R}^3$.
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Barut & Raczka; Dobrev & all: Levi-Malcev decomposition of the Schrödinger group in $(n+1)$-space-time dimensions (it has $(n^2+3n+6)/2$-dim):

$$\mathrm{Sch}(n) = \Big[\underbrace{\mathbb{R}^n \times G^G_n}_{\text{radical}}\Big] \rtimes \Big[\underbrace{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(n)}_{\text{SS-Levi part}}\Big]$$

$$G^G_n = (\mathrm{SO}(n) \rtimes \mathbb{R}^n) \rtimes \mathbb{R}^{n+1}$$

In the case $n = 1$ with $t \in \mathbb{C}, \Im(t) > 0$, $x \in \mathbb{C}$, corresponds to a formula considered by Eichler & Zagier, Kähler, Berndt for $G^J_1(\mathbb{R})$.

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• K. B. Wolf: Weyl-symplectic group:

K. B. Wolf, *The Heisenberg-Weyl ring in quantum mechanics*, in *Group theory and its applications*, Vol. 3, Ed. E M Loebl, Academic Press, New York (1975) 189-247
K. B. Wolf, Integral transforms in science and engineering, Plenum Publ. Corp. New York (1979)
K. B. Wolf, Geometric Optics on Phase Space, Springer (2004)

The Jacobi group is an important object in connection with Quantum Mechanics, Geometric Quantization, Optics: Guillemin & Sterenberg (1977, 1984); Bacry & Cadilac, Stoler, Nazarthy & Shamir, Simon & Wolf.

Jacobi group describes *squeezed states* in Quantum Optics (see Stoler).

Applications of squeezed states: detection of gravitational waves, spectroscopy with two and three-level atoms in squeezed fields, quantum communications, Einstein-Podolsky-Rosen correlations, entanglement, quantum cryptography, teleportation, ...

For the harmonic oscillator: $\Delta x = \Delta p = 1/\sqrt{2}$ (in units of $\hbar$). "The squeezed states": $\Delta x < 1/\sqrt{2}$. The squeezed states are a particular class of *minimum uncertainty states* (MUS) — states which saturates the Heisenberg uncertainty relation.

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Appendix: CS (à la Perelomov)

CS: $(G, \pi, \mathfrak{H})$ G = Lie group, π = unitary irreducible representation of G on the complex separable Hibert space $\mathfrak{H}$. A common realization of coherent states as space of holomorphic functions defined on the homogeneous manifold $M = G/H$, square integrable with respect to a scalar product determined by the reproducing kernel K. Usually M is a Kähler manifold.

Notation: $\boldsymbol{X} := d\pi(X), X \in \mathfrak{g}$

$$\underline{e}_x = \exp\Big(\sum_{\varphi\in\Delta_+} x_\varphi \boldsymbol{X}^+_\varphi - \bar{x}_\varphi \boldsymbol{X}^-_\varphi\Big)e_0,\ \ e_z = \exp\Big(\sum_{\varphi\in\Delta_+} z_\varphi \boldsymbol{X}^+_\varphi\Big)e_0$$

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Reductive manifolds, CS-manifolds

M = CS-*orbits*: admit a holomorphic embedding $\iota_M : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$.

Remark

For reductive homogeneous manifolds the tangent space to M at o can be identified with $\mathfrak{m}$ and if $\exp : \mathfrak{g} \rightarrow G$, then $G/H = \exp(\mathfrak{m})$, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m},\ \mathfrak{h} \cap \mathfrak{m} = 0, [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
CS-manifolds - reductive spaces.
For symmetric spaces: $z(t) = FC(tx)$ gives geodesics in M s.t. $z(0) = p$, $\dot{z}(0) = x$.

Recall

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The (6-dim) Jacobi algebra

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{2.1}$$

$\mathfrak{h}_1$ is an ideal in $\mathfrak{g}_1^J$

$$[a, a^{\dagger}] = 1,\ [K_0, K_{\pm}] = \pm K_{\pm}\ ,\ [K_-, K_+] = 2K_0,$$
$$[a, K_+] = a^{\dagger}\ ,\ \left[K_-, a^{\dagger}\right] = a,$$
$$2\left[K_0,\ a^{\dagger}\right] = a^{\dagger},\ 2\left[K_0, a\right] = -a.$$

Perelomov's CS vectors

$$ae_0 = 0, \;\; \boldsymbol{K}_- e_0 = 0, \;\; \boldsymbol{K}_0 e_0 = k e_0; \;\; k > 0, 2k = 2, 3, ...$$

For SU(1, 1), D_k^+ the positive discrete series representations (Bargmann).
To G_1^J we associate Perelomov's CS vectors

$$e_{z,w} := \mathrm{e}^{\sqrt{\mu} z a^{\dagger} + w \boldsymbol{K}_+} e_0, \;\; z, w \in \mathbb{C}, \;\; |w| < 1. \qquad (2.3)$$

on (4-dim) manifold

$$M := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C} \times \mathcal{D}_1 (:= \mathcal{D}_1^J)$$

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The *displacement operator*

$$D_\mu(\alpha) := \exp\sqrt{\mu}(\alpha a^\dagger - \bar{\alpha} a)$$

S_k – *the unitary squeezed operator* – the D^k_+ representation of the group SU(1, 1), $\underline{S}_k(z) = S(w)$, $z, w \in \mathbb{C}$, $|w| < 1$:

$$\begin{aligned} \underline{S}_k(z) &:= \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-),\ w = \frac{z}{|z|}\tanh(|z|),\ (w = FC(z)); \\ S_k(w) &= \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-); \end{aligned}$$

The normalized (squeezed) CS vector

$$\underline{e}_{\eta,w} := D_\mu(\eta)S_k(w)e_0. \tag{2.5}$$

The *displacement operator*

$$D_\mu(\alpha) := \exp\sqrt{\mu}(\alpha a^\dagger - \bar{\alpha} a)$$

S_k – *the unitary squeezed operator* – the D_+^k representation of the group SU(1, 1), $\underline{S}_k(z) = S(w)$, $z, w \in \mathbb{C}$, $|w| < 1$:

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Lemma

The differential action of the generators of the Jacobi algebra $\mathfrak{g}_1^J$:

$$\boldsymbol{a} = \frac{\partial}{\sqrt{\mu}\partial z}; \ \boldsymbol{a}^{\dagger} = \sqrt{\mu} z + w \frac{\partial}{\sqrt{\mu}\partial z}; \tag{2.6a}$$

$$\mathbb{K}_- = \frac{\partial}{\partial w}; \ \mathbb{K}_0 = k + \frac{1}{2} z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}; \tag{2.6b}$$

$$\mathbb{K}_+ = \frac{1}{2}\mu z^2 + 2kw + zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}, \tag{2.6c}$$

Siegal-Bargmann-Fock representation

Segal-Bargmann-Fock space $\mathfrak{F}_\mu = L^2_{\text{hol}}(\mathbb{C}, \rho_\mu)$, $\rho_\mu = \frac{\mu}{\pi}\exp(-\mu|z|^2)$:
$(f,g)_\mu = \int_{\mathbb{C}} \bar{f} g \rho_\mu d\Re z d\Im z$, $K_\mu(z,\bar{z}') = e^{\mu z\bar{z}'}$, $-\mathrm{i}\omega_\mu(z) = \mu dz \wedge d\bar{z}$.

Remark

The differential realization on $\mathfrak{F}_\mu$: $\boldsymbol{a}^\dagger = \sqrt{\mu} z$, $\boldsymbol{a} = \frac{1}{\sqrt{\mu}}\frac{\partial}{\partial z}$
$\boldsymbol{q} = q$, $\boldsymbol{p} = -i\hbar\frac{\partial}{\partial q}$ *in* $\mathfrak{H} = L^2(\mathbb{R}, dx)$, $\boldsymbol{a} = \lambda(\boldsymbol{q} + \mathrm{i}\boldsymbol{p})$, $\boldsymbol{a}^\dagger = \lambda(\boldsymbol{q} - \mathrm{i}\boldsymbol{p})$,
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Proposition

$$G_1^J := HW \rtimes SU(1,1)$$

$$(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),$$

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, |a|^2 - |b|^2 = 1, \ \alpha_g = a\,\alpha + b\,\bar{\alpha}, \ g^{-1} \cdot \alpha = \bar{a}\alpha - b\bar{\alpha}.$$

$h := (g, \alpha) \in G_1^J$, $\pi_{k\mu}(h) := S_k(g) D_\mu(\alpha)$, $g \in SU(1,1)$,$\alpha \in \mathbb{C}$,
$x := (z, w) \in \mathcal{D}_1^J = \mathbb{C} \times \mathcal{D}_1$.

$$(\pi)_{k\mu}(h) \cdot f)(x) = (\bar{a} + \bar{b}w)^{-2k} \exp(-\mu\lambda_1) f(x_1) \quad (2.7)$$

$$z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{b}w + \bar{a}}; \ w_1 = g \cdot w = \frac{aw + b}{\bar{b}w + \bar{a}}. \quad (2.8)$$

$$\lambda_1 = \frac{\bar{b}(z + z_0)^2}{2(\bar{a} + \bar{b}w)} + \bar{\alpha}(z + \frac{z_0}{2}), \ z_0 = \alpha - \bar{\alpha}w.$$

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Reproducing Bergman kernel

$$\mathcal{D}_1^J \ni \zeta := (z, w) \in (\mathbb{C} \times \mathcal{D}_1),\ K_{k\mu}(\zeta; \bar{\zeta}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}):$$

$$K_{k\mu}(\zeta, \bar{\zeta}') = (1 - w\bar{w}')^{-2k} \exp \mu F(\zeta; \bar{\zeta}'),\ \ F(\zeta; \bar{\zeta}') = \frac{2\bar{z}' z + z^2 \bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}.$$

$$K_{k\mu}(z, w) = (e_{z,w}, e_{z,w}) > 0 :$$

$$K_{k\mu}(z, w) = (1 - w\bar{w})^{-2k} \exp \mu F(z, w),\ \ F(z, w) = \frac{2z\bar{z} + z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}$$

Orhonormal system

$$f_{|n>;e_{k',k'+m}}(z,w)=\sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}w^n\frac{P_n(\sqrt{\mu}z,w)}{\sqrt{n!}},k=k'+\frac{1}{4},2k'\in\mathbb{Z}_+$$

$$P_n(z,w)=n!\sum_{k=0}^{[\frac{n}{2}]}(\frac{w}{2})^k\frac{z^{n-2k}}{k!(n-2k)!},z,w\in\mathcal{D}_1^J. \qquad (3.1)$$

$$\underline{e}_{\eta,w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\eta}}{2}z)e_{z,w}, (z,w) = FC(\eta, w). \qquad (3.2)$$

$$z = \eta - w\bar{\eta};\ (\eta = \frac{z + \bar{z}w}{1 - w\bar{w}})$$

Appendix: Berezin's quantization on Kähler manifolds

$\mathfrak{F}_{\mathfrak{H}} = L^2_{\text{hol}}(M, d\nu_M)$ (Rawnsley: sections of hol. line bundle (L, h, ∇) associated with (homogeneous) Kähler manifold (M, ω))

$$(f,g)_{\mathfrak{F}_{\mathfrak{H}}} = \int_M \bar{f}(z)g(z)d\nu_M(z,\bar{z}), d\nu_M(z,\bar{z}) = \frac{\Omega_M(z,\bar{z})}{K_M},\ K_M = (e_{\bar{z}}, e_{\bar{z}}).$$

$$\Omega_M := (-1)^{\binom{n}{2}}\frac{1}{n!}\underbrace{\omega\wedge\ldots\wedge\omega}_{n \text{ times}}$$

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Parseval overcompletness identity (Rawnsley: on **quantizable** M):

$$(\psi_1, \psi_2) = \int_{M=G/H} (\psi_1, e_{\bar{z}})(e_{\bar{z}}, \psi_2)d\nu_M(z,\bar{z}).$$

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$$(\phi,\psi)_{k\mu} = \Lambda_1\int_{z\in\mathbb{C};|w|<1}\bar{f}_\phi(z,w)f_\psi(z,w)\frac{d\nu_1}{K_{k\mu}},$$

$$d\nu_1=\mu\frac{d\Re w d\Im w}{(1-w\bar{w})^3}d\Re z d\Im z,\ \text{the } G_1^J\text{-invariant measure}$$

$$\Lambda_1=\frac{4k-3}{2\pi^2}.$$

$$-\mathrm{i}\,\omega^1_{k,\mu}=\frac{2k}{(1-w\bar{w})^2}dw\wedge d\bar{w}+\mu\frac{A\wedge\bar{A}}{1-w\bar{w}},\ A=dz+\bar{\eta}dw,$$

$$\eta=\frac{z+\bar{z}w}{1-w\bar{w}},(z,w)=FC(\eta,w).$$

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Remark

The Jacobi groups are unimodular, non-reductive, algebraic groups of Harish -Chandra type. The Siegel-Jacobi domains are reductive, non-symmetric manifolds associated to the Jacobi groups by the generalized Harish-Chandra embedding, quantizable homogeneous Kähler manifolds.

The Ricci form is

$$\rho_{\mathcal{D}_1^J}(z,w) = -3i\frac{dw \wedge d\bar{w}}{(1-w\bar{w})^2}.$$

The scalar curvature is

$$s_{\mathcal{D}_1^J}(p) = -\frac{3}{2k}, p \in \mathcal{D}_1^J.$$

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Ricci form of the Bergman metric

$$\rho_M(z) := \mathrm{i}\sum_{\alpha,\beta=1}^{n} \mathrm{Ric}_{\alpha\bar{\beta}}(z) dz_\alpha \wedge d\bar{z}_\beta, \ \mathrm{Ric}_{\alpha\bar{\beta}}(z) = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln G(z).$$

$$G(z) := \det(h_{\alpha\bar{\beta}})_{\alpha,\beta=1,\dots,n},$$

The scalar curvature at a point $p \in M$ of coordinates z is

$$s_M(p) = \sum_{\alpha,\bar{\beta}=1}^{n} (h_{\alpha,\bar{\beta}})^{-1} \mathrm{Ric}_{\alpha\bar{\beta}}(z)$$

The fundamental conjecture (Gindikin-Vinberg)

Every homogenous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogenous manifold (generalized flag manifold), a homogenous bounded domain and $\mathbb{C}^n/\Gamma$, where Γ denotes a discrete subgroup of translations of $\mathbb{C}^n$.

$$\begin{array}{ccccc} M = (G^{\mathbb{C}}/P) & \times & D & \times & (\mathbb{C}^n/\Gamma) \\ \swarrow & & \downarrow & & \downarrow \\ \text{flag manifold} & & \text{homogeneous} & & \text{Kähler} \\ P - \text{parabolic} & & \text{bounded domain} & & \text{flat} \end{array}$$

Proposition

The FC-transform, $FC(\eta, w) = (z, w),\ z = \eta - w\bar{\eta}$, is a homogeneous Kähler diffeomorphism, $FC^*\omega_{k\mu}(z, w) = \omega_{k\mu}(\eta, w)$:

$$\omega_{k\mu}(\eta, w) = \omega_k(w) + \omega_\mu(\eta).$$

$\omega_{k\mu}(\eta, w)$ is invariant to the action of G^J_1 on $\mathbb{C} \times \mathcal{D}_1$, $((g, \alpha), (\eta, w)) \rightarrow (\eta_1, w_1)$,

$$\eta_1 = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}),\ w_1 = \frac{aw + b}{\bar{b}w + \bar{a}},\ g = \left(\begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array}\right) \in SU(1, 1). \tag{3.3}$$

$$\mu G_1^2 = -2kG_2, \;\; \mu\bar{\eta}G_1^2 = 2kG_3, \; \eta = \frac{z + \bar{z}w}{P};$$
$$G_1 = \frac{\mathrm{d}\, z}{\mathrm{d}\, t} + \bar{\eta}\frac{\mathrm{d}\, w}{\mathrm{d}\, t}, \; G_2 = \frac{\mathrm{d}^2\, w}{\mathrm{d}\, t^2} + 2\frac{\bar{w}}{P}(\frac{\mathrm{d}\, w}{\mathrm{d}\, t})^2,$$
$$G_3 = \frac{\mathrm{d}^2\, z}{\mathrm{d}\, t^2} + 2\frac{\bar{w}}{P}\frac{\mathrm{d}\, z}{\mathrm{d}\, t}\frac{\mathrm{d}\, w}{\mathrm{d}\, t}, \; P = 1 - w\bar{w}.$$

Remark

The FC transform gives geodesics $(z(t), w) = FC(\eta, w)$ *on the non-symmetric space* $\mathcal{D}_1^J$ *with* $w = FC(B), \; w(t) = \frac{B}{|B|}\tanh(t|B|)$ (2.4a) *and* $\eta = \eta_0$.

Notation

“Normalized” Bergman kernel:

$$\kappa_M(z,\bar{z}') := \frac{K_M(z,\bar{z}')}{\sqrt{K_M(z)K_M(z')}} = (\tilde{e}_{\bar{z}},\tilde{e}_{\bar{z}'}) = \frac{(e_{\bar{z}},e_{\bar{z}'})}{||e_{\bar{z}}||||e_{\bar{z}'}||}$$

Berezin kernel $b_M : M\times M \rightarrow [0,1]\in\mathbb{R}$:

$$b_M(z,z') = |\kappa_M(z,\bar{z}')|^2.$$

$\xi : \mathfrak{H}\setminus 0 \rightarrow \mathbb{P}(\mathfrak{H})$ $\xi(\boldsymbol{z}) = [\boldsymbol{z}]$, *Fubini-Study metric* on $\mathbb{CP}^\infty$:

$$\mathrm{d}\, s^2|_{FS}([z]) = \frac{(\mathrm{d}\,\boldsymbol{z},\mathrm{d}\,\boldsymbol{z})(\boldsymbol{z},\boldsymbol{z}) - (\mathrm{d}\,\boldsymbol{z},\boldsymbol{z})(\boldsymbol{z},\mathrm{d}\,\boldsymbol{z})}{(\boldsymbol{z},\boldsymbol{z})^2}.$$

Cayley distance between points in $\mathbb{P}(\mathfrak{H})$:

$$\mathrm{d}_C([z_1],[z_2]) = \arccos\frac{|(z_1,z_2)|}{||z_1||||z_2||}.$$

Calabi's diastasis (Cahen, Gutt and Rawnsley):

$$D_M(z,z') = -\ln b_M(z,z') = -2\ln|(\tilde{e}_z,\tilde{e}_{z'})|.$$

M - homogeneous Kähler manifold $M = G/H$ with $\mathcal{F}_{\mathfrak{H}}$. K admits the series expansion in the orthonormal basis $\Phi_M = (\varphi_0, \varphi_1, \dots)$

Remark

Suppose that the Kähler manifold M admits a holomorphic embedding

$$\iota_M : M \hookrightarrow \mathbb{CP}^\infty, \iota_M(z) = [\varphi_0(z) : \varphi_1(z) : \dots]$$

The hermitian metric on M = the pullback of the Fubini-Study (Kobayashi)

$$\mathrm{d}\, s^2_M(z) = \iota^*_M \mathrm{d}\, s^2_{FS}(z) = \mathrm{d}\, s^2_{FS}(\iota_M(z)).$$

The angle defined by the normalized Bergman kernel:

$$\theta_M(z_1, z_2) = \arccos |\kappa_M(z_1, \bar{z}_2)| = \arccos |(\tilde{e}_{z_1}, \tilde{e}_{z_2})_M| = d_C(\iota_M(z_1), \iota_M(z_2))$$

The Cauchy formula:

$$(\tilde{e}_{z_1}, \tilde{e}_{z_2})_M = (\iota_M(z_1), \iota_M(z_2))_{\mathbb{CP}^\infty}.$$

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Remark

$\mathcal{D}^J_1$ *is a coherent state manifold;* G^J_1 *is a coherent-state group.*
$\mathcal{F}_{\mathfrak{H}} = \mathfrak{F}_{k\mu} = L^2_{hol}(\mathcal{D}^J_1, \rho_{k\mu})$. *The Kählerian embedding* $\iota_{\mathcal{D}^J_1} : \mathcal{D}^J_1 \hookrightarrow \mathbb{CP}^\infty$
$\iota_{\mathcal{D}^J_1} = [\Phi] = \left\{ f_{|n>;e_{k',k'+m}}(z,w) \right\}$,

$$\omega_{k\mu} = \iota^*_{\mathcal{D}^J_1}\omega_{FS}|_{\mathbb{CP}^\infty}, \ \omega_{k\mu}(z,w) = \omega_{FS}([\varphi_N(z,w)]).$$

The normalized Bergman kernel of the Siegel-Jacobi disk

$$\kappa_{k\mu}(\zeta,\bar{\zeta}') = \kappa_k(w,\bar{w}')\exp[\mu(F(\zeta,\bar{\zeta}') - \frac{1}{2}(F(\zeta)+F(\zeta'))],$$

$$\kappa_k(w,\bar{w}') = \left[\frac{(1-|w|^2)(1-|w'|^2)}{(1-w\bar{w}')^2}\right]^k .$$

The Berezin kernel of $\mathcal{D}^J_1$

$$b_{k\mu}(\zeta,\zeta') = b_k(w,w')\exp[2\Re F(\zeta,\bar{\zeta}') - F(\zeta) - F(\zeta')],$$

Diastasis function on the Siegel-Jacobi disk:

$$\frac{D_{k\mu}(\zeta,\zeta')}{2} = k\ln\frac{|1-w\bar{w}'|^2}{(1-|w|^2)(1-|w'|^2)} + \mu[\frac{F(\zeta)+F(\zeta')}{2} - \Re F(\zeta,\bar{\zeta}')].$$

$\mathfrak{h}_n$ - the Heisenberg algebra

$$\mathfrak{h}_n =< \mathrm{i}s1 + \sum_{i=1}^{n}(x_i a_i^{\dagger} - \bar{x}_i a_i) >_{s\in\mathbb{R}, x_i\in\mathbb{C}}, \tag{4.1}$$

$$[a_i, a_j^{\dagger}] = \delta_{ij};\ [a_i, a_j] = [a_i^{\dagger}, a_j^{\dagger}] = 0. \tag{4.2}$$

The algebra $\mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}} = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(n,n)$

$$
\begin{aligned}
[K^-_{ij}, K^-_{kl}] &= [K^+_{ij}, K^+_{kl}] = 0,\; 2[K^-_{ij}, K^0_{kl}] = K^-_{il}\delta_{kj} + K^-_{jl}\delta_{ki},\\
2[K^-_{ij}, K^+_{kl}] &= K^0_{kj}\delta_{li} + K^0_{lj}\delta_{ki} + K^0_{ki}\delta_{lj} + K^0_{li}\delta_{kj},\\
2[K^+_{ij}, K^0_{kl}] &= -K^+_{ik}\delta_{jl} - K^+_{jk}\delta_{li},\; 2[K^0_{ji}, K^0_{kl}] = K^0_{jl}\delta_{ki} - K^0_{ki}\delta_{lj}.
\end{aligned}
$$

The algebra $\mathfrak{g}^J_n := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}$

$\mathfrak{h}_n$ - ideal in $\mathfrak{g}^J_n$, i.e. $[\mathfrak{h}_n, \mathfrak{g}^J_n] = \mathfrak{h}_n$,

$$
\begin{aligned}
[a^{\dagger}_k, K^+_{ij}] &= [a_k, K^-_{ij}] = 0,\\
[a_i, K^+_{kj}] &= \frac{1}{2}\delta_{ik}a^{\dagger}_j + \frac{1}{2}\delta_{ij}a^{\dagger}_k,\ [K^-_{kj}, a^{\dagger}_i] = \frac{1}{2}\delta_{ik}a_j + \frac{1}{2}\delta_{ij}a_k,\\
[K^0_{ij}, a^{\dagger}_k] &= \frac{1}{2}\delta_{jk}a^{\dagger}_i,\ [a_k, K^0_{ij}] = \frac{1}{2}\delta_{ik}a_j.
\end{aligned}
$$

Correspondence

Under the identification $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, $(p,q)\mapsto \alpha = p + \mathrm{i}q,\ p,q\in\mathbb{R}^n$, we have the correspondence

$$M=\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in M(2n,\mathbb{R})\leftrightarrow M_{\mathbb{C}}=\mathcal{C}^{-1}M\mathcal{C}=\left(\begin{array}{cc} p & q\\ \bar{q} & \bar{p}\end{array}\right), \tag{4.3}$$

$$\mathcal{C}=\left(\begin{array}{cc} \mathrm{i}\mathbb{1}_n & \mathrm{i}\mathbb{1}_n\\ -\mathbb{1}_n & \mathbb{1}_n\end{array}\right),$$

$$2a=p+q+\bar{p}+\bar{q},\ 2b=\mathrm{i}(\bar{p}-\bar{q}-p+q),\ p,q\in M(n,\mathbb{C}). \tag{4.4}$$

Coherent states on $\mathcal{D}_n^J$

$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} := \sqrt{\mu}\sum_i z_i a_i^\dagger + \sum_{ij} w_{ij}\boldsymbol{K}_{ij}^+,\ z = (z_i)$$

$$\boldsymbol{a}_i e_o = 0,\ \boldsymbol{K}_{ij}^+ e_0 \neq 0,\ \boldsymbol{K}_{ij}^- e_0 = 0,\ \boldsymbol{K}_{ij}^0 e_0 = \frac{k_i}{4}\delta_{ij}e_0,\ i,j = 1,\ldots,n.$$

$\mathcal{D}_n^J = H_n/\mathbb{R} \times \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\mathrm{U}(n) = \mathbb{C}^n \times \mathcal{D}_n$, dimension $\frac{n(n+3)}{2}$

$$\mathcal{D}_n := \{W \in M(n,\mathbb{C}) | W = W^t,\ \mathbb{1}_n - W\bar{W} > 0\}$$

$$(e_{x,V}, e_{y,W})_{k\mu} = \det(U)^{k/2}\exp \mu F(\bar{x},\bar{V};y,W),\ U = (\mathbb{1}_n - W\bar{V})^{-1};$$
$$2F(\bar{x},\bar{V};y,W) = 2\langle x, Uy\rangle + \langle V\bar{y}, Uy\rangle + \langle x, UW\bar{x}\rangle.$$

$$K_{k\mu} = (e_{z,W}, e_{z,W}) = \det(M)^{\frac{k}{2}}\exp \mu F, M = (\mathbb{1}_n - W\bar{W})^{-1},$$

$$2F = 2\bar{z}^t Mz + z^t\bar{W}Mz + \bar{z}^t MW\bar{z}.$$

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$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} := \sqrt{\mu}\sum_i z_i a_i^{\dagger} + \sum_{ij} w_{ij}\boldsymbol{K}_{ij}^{+},\ z = (z_i)$$

$$\boldsymbol{a}_i e_o = 0,\ \boldsymbol{K}_{ij}^{+} e_0 \neq 0,\ \boldsymbol{K}_{ij}^{-} e_0 = 0,\ \boldsymbol{K}_{ij}^{0} e_0 = \frac{k_i}{4}\delta_{ij}e_0,\ i,j = 1,\dots,n.$$

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$$K_{k\mu} = (e_{z,W}, e_{z,W}) = \det(M)^{\frac{k}{2}}\exp\mu F, M = (\mathbb{1}_n - W\bar{W})^{-1},$$
$$2F = 2\bar{z}^t Mz + z^t\bar{W}Mz + \bar{z}^t MW\bar{z}.$$

The differential action

$$w_{ij} = w_{ji} \, , \chi_{ij} = \frac{1+\delta_{ij}}{2}, \quad \nabla_{ij} = \chi_{ij} \frac{\partial}{\partial w_{ij}}$$

Lemma

$$\begin{aligned}
\boldsymbol{a} &= \frac{\partial}{\sqrt{\mu}\partial z}; \; \boldsymbol{a}^{\dagger} = \sqrt{\mu} z + W \frac{\partial}{\sqrt{\mu}\partial z}; \; z \in \mathbb{C}^n; \; W \in \mathcal{D}_n \\
\mathbb{K}^{-} &= \nabla_W; \; \mathbb{K}^0 = \frac{k}{4} + \frac{1}{2} \frac{\partial}{\partial z} \otimes z + \nabla_W W; \\
\mathbb{K}^{+} &= \frac{W'}{4} + \frac{\mu}{2} z \otimes z + \frac{1}{2} (W \frac{\partial}{\partial z} \otimes z + z \otimes \frac{\partial}{\partial z} W) + W \nabla_W W.
\end{aligned}$$

$k = diag(k_1, \ldots, k_n)$, $w'_{kl} = (k_k + k_l) w_{kl}$, $k, l = 1, \ldots, n$.

Continuation

Lemma

The operators $\mathbf{a}^\dagger$, $\mathbf{a}$; $\mathbf{K}^+_{kl}$, $\mathbf{K}^-_{kl}$; $\mathbf{K}^0_{kl}$, $\mathbf{K}^0_{lk}$ are respectively hermitian conjugate w. r. the scalar product ($k_i = k$):

$$(\phi,\psi)=\Lambda_n\int_{z\in\mathbb{C}^n;W\in\mathcal{D}_n}\bar{f}_\phi(z,W)f_\psi(z,W)\rho_1\,\mathrm{d}\,z\,\mathrm{d}\,W, \tag{4.5}$$

$$\rho_1=\det(\mathbb{1}_n-W\bar{W})^p\exp-\mu F,\ p=k/2-n-2,\ f_\psi(z)=(e_{\bar{z}},\psi).$$

Proof: $\frac{\partial w_{ij}}{\partial w_{pq}}=\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{iq}-\delta_{ij}\delta_{pq}\delta_{ip},\ w_{ij}=w_{ji}.$

Comment: Berezin's quantization

$$\Lambda_n = \frac{k-3}{2\pi^{\frac{n(n+3)}{2}}} \prod_{i=1}^{n-1} \frac{(\frac{k-3}{2} - n + i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

Compare with the case of the symplectic group: *a shift of* p to $p - 1/2$ *in the normalization constant* $\Lambda_n = \pi^{-n} J^{-1}(p)$.

Composition law in G^J_n and action on $\mathcal{D}^J_n$

$$\mathrm{Sp(n,\mathbb{R})_{\mathbb{C}}} \ni \mathrm{g} = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \ \mathrm{pp^* - qq^*} = \mathbb{1}_{\mathrm{n}}, \quad \mathrm{pq^t = qp^t}; \tag{4.6}$$

$$(g_1,\alpha_1,t_1)\circ(g_2,\alpha_2,t_2) = \left(g_1\circ g_2, g_2^{-1}\cdot\alpha_1+\alpha_2, t_1+t_2+\Im(g_2^{-1}\cdot\alpha_1\bar{\alpha}_2)\right),$$

$g^{-1}\cdot\alpha = p^*\alpha - q^t\bar{\alpha}$.

Action: $G^J_n \ni (g,\alpha)\circ(z,W) \to (z_1,W_1) \in \mathcal{D}^J_n$:

$$W_1 = (pW+q)(\bar{q}W+\bar{p})^{-1} = (Wq^*+p^*)^{-1}(q^t+Wp^t), \tag{4.7a}$$

$$z_1 = (Wq^*+p^*)^{-1}(z+\alpha-W\bar{\alpha}), \tag{4.7b}$$

Kähler two-form on $\mathcal{D}_n^J$

The G_n^J-invariant Kähler two-form $\omega_n = \mathrm{i}\partial\bar{\partial} f$, deduced from the Kähler potential $f = \log K$, $K = K_{k\mu}$- Bergman kernel

$$-\mathrm{i}\omega_{k\mu} = \frac{k}{2}\mathrm{Tr}(B \wedge \bar{B}) + \mu \mathrm{Tr}(A^t \bar{M} \wedge \bar{A}), A = \mathrm{d}\, z + \mathrm{d}\, W \bar{\eta},$$
$$B = M \mathrm{d}\, W, M = (\mathbb{1}_n - W\bar{W})^{-1}, \ \eta = M(z + W\bar{z}). \tag{4.8}$$

$G_n^J(\mathbb{R}) = \mathbf{Sp}(n,\mathbb{R}) \ltimes H_n$

Let $g = (M, X, k), g' = (M', X', k') \in G_n^J(\mathbb{R})$, $X = (\lambda, \mu) \in \mathbb{R}^{2n}$, $(X, k) \in H_n$. The composition law

$$gg' = (MM', XM' + X', k + k' + XM'JX'^t).$$

The restricted real Jacobi group $G_n^J(\mathbb{R})_0$- elements of the form above, but $g = (M, X)$.

The Siegel-Jacobi upper half-plane $\mathcal{X}_n^J := \mathcal{X}_n \times \mathbb{R}^{2n}$,
$\mathcal{X}_n = \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$ - the Siegel upper half-plane realized as

$$\mathcal{X}_n := \{v \in M(n,\mathbb{C}) | v = s + \mathrm{i}r, s, r \in M(n,\mathbb{R}), r > 0, s^t = s; r^t = r\}.$$

Partial Cayley transform

Let us consider an element $h = (g, l)$ in $G_n^J(\mathbb{R})_0$, i.e.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}),\ l = (n, m) \in \mathbb{R}^{2n}, \tag{4.9}$$

$v \in \mathcal{X}_n,\ u \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$.
Partial Cayley transform $\Phi : \mathcal{X}_n^J \to \mathcal{D}_n^J,\ \Phi(v, u) = (W, z)$

$$W = (v - \mathrm{i}\mathbb{1}_n)(v + \mathrm{i}\mathbb{1}_n)^{-1}, \tag{4.10a}$$

$$z = 2\mathrm{i}(v + \mathrm{i}\mathbb{1}_n)^{-1}u, \tag{4.10b}$$

Inverse partial Cayley transform $\Phi^{-1} : \mathcal{D}_n^J \to \mathcal{X}_n^J$, $\Phi^{-1}(W, z) = (v, u)$

$$v = \mathrm{i}(\mathbb{1}_n - W)^{-1}(\mathbb{1}_n + W), \tag{4.11a}$$

$$u = (\mathbb{1}_n - W)^{-1}z. \tag{4.11b}$$

Θ- isomorphism

$\Theta : G_n^J(\mathbb{R})_0 \to G_n^J$, $\Theta(h) = h_*$, $h = (g, n, m)$, $h_* = (g_{\mathbb{C}}, \alpha)$.

Proposition

Θ is an group isomorphism and the action of G_n^J on $\mathcal{D}_n^J$ is compatible with the action of $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$ through Φ, i.e. if $\Theta(h) = h_*$, then $\Phi h = h_*\Phi$. More exactly, if the action of G_n^J on $\mathcal{D}_n^J$ is given by (4.7), then the action of $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$ is given by
$(g, l) \times (v, u) \to (v_1, u_1) \in \mathcal{X}_n^J$,

$$v_1 = (av + b)(cv + d)^{-1} = (vc^t + d^t)^{-1}(va^t + b^t); \tag{4.12a}$$

$$u_1 = (vc^t + d^t)^{-1}(u + vn + m). \tag{4.12b}$$

The matrices g in (4.9) and $g_{\mathbb{C}}$ in (4.6) are related by (4.3), (4.4), while $\alpha = m + \mathrm{i}n$, $m, n \in \mathbb{R}^n$.

The Kähler two-form

Proposition

The partial Cayley transform is a Kähler homogeneous diffeomorfism, $\Phi^*\omega_{k\mu} = \omega'_{k\mu} = \omega_{k\mu} \circ \Phi$, *i.e. the Kähler two-form* (4.8) *on* $\mathcal{D}_n^J$, G_n^J*-invariant under the action* (4.7), *becomes the Kähler two-form* $\omega'_{k\mu}$ (4.13) *on* $\mathcal{X}_n^J$, $G_n^J(\mathbb{R})_0$*-invariant*

$$-\mathrm{i}\omega'_{k\mu} = \frac{k}{2}\mathrm{Tr}(H \wedge \bar{H}) + \mu\frac{2}{\mathrm{i}}\mathrm{Tr}(G^t D \wedge \bar{G}), \qquad (4.13)$$
$$D = (\bar{v} - v)^{-1}, H = D\,\mathrm{d}\,v;\ G = \mathrm{d}\,u - \mathrm{d}\,vD(\bar{u} - u).$$

"n"-dimensional generalization of Berndt-Kähler two-form ω'_1.

Remark : ω_n and ω'_n also by Yang. $\mathcal{D}_n^J$ and $\mathcal{X}_n^J$ are called by Jae-Hyun Yang *Siegel-Jacobi spaces*. Kähler calls $\mathcal{X}_1^J$ *Phasenraum der Materie*, v is *Pneuama*, u is *Soma*.

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Proposition

Proposition

Under the homogeneous Kähler transform FC

$$\mathbb{C}^n \times \mathcal{D}_n \ni (\eta, W) \xrightarrow{FC} (z, W) \in \mathcal{D}_n^J, \; z = \eta - W\bar{\eta}, \tag{4.14}$$

$$FC^{-1} : \eta = (\mathbb{1}_n - W\bar{W})^{-1}(z + W\bar{z}). \tag{4.15}$$

the G_n^J-invariant Kähler two-form on $\mathcal{D}_n^J$, becomes the Kähler two-form on $\mathcal{D}_n \times \mathbb{C}^n$, $FC^*\omega_n = \omega_{n,0}$,

$$-\mathrm{i}\omega_{n,0} = \frac{k}{2}\mathrm{Tr}(B \wedge \bar{B}) + \mu\mathrm{Tr}(\mathrm{d}\,\eta^t \wedge \mathrm{d}\,\bar{\eta}), \tag{4.16}$$

invariant to the G_n^J-action on $\mathcal{D}_n \times \mathbb{C}^n$, $(g, \alpha) \cdot (\eta, W) \to (\eta_1, W_1)$, with W_1 given in (4.7) and

Main result- continuation

$$\eta_1 = p(\eta+\alpha) + q(\bar{\eta}+\bar{\alpha}). \qquad (4.17)$$

Under the homogenous Kähler transform FC_1^{-1}:

$$\eta = (\bar{v} - \mathrm{i}\mathbb{1}_n)(\bar{v}-v)^{-1}(v-\mathrm{i}\mathbb{1}_n)[(v-\mathrm{i}\mathbb{1}_n)^{-1}u - (\bar{v}-\mathrm{i}\mathbb{1}_n)^{-1}\bar{u}],$$

$$FC_1 : u = \frac{1}{2\mathrm{i}}[(v+\mathrm{i}\mathbb{1}_n)\eta - (v-\mathrm{i}\mathbb{1}_n)\bar{\eta}],$$

the Kähler two-form (4.13) becomes a Kähler two-form on $\mathcal{X}_n \times \mathbb{C}^n$, $FC_1^*\omega'_n = \omega'_{n,0}$,

$$-\mathrm{i}\omega'_{n,0} = \tfrac{k}{2}\mathrm{Tr}(H\wedge\bar{H}) + \mu\mathrm{Tr}(\mathrm{d}\,\eta^t\wedge\mathrm{d}\,\bar{\eta}),\ H = (\bar{v}-v)^{-1}\mathrm{d}\,v. \qquad (4.18)$$

ω'_n is $G_n^J(\mathbb{R})_0$-invariant $(g,\alpha)\circ(v,\eta)\to(v_1,\eta_1)\in\mathcal{X}_n\times\mathbb{C}^n$, g given by (4.9)*, v_1 given by* (4.12a)*, while*

$$\eta_1 = [a+d+\mathrm{i}(b-c)](\eta+\alpha)/2 + [a-d-\mathrm{i}(b+c)](\bar{\eta}+\bar{\alpha})/2. \qquad (4.19)$$

General considerations

We consider an algebraic Hamiltonian linear in the generators of the group of symmetry G

$$\boldsymbol{H} = \sum_{\lambda \in \Delta} \epsilon_\lambda \boldsymbol{X}_\lambda. \tag{5.1}$$

Passing on from the dynamical system problem in the Hilbert space $\mathfrak{H}$ to the corresponding one on $M = G/H$ is called sometimes *dequantization*, and the system on M is a classical one. Following Berezin, the motion on the classical phase space can be described by the local equations of motion. The *classical & quantum equations of motion on $M = G/H$ are*

$$i\dot{z}_\alpha = \sum_{\lambda} \epsilon_\lambda Q_{\lambda,\alpha}, \tag{5.2}$$

$$\mathbb{X}_\lambda = P_\lambda + \sum_{\beta} Q_{\lambda,\beta}\partial_\beta.$$

The Hamiltonian $\boldsymbol{H}$

$$\boldsymbol{H} = \epsilon_i \boldsymbol{a}_i + \bar{\epsilon}_i \boldsymbol{a}_i^{\dagger} + \epsilon^0_{ij}\boldsymbol{K}^0_{ij} + \epsilon^-_{ij}\boldsymbol{K}^-_{ij} + \epsilon^+_{ij}\boldsymbol{K}^+_{ij}. \tag{5.3}$$

$$\epsilon_0^{\dagger} = \epsilon_0;\ \epsilon_- = \epsilon^t_-;\ \epsilon_+ = \epsilon^t_+;\ \epsilon^{\dagger}_+ = \epsilon_-. \tag{5.4}$$

$$\epsilon_- = m + \mathrm{i}n,\ \epsilon^t_0/2 = p + \mathrm{i}q; p, m, n \in \mathrm{Sym}(n, \mathbb{R}); q^t = -q. \tag{5.5}$$

$$\dot{W} = AW + WD + B + WCW,\ A, B, C, D \in M(n, \mathbb{C}); \tag{5.6a}$$

$$\dot{z} = M + Nz;\ M = E + WF;\ N = A + WC,\ E, F \in C^n. \tag{5.6b}$$

Equations on motion

Proposition

a) on $\mathcal{D}^J_n$, $(z, W) \in \mathbb{C}^n \times \mathcal{D}_n$ *verifies* (5.6), *with coefficients*

$$A_c = -\frac{\mathrm{i}}{2}\epsilon^t_0,\ B_c = -\mathrm{i}\epsilon_-,\ C_c = -\mathrm{i}\epsilon_+,\ D_c = A^t_c; \tag{5.7a}$$

$$E_c = -\mathrm{i}\epsilon,\ F_c = -\mathrm{i}\bar{\epsilon}. \tag{5.7b}$$

b) on $\mathcal{X}^J_n$, $(u, v) \in \mathbb{C}^n \times \mathcal{X}_n$, *verifies* (5.6), *with coefficients*

$$A_r = n + q,\ B_r = m - p,\ C_r = -(m + p),\ D_r = n - q; \tag{5.8a}$$

$$E_r = \Im\epsilon;\ F_r = -\Re\epsilon. \tag{5.8b}$$

Decoupling of motions under *FC*

c) under the FC transform, the equations in $\eta \in \mathbb{C}^n$, $W \in \mathcal{D}_n$ *become independent: W verifies* (5.6a) *with coefficients* (5.7a) *and* η *verifies*

$$\mathrm{i}\dot{\eta} = \epsilon + \epsilon_- \bar{\eta} + \frac{1}{2}\epsilon_0^t \eta, \; \eta \in \mathbb{C}^n. \tag{5.9}$$

d) under the FC_1 *transform, the equations in* $\eta \in \mathbb{C}^n$, $v \in \mathcal{X}_n$ *become independent:* η *verifies* (5.9)*, while v verifies* (5.6a) *with coefficients* (5.8a)*.*

a. Solving m. Riccati equation by linearization

$W = XY^{-1}$, $X, Y \in M(n, \mathbb{C}) \Rightarrow$ *a linear system*

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h \begin{pmatrix} X \\ Y \end{pmatrix}, \; h = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}. \tag{5.10}$$

Every solution of (5.10) *is a solution of* (5.6a), $\det(Y) \neq 0$.

$$h_c = \begin{pmatrix} -\mathrm{i}(\frac{\epsilon_0}{2})^t & -\mathrm{i}\epsilon_- \\ \mathrm{i}\epsilon_+ & \mathrm{i}\frac{\epsilon_0}{2} \end{pmatrix}, \; h_r = \begin{pmatrix} A_r & B_r \\ -C_r & -D_r \end{pmatrix}.$$

$W(v) = X/Y \in \mathcal{D}_n(\mathcal{X}_n)$; $h_c \in \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$; $h_r \in \mathfrak{sp}(n, \mathbb{R})$, $h_c = (h_r)_{\mathbb{C}}$;

Remarks

Remark

The linear system (5.10) *associated to the matrix Riccati eq. describes the time-dependent vector field induced by the infinitesimal action of the group* $Sp(n,\mathbb{R})_{\mathbb{C}}$ *(*$Sp(n,\mathbb{R})$*) - a linear Hamiltonian system on* $\mathcal{D}_n$ *(*$\mathcal{X}_n$*).*

The infinitesimal group action of $\mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}$ *(*$\mathfrak{sp}(n,\mathbb{R})$*) is given by the Lie algebras homomorphism*

$$\nu_c : \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}} \rightarrow \mathrm{Ham}(\mathcal{D}_n), \quad \nu_r : \mathfrak{sp}(n,\mathbb{R}) \rightarrow \mathrm{Ham}(\mathcal{X}_n), \tag{5.11}$$

$$\nu \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = -(B + AZ + ZD + ZCZ)_{im} \frac{\partial}{\partial w_{im}}. \tag{5.12}$$

$$U(t,t_0) = \begin{pmatrix} U_1(t,t_0) & U_2(t,t_0) \\ U_3(t,t_0) & U_4(t,t_0) \end{pmatrix} \tag{5.13}$$

Continuation

- fundamental matrix of the ordinary differential equation (5.10), $\dot{U} = hU$, $U(t_0, t_0) = 1$.
The fundamental solution $U_c(t, t_0)$ ($U_r(t, t_0)$) is a $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively $\mathrm{Sp}(n, \mathbb{R})$)-matrix and

$$W(t, t_0) = [U_1(t, t_0)W(t_0) + U_2(t, t_0)][U_3(t, t_0)W(t_0) + U_4(t, t_0)]^{-1}$$

- solution of (5.6a), $W(t_0, t_0) = W(t_0)$.

• $\dot{z} = Az$, $A \in \mathfrak{sp}(n, \mathbb{R})$- Hamiltonian linear system- in N-body systems; Eigenvalues- Laub & Mayer, *Celestial Mechanics*;

• Yakulovich & Starzhinskii: Floquet-Lyapunov Thm, Krein Gel'fand Thm in the case of periodic coefficients. The linear autonomous system is stable iff A has only pure imaginary eigenvalues and is diagonalizable. Parametrically stable ....

Continuation

- fundamental matrix of the ordinary differential equation (5.10), $\dot{U} = hU$, $U(t_0, t_0) = 1$.
The fundamental solution $U_c(t, t_0)$ ($U_r(t, t_0)$) is a $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively $\mathrm{Sp}(n, \mathbb{R})$)-matrix and

$$W(t, t_0) = [U_1(t, t_0)W(t_0) + U_2(t, t_0)][U_3(t, t_0)W(t_0) + U_4(t, t_0)]^{-1}$$

- solution of (5.6a), $W(t_0, t_0) = W(t_0)$.

• $\dot{z} = Az, A \in \mathfrak{sp}(n, \mathbb{R})$- Hamiltonian linear system- in *N*-body systems; Eigenvalues- Laub & Mayer, *Celestial Mechanics*;

• Yakulovich & Starzhinskii: Floquet-Lyapunov Thm, Krein Gel'fand Thm in the case of periodic coefficients. The linear autonomous system is stable iff A has only pure imaginary eigenvalues and is diagonalizable. Parametrically stable ....

Continuation

- fundamental matrix of the ordinary differential equation (5.10), $\dot{U} = hU$, $U(t_0, t_0) = 1$.
The fundamental solution $U_c(t, t_0)$ ($U_r(t, t_0)$) is a $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively $\mathrm{Sp}(n, \mathbb{R})$)-matrix and

$$W(t, t_0) = [U_1(t, t_0)W(t_0) + U_2(t, t_0)][U_3(t, t_0)W(t_0) + U_4(t, t_0)]^{-1}$$

- solution of (5.6a), $W(t_0, t_0) = W(t_0)$.
• $\dot{z} = Az, A \in \mathfrak{sp}(n, \mathbb{R})$- Hamiltonian linear system- in *N*-body systems; Eigenvalues- Laub & Mayer, *Celestial Mechanics*;
• Yakulovich & Starzhinskii: Floquet-Lyapunov Thm, Krein Gel'fand Thm in the case of periodic coefficients. The linear autonomous system is stable iff *A* has only pure imaginary eigenvalues and is diagonalizable. Parametrically stable ....

b. The decoupled system

Introduce in (5.9) $\eta = \xi - \mathrm{i}\zeta, \xi, \zeta \in \mathbb{R}^n$, $\epsilon = b + \mathrm{i}a$, where $a, b \in \mathbb{R}^n$. The first order complex differential equation equation (5.9) is equivalent with system of first order real differential equations with real coefficients

$$\dot{Z} = h_r Z + F,\ Z = \begin{pmatrix} \xi \\ \zeta \end{pmatrix},\ F = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{5.14}$$

Berry phase on Siegel-Jacobi ball in (W, η)

$$\varphi_B = \frac{\mathrm{i}}{2}\int_0^t \sum_{\alpha\in\Delta_+} (\dot{z}_\alpha\partial_\alpha - \dot{\bar{z}}_\alpha\bar{\partial}_\alpha) \ln < e_{\bar{z}}, e_{\bar{z}} > .$$

$$\begin{aligned}\frac{2}{\mathrm{i}}\,\mathrm{d}\,\varphi_B = &\{\frac{k}{2}[2\mathrm{Tr}(X\,\mathrm{d}\,W) - \mathrm{Tr}(\mathrm{diag}(X)\mathrm{diag}(\mathrm{d}\,W))]\\ &-\frac{1}{2}\bar{\eta}^t\mathrm{diag}(\mathrm{d}\,W)\bar{\eta} - cc\} + [\mathrm{d}\,\bar{\eta}^t(\bar{\eta} + \bar{W}\eta) - cc];\\ X = &\bar{W}(\mathbb{1}_n - W\bar{W})^{-1}.\end{aligned}$$

Dynamical phase

$$\mathcal{H} = \mathcal{H}_\eta + \mathcal{H}_w,$$
$$\mathcal{H}_\eta = \epsilon^t\eta + \bar{\epsilon}^t\bar{\eta} + \frac{1}{2}(\eta^t\epsilon_-\eta + \bar{\eta}^t\epsilon_+\bar{\eta} + \bar{\eta}^t\epsilon_0\eta),$$
$$\mathcal{H}_w = \frac{k}{2}\mathrm{Tr}\{(\epsilon_0)^s + [W\epsilon_- + \epsilon_+\bar{W} + (\epsilon_0 W)^s\bar{W}](1\!\!1_n - W\bar{W})^{-1}\}.$$

$$\nabla\mathcal{H}_w = 2(1\!\!1_n - \bar{W}W)^{-1}\bar{\Lambda}(1\!\!1_n - W\bar{W})^{-1}, \tag{5.16a}$$
$$\frac{\partial\mathcal{H}_\eta}{\partial\eta} = \epsilon + \epsilon_-\eta + \frac{1}{2}\epsilon_0^t\bar{\eta};\ \Lambda = \epsilon_+ + (\epsilon_0 W)^s + W\epsilon_- W \tag{5.16b}$$

Critical points of $\mathcal{H}$: W_c: $\Lambda = 0$ ($\dot{W} = 0$); η_c: $\dot{\xi} = 0; \dot{\zeta} = 0$

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