Open-closed B-type Landau-Ginzburg models with non-compact Kählerian target space

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Outline

We study general open-closed B-type Landau-Ginzburg models (including their coupling to topological D-branes), without making unnecessary assumptions.

Classical oriented open-closed topological Landau-Ginzburg theories of type B are classical field theories, defined on compact oriented Riemann surfaces Σ with corners and parameterized by pairs (X, W), where X is a non-compact Kählerian manifold and $W : X \to \mathbb{C}$ is a non-constant holomorphic function defined on X and called the superpotential. It is expected that such theories admit a non-anomalous quantization when X is a Calabi-Yau manifold. A physically acceptable quantization procedure must produce a quantum oriented open-closed topological field theory which can be described equivalently by an algebraic structure called a *TFT datum*.

Limitations of previous work

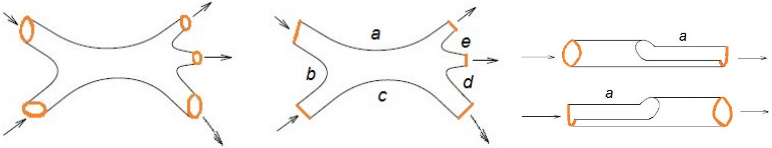
All previous work assumed algebraicity of X and W and most of it was limited to very simple examples such as $X = \mathbb{C}^d$. It was also assumed that the critical points of the superpotential W are isolated.

We do not require that X is algebraic, since there is no Physics reason to do so. Moreover, we require only that the critical locus of W is compact.

Fact [Lazaroiu (2001)] A non-anomalous oriented 2-dimensional open-closed topological field theory (TFT) can be described axiomatically as a monoidal functor from a certain category Cob_2 of oriented open-closed cobordisms with corners to the category of finite-dimensional vector spaces over $\mathbb{C}$.

$$Z : (\mathrm{Cob}_2, \sqcup, \emptyset) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$$

The objects of Cob_2 are disjoint unions of compact oriented smooth 1-manifolds with and without boundary, i.e. disjoint unions of oriented circles and oriented closed intervals. The morphisms are certain compact oriented smooth 2-manifolds with corners (corresponding to the worldsheets of open and closed strings). The corners coincide with the boundary points of the intervals. The labels associated to the ends of the open strings indicate the D-branes which determine the corresponding boundary conditions.



Theorem (Lazaroiu (2001))

A (non-anomalous) oriented 2-dimensional open-closed TFT can be described equivalently by an algebraic structure known as a ***TFT datum****.*

Definition

A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

- $\mathcal{H}$ = **bulk algebra**, a finite-dimensional supercommutative $\mathbb{C}$-superalgebra with unit $1_{\mathcal{H}}$ (the space of on-shell states of the *closed* oriented topological string)
- $\mathcal{T}$ = **category of topological D-branes**, a Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category, with composition of morphisms denoted by $\circ$ and units:

  $$1_a \in \mathrm{Hom}_{\mathcal{T}}(a, a) \ , \ \forall a \in \mathrm{Ob}\mathcal{T}$$

  Here $\mathrm{Hom}_{\mathcal{T}}(a, b)$ is the space of on-shell states of the *open* oriented topological string stretching from the D-brane a to the D-brane b
- $e = (e_a)_{a\in \mathrm{Ob}\mathcal{T}}$, a family of even $\mathbb{C}$-linear **bulk-boundary maps**, defined for each object a of $\mathcal{T}$:

  $$e_a : \mathcal{H} \to \mathrm{Hom}_{\mathcal{T}}(a, a)$$

  such that the following conditions are satisfied:
  - For any $a \in \mathrm{Ob}\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the algebra $(\mathrm{End}_{\mathcal{T}}(a), \circ)$, where $\mathrm{End}_{\mathcal{T}}(a) \stackrel{\text{def.}}{=} \mathrm{Hom}_{\mathcal{T}}(a, a)$.
  - For any $a, b \in \mathrm{Ob}\mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous bulk state $h \in \mathcal{H}$ and any $\mathbb{Z}_2$-homogeneous elements $t \in \mathrm{Hom}_{\mathcal{T}}(a, b)$, we have:

    $$e_b(h) \circ t = (-1)^{\deg h \deg t}\, t \circ e_a(h).$$

Definition

A Calabi-Yau supercategory of **parity** $\mu \in \mathbb{Z}_2$ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in Ob\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-degree μ

$$\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$$

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$, the $\mathbb{C}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a, b) \times \mathrm{Hom}_{\mathcal{T}}(b, a) \to \mathbb{C}$$

defined through:

$$\langle t_1, t_2 \rangle_{a,b} = \mathrm{tr}_b(t_1 \circ t_2), \ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b), \ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$$

is non-degenerate.

- For any two objects $a, b \in \mathrm{Ob}\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b, a)$, we have:

$$\langle t_1, t_2 \rangle_{a,b} = (-1)^{\deg t_1, \deg t_2} \langle t_2, t_1 \rangle_{b,a}$$

Definition

A TFT datum of **parity** $\mu \in \mathbb{Z}_2$ is a system $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a **pre-TFT datum**
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the **bulk trace** and representing the one-point function on the sphere)
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{Hom}_{\mathcal{T}}(a, a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called **boundary traces** and representing the one-point function on the disk with boundary condition a)

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra, meaning that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate (i.e. the condition $\mathrm{Tr}(hh') = 0$ for all $h' \in \mathcal{H}$ implies $h = 0$)
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi-Yau supercategory of parity μ.
- The *topological Cardy constraint* holds for all $a, b \in \mathrm{Ob}\mathcal{T}$.

The topological Cardy constraint has the form:

$$\boxed{\mathrm{Tr}(f_a(t_1)f_b(t_2)) = \mathrm{str}(\Phi_{ab}(t_1,t_2)) \ \ , \ \ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b,b)}$$

where:

- "str" is the supertrace on the $\mathbb{Z}_2$-graded vector space $End_{\mathbb{C}}(Hom_{\mathcal{T}}(a,b))$
- $f_a : Hom_{\mathcal{T}}(a,a) \rightarrow \mathcal{H}$ is the *boundary-bulk map of a*, which has $\mathbb{Z}_2$-degree μ and is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \rightarrow Hom_{\mathcal{T}}(a,a)$ with respect to Tr and tr:

$$\mathrm{Tr}(hf_a(t)) = \mathrm{tr}_a(e_a(h)\circ t),\ \forall h \in \mathcal{H},\ \forall t \in \mathrm{Hom}_{\mathcal{T}}(a,a)$$

- $\Phi_{ab}(t_1,t_2) : \mathrm{Hom}_{\mathcal{T}}(a,b) \rightarrow \mathrm{Hom}_{\mathcal{T}}(a,b)$ is the $\mathbb{C}$-linear map defined through:

$$\Phi_{ab}(t_1,t_2)(t) = t_2 \circ t \circ t_1 \ ,$$

$$\forall t \in \mathrm{Hom}_{\mathcal{T}}(a,b) \ , \ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a,a) \ , \ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b,b)$$

Definition

A Landau-Ginzburg (LG) pair of dimension d is a pair (X, W), where:

1. X is a non-compact Kählerian manifold of complex dimension d which is *Calabi-Yau* in the sense that the canonical line bundle $K_X = \wedge^d T^* X$ is holomorphically trivial.
2. $W : X \to \mathbb{C}$ is a *non-constant* complex-valued holomorphic function defined on X.

The **signature** $\mu(X, W)$ of a Landau-Ginzburg pair (X, W) is defined as the mod 2 reduction of the complex dimension of X:

$$\mu(X, W) = \hat{d} \in \mathbb{Z}_2$$

Definition

The *critical set of* W is the set:

$$Z_W = \{p \in X | (\partial W)(p) = 0\}$$

of critical points of W.

Definition

Let (X, W) be a Landau-Ginzburg pair with $\dim_{\mathbb{C}} X = d$. The space of polyvector - valued forms is defined through:

$$\mathrm{PV}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathrm{PV}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathcal{A}^{j}(X, \wedge^{|i|} TX)$$

where $\mathcal{A}^{j}(X, \wedge^{|i|} TX) \equiv \Omega^{0,j}(X, \wedge^{|i|} TX)$.

We denote by TX and $\bar{T}X$ are the holomorphic and antiholomorphic tangent bundles of X and by T^*X and $\bar{T}^*X$ the corresponding cotangent bundles. Let $z = (z_1, \ldots, z_d)$ be local holomorphic coordinates defined on $U \subset X$ and $\partial_k := \frac{\partial}{\partial z_k}$, $\bar{\partial}_k := \frac{\partial}{\partial \bar{z}_k}$, then:

$$TX|_U = \mathrm{Span}_{\mathbb{C}}\left\{\partial_1, \ldots, \partial_d\right\} \quad , \quad \overline{T}X|_U = \mathrm{Span}_{\mathbb{C}}\left\{\bar{\partial}_1, \ldots, \bar{\partial}_d\right\} \quad ,$$
$$T^*X|_U = \mathrm{Span}_{\mathbb{C}}\{\mathrm{d}z_1, \ldots, \mathrm{d}z_d\} \quad , \quad \overline{T}^*X|_U = \mathrm{Span}_{\mathbb{C}}\{\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_d\} \quad .$$

A polyvector-valued form $\omega \in \mathrm{PV}^{i,j}(X)$ expands as:

$$\omega =_U \sum_{|I|=-i, |J|=j} \omega^I{}_J \mathrm{d}\bar{z}_J \otimes \partial_I \quad , \quad \omega^I{}_J \in C^\infty(X)$$

$$\mathrm{d}\bar{z}_J \overset{\text{def.}}{=} \mathrm{d}\bar{z}_{t_1} \wedge \mathrm{d}\bar{z}_{t_2} \wedge \cdots \wedge \mathrm{d}\bar{z}_{t_j} \quad , \quad \partial_I \overset{\text{def.}}{=} \partial_{t_1} \wedge \ldots \wedge \partial_{t_{|i|}}$$

$(\mathrm{PV}(X), \delta_W)$

The **twisted Dolbeault differential** determined by W on $\mathrm{PV}(X)$:

$$\delta_W : \mathrm{PV}(X) \to \mathrm{PV}(X)$$

is defined through $\boxed{\delta_W = \boldsymbol{\bar{\partial}} + \boldsymbol{\iota}_W}$ where:

- $\boldsymbol{\bar{\partial}}$ is the antiholomorphic Dolbeault operator of $\wedge TX$, which satisfies $\boldsymbol{\bar{\partial}}(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i,j+1}(X)$

$$\bar{\partial}\omega =_U \sum_{|I|=-i,|J|=j} [(\bar{\partial}\omega^I_J) \wedge \mathrm{d}\bar{z}_J] \otimes \partial_I = \sum_{|I|=-i,|J|=j} \sum_{r=1}^{d} (\bar{\partial}_r \omega^I_J)(\mathrm{d}\bar{z}_r \wedge \mathrm{d}\bar{z}_J) \otimes \partial_I$$

- $\boldsymbol{\iota}_W \stackrel{\mathrm{def.}}{=} -\mathrm{i}(\partial W)\lrcorner$, which satisfies $\boldsymbol{\iota}_W(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i+1,j}(X)$

$$\boldsymbol{\iota}_W \omega = -\mathrm{i}\, \iota_{\partial W} \omega =_U -\mathrm{i} \sum_{r=1}^{d} (\partial_r W) \mathrm{d}z^r \lrcorner \omega$$

Notice that $(\mathrm{PV}(X), \boldsymbol{\bar{\partial}}, \boldsymbol{\iota}_W)$ is a bicomplex since:

$$\boldsymbol{\bar{\partial}}^2 = \boldsymbol{\iota}_W^2 = \boldsymbol{\bar{\partial}}\boldsymbol{\iota}_W + \boldsymbol{\iota}_W \boldsymbol{\bar{\partial}} = 0$$

Definition

The twisted Dolbeault algebra of polyvector-valued forms of the LG pair (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear dg-algebra $(\mathrm{PV}(X), \delta_W)$, where $\mathrm{PV}(X)$ is endowed with the canonical $\mathbb{Z}$-grading.

Definition

The cohomological twisted Dolbeault algebra of (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear algebra defined through:

$$\mathrm{HPV}(X, W) = \mathrm{H}(\mathrm{PV}(X), \delta_W)$$

We use the following notations:
$\mathrm{O}(X)=$ the ring of complex-valued holomorphic functions defined on X,
$O_X=$ the sheaf of holomorphic complex-valued functions defined on X.

In our terminology "off-shell" refers to an object defined at cochain level while "on-shell" refers to an object defined at cohomology level.

An analytic model for the off-shell bulk algebra

Definition

The *sheaf Koszul complex* of W is the following complex of locally-free sheaves of $\mathcal{O}_X$-modules:

$$(\mathcal{Q}_W): \quad 0 \to \wedge^d TX \stackrel{\iota_W}{\to} \wedge^{d-1} TX \stackrel{\iota_W}{\to} \cdots \stackrel{\iota_W}{\to} \mathcal{O}_X \to 0$$

where $\mathcal{O}_X$ sits in degree zero and we identify the exterior power $\wedge^k TX$ with its locally-free sheaf of holomorphic sections.

Proposition

Let $\mathbb{H}(\mathcal{Q}_W)$ denote the hypercohomology of the Koszul complex $\mathcal{Q}_W$. There exists a natural isomorphism of $\mathbb{Z}$-graded $\mathrm{O}(X)$-modules:

$$\mathrm{HPV}(X, W) \cong_{\mathrm{O}(X)} \mathbb{H}(Q_W)$$

where $\mathrm{HPV}(X, W)$ is endowed with the canonical $\mathbb{Z}$-grading. Thus:

$$\mathrm{H}^k(\mathrm{PV}(X), \delta_W) \cong_{\mathrm{O}(X)} \mathbb{H}^k(\mathcal{Q}_W), \ \forall k \in \{-d, \ldots, d\}$$

Moreover, we have:

$$\mathbb{H}^k(\mathcal{Q}_W) = \bigoplus_{i+j=k} \mathbf{E}_\infty^{i,j}$$

where $(\mathbf{E}_r^{i,j}, d_r)_{r \geq 0}$ is a spectral sequence which starts with:

$$\mathbf{E}_0^{i,j} := \mathrm{PV}^{i,j}(X) = \mathcal{A}^j(X, \wedge^{|i|} TX), \ \ \mathrm{d}_0 = \overline{\boldsymbol{\partial}} \ \ , \ (i = -d, \ldots, 0, \ j = 0, \ldots, d)$$

$$
\begin{array}{ccccccc}
\mathbf{E}_0^{-d,d} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,d} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,d} & \cdots\cdots & \mathbf{E}_0^{0,d} \\
\vdots & & \vdots & & \vdots & \cdots & \vdots \\
\mathbf{E}_0^{-d,2} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,2} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,2} & \cdots\cdots & \mathbf{E}_0^{0,2} \\
\uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
\mathbf{E}_0^{-d,1} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,1} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,1} & \cdots\cdots & \mathbf{E}_0^{0,1} \\
\uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
\mathbf{E}_0^{-d,0} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+1,0} & \xrightarrow{\iota_W} & \mathbf{E}_0^{-d+2,0} & \cdots\cdots & \mathbf{E}_0^{0,0}
\end{array}
$$

The zeroth page of the spectral sequence.

Definition

A holomorphic vector superbundle on X is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on X, i.e. a complex holomorphic vector bundle E endowed with a direct sum decomposition $E = E^{\hat{0}} \oplus E^{\hat{1}}$, where $E^{\hat{0}}$ and $E^{\hat{1}}$ are holomorphic sub-bundles of E.

Definition

A holomorphic factorization of W is a pair $a = (E, D)$, where $E = E^{\hat{0}} \oplus E^{\hat{1}}$ is a holomorphic vector superbundle on X and $D \in \Gamma(X, End^{\hat{1}}(E))$ is a holomorphic section of the bundle $End^{\hat{1}}(E) = Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \subset End(E)$ which satisfies the condition $D^2 = W\mathrm{id}_E$.

Let $a = (E, D)$ be a holomorphic factorization of W. Decomposing $E = E^{\hat{0}} \oplus E^{\hat{1}}$, the condition that D is odd implies:

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$$

where $u \in \Gamma(X, Hom(E^{\hat{0}}, E^{\hat{1}}))$ and $v \in \Gamma(X, Hom(E^{\hat{1}}, E^{\hat{0}}))$. The condition $D^2 = W\mathrm{id}_E$ amounts to:

$$v \circ u = W\mathrm{id}_{E^{\hat{0}}} \ , \quad u \circ v = W\mathrm{id}_{E^{\hat{1}}}$$

The category of topological D-branes

Definition

The twisted Dolbeault category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category $\mathrm{DF}(X, W)$ defined as follows:

- The objects of $\mathrm{DF}(X, W)$ are the holomorphic factorizations of W.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$:

  $$\mathrm{Hom}_{\mathrm{DF}(X,W)}(a_1, a_2) \stackrel{\mathrm{def.}}{=} \mathcal{A}(X, Hom(E_1, E_2)) = \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, Hom(E_1, E_2))$$

  endowed with the total $\mathbb{Z}_2$-grading and with the twisted differentials δ_{a_1,a_2}:

  $$\delta_{a_1,a_2} \stackrel{\mathrm{def.}}{=} \overline{\boldsymbol{\partial}}_{a_1,a_2} + \mathfrak{d}_{a_1,a_2} \ , \quad \textit{where} \quad \overline{\boldsymbol{\partial}}_{a_1,a_2} := \overline{\boldsymbol{\partial}}_{Hom(E_1,E_2)} \ ,$$

  $$\mathfrak{d}_{a_1,a_2}(\rho \otimes f) = (-1)^{\mathrm{rk}\rho} \rho \otimes (D_2 \circ f) - (-1)^{\mathrm{rk}\rho + \sigma(f)} \rho \otimes (f \circ D_1)$$

- The composition of morphisms $\circ : \mathcal{A}(X, Hom(E_2, E_3)) \times \mathcal{A}(X, Hom(E_1, E_2)) \to \mathcal{A}(X, Hom(E_1, E_3))$ is determined uniquely through the condition:

  $$(\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f)\,\mathrm{rk}\eta} (\rho \wedge \eta) \otimes (f \circ g)$$

  for all pure rank forms $\rho, \eta \in \mathcal{A}(X)$ and all pure $\mathbb{Z}_2$-degree elements $f \in \Gamma_\infty(X, Hom(E_2, E_3))$ and $g \in \Gamma_\infty(X, Hom(E_1, E_2))$.

$$\delta^2 = \overline{\boldsymbol{\partial}}^2 = \mathfrak{d}^2 = \overline{\boldsymbol{\partial}} \circ \mathfrak{d} + \mathfrak{d} \circ \overline{\boldsymbol{\partial}} = 0$$

The full TFT datum

Definition

The cohomological twisted Dolbeault category of holomorphic factorizations of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear algebra defined through:

$$\mathrm{HDF}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{DF}(X))$$

Theorem

Suppose that the critical set Z_W is compact. Then the cohomology algebra $\mathrm{HPV}(X, W)$ of $(\mathrm{PV}(X), \delta_W)$ is finite-dimensional over $\mathbb{C}$ while the total cohomology category $\mathrm{HDF}(X, W)$ of $\mathrm{DF}(X, W)$ is Hom-finite over $\mathbb{C}$. Moreover, the system:

$$(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$$

obeys all defining properties of a TFT datum (up to non-degeneracy of the bulk and boundary traces and up to the topological Cardy constraint, the proof of which is ongoing work).

Conjecture

Suppose that Z_W is compact. Then $(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$ is a TFT datum and hence defines a quantum open-closed TFT.

Off-shell bulk traces

Let Ω be a holomorphic volume form on X.

Definition

The *Serre trace* induced by Ω on $\mathcal{A}_c(X)$ is the $\mathbb{C}$-linear map $\int_\Omega : \mathcal{A}_c(X) \to \mathbb{C}$ defined through:

$$\int_\Omega \rho \stackrel{\text{def.}}{=} \int_X \Omega \wedge \rho \ , \ \ \forall \rho \in \mathcal{A}_c(X) \ .$$

Definition

The *canonical off-shell trace induced by* Ω *on* $\mathrm{PV}_c(X)$ is the $\mathbb{C}$-linear map $\mathrm{Tr}_B := \mathrm{Tr}_B^\Omega : \mathrm{PV}_c(X) \to \mathbb{C}$ defined through:

$$\mathrm{Tr}_B^\Omega(\omega) = \int_X \Omega \wedge (\Omega \lrcorner \omega) \ , \ \ \forall \omega \in \mathrm{PV}_c(X) \ .$$

Proposition

For any $\eta \in \mathrm{PV}_c(X)$*, we have:*

$$\mathrm{Tr}_B(\delta_W \eta) = \mathrm{Tr}_B(\boldsymbol{\bar\partial} \eta) = \mathrm{Tr}_B(\boldsymbol{\iota}_W \eta) = 0 \ .$$

In particular, Tr_B *descends to* $\mathrm{HPV}_c(X, W)$.

Definition

The *cohomological trace induced by* Ω *on* $\mathrm{HPV}_c(X, W)$ is the $\mathbb{C}$-linear map $\mathrm{Tr}_c := \mathrm{Tr}_c^\Omega : \mathrm{HPV}_c(X, W) \to \mathbb{C}$ induced by Tr_B^Ω.

Definition

Assume that the critical set Z_W is compact. In this case, the *cohomological trace induced by* Ω *on* $\mathrm{HPV}(X, W)$ is the $\mathbb{C}$-linear map $\mathrm{Tr} := \mathrm{Tr}^\Omega \stackrel{\text{def.}}{=} \mathrm{Tr}_c^\Omega \circ i_*^{-1} : \mathrm{HPV}(X, W) \to \mathbb{C}$ obtained by composing Tr_c with the inverse of the linear isomorphism $i_* : \mathrm{HPV}_c(X, W) \xrightarrow{\sim} \mathrm{HPV}(X, W)$ induced on cohomology by the inclusion map.

Let $a = (E, D)$ be a holomorphic factorization of W. Let $\delta_a := \delta_{a,a}$ and $\mathfrak{d}_a := \mathfrak{d}_{a,a}$ denote the twisted Dolbeault and defect differentials on $\mathrm{End}_{\mathrm{DF}(X,W)}(a)$. Let $\overline{\boldsymbol{\partial}}_a := \overline{\boldsymbol{\partial}}_{a,a} = \overline{\boldsymbol{\partial}}_{End(E)}$ denote the Dolbeault operator of $End(E)$. We have:

$$\delta_a = \overline{\boldsymbol{\partial}}_a + \mathfrak{d}_a \quad , \quad \mathfrak{d}_a = [D, \cdot] \quad ,$$

where $[\cdot, \cdot]$ denotes the graded commutator.

Definition

The *canonical off-shell boundary trace* induced by Ω on $\mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}_a^B := \mathrm{tr}_a^{B,\Omega} : \mathrm{End}_{\mathrm{DF}_c(X,W)}(a) \to \mathbb{C}$ defined through:

$$\mathrm{tr}_a^{B,\Omega}(\alpha) = \int_X \Omega \wedge \mathrm{str}(\alpha) = \int_\Omega \mathrm{str}(\alpha) \quad ,$$

for all $\alpha \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a) = \mathcal{A}_c(X, End(E))$, where str denotes the extended supertrace.

Proposition

For any holomorphic factorizations a_1 and a_2 of W, we have:

$$\mathrm{tr}^B_{a_2}(\alpha\beta) = (-1)^{\deg\alpha \deg\beta}\mathrm{tr}^B_{a_1}(\beta\alpha) \quad ,$$

when $\alpha \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_1, a_2)$ and $\beta \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_2, a_1)$ have pure total $\mathbb{Z}_2$-degree.

Proposition

For any $\alpha \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$, we have:

$$\mathrm{tr}^B_a(\delta_a\alpha) = \mathrm{tr}^B_a(\boldsymbol{\overline{\partial}}_a\alpha) = \mathrm{tr}^B_a(\mathfrak{d}_a\alpha) = 0 \quad .$$

In particular, tr^B_a descends to $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) = \mathrm{H}^*(\mathcal{A}_c(X, End(E)), \delta_a)$.

Definition

The *cohomological boundary trace induced by Ω on* $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}^c_a := \mathrm{tr}^{c,\Omega}_a : \mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) \to \mathbb{C}$ induced by $\mathrm{tr}^{B,\Omega}_a$ on $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a)$.

Definition

Assume that the critical locus Z_W is compact. Then the *cohomological boundary trace induced by* Ω *on* $\mathrm{End}_{\mathrm{HDF}(X,W)}(a)$ is the $\mathbb{C}$-linear map $\mathrm{tr}_a \stackrel{\mathrm{def.}}{=} \mathrm{tr}_a^c \circ j_{*,a}^{-1} : \mathrm{End}_{\mathrm{HDF}(X,W)}(a) \to \mathbb{C}$, where $j_{*,a} : \mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) \xrightarrow{\sim} \mathrm{End}_{\mathrm{HDF}(X,W)}(a)$ is the linear isomorphism induced by the inclusion functor.

Thus $(\mathrm{HDF}_c(X,W), \mathrm{tr}^c)$ is a pre-Calabi-Yau supercategory. When the critical set Z_W is compact, this implies that $(\mathrm{HDF}(X,W), \mathrm{tr})$ is also a pre-Calabi-Yau supercategory.

Definition

A Hermitian metric h on E is called *admissible* if the sub-bundles $E^{\hat{0}}$ and $E^{\hat{1}}$ of E are h-orthogonal:

$$h|_{E^{\hat{0}}\times E^{\hat{1}}} = h|_{E^{\hat{1}}\times E^{\hat{0}}} = 0 \quad .$$

Definition

A *Hermitian holomorphic factorization* of W is a triplet $\mathbf{a} = (E, h, D)$, where $a = (E, D)$ is a holomorphic factorization of W and h is an admissible Hermitian metric on E.

Fix a Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ of W and let $a = (E, D)$. Let $\nabla := \nabla_{\mathbf{a}}$ denote the Chern connection of (E, h). Let $\boldsymbol{\partial}_E^h : \Omega(X, E) \to \Omega(X, E)$ be the unique $\mathbb{C}$-linear operator which satisfies the Leibniz rule:

$$\boldsymbol{\partial}_E^h(\rho \otimes s) = (\partial\rho) \otimes s + (-1)^k \rho \wedge \nabla_{\mathbf{a}}^{1,0}(s)$$

for all $\rho \in \Omega^k(X)$ and all $s \in \Gamma_\infty(X, E)$. Let $F_{\mathbf{a}}$ denote the curvature form of $\nabla_{\mathbf{a}}$. We have:

$$(\boldsymbol{\partial}_E^h)^2 = \overline{\boldsymbol{\partial}}_E^2 = 0 \quad , \quad \boldsymbol{\partial}_E^h \overline{\boldsymbol{\partial}}_E + \overline{\boldsymbol{\partial}}_E \boldsymbol{\partial}_E^h = \mathrm{id}_{\Omega(X)} \otimes F_{\mathbf{a}} \quad .$$

Let $\boldsymbol{\partial}_{\mathbf{a}} : \Omega(X, End(E)) \to \Omega(X, End(E))$ denote the differential induced by $\boldsymbol{\partial}_E^h$ on $\Omega(X, End(E))$.

The natural isomorphism $End(TX) \simeq T^*X \otimes TX$ maps the identity endomorphism into a holomorphic section $\theta \in \Gamma(X, T^*X \otimes TX)$. Let G be a Kähler metric on X and $\omega_G \in \Omega^{1,1}(X)$ be the Kähler form of G. Let $\mathbf{a} = (E, h, D)$ be a Hermitian factorization of W. Define:

$$V_{\mathbf{a}}^G \stackrel{\text{def.}}{=} \partial_{\mathbf{a}} D + F_{\mathbf{a}} - \omega_G \mathrm{id}_E \in \Omega^{1,0}(X, End^{\hat{1}}(E)) \oplus \Omega^{1,1}(X, End^{\hat{0}}(E)) \quad ,$$

where $F_{\mathbf{a}} \in \Omega^{1,1}(X, End^{\hat{0}}(E))$ is the Chern curvature of (E, h).

Definition

The *twisted curvature* of the Hermitian holomorphic factorization $\mathbf{a}$ determined by G is defined through:

$$A_{\mathbf{a}}^G \stackrel{\text{def.}}{=} \theta \otimes \mathrm{id}_E + \mathrm{i} V_{\mathbf{a}}^G \in \Omega^{1,0}(X, TX \otimes End^{\hat{0}}(E)) \oplus \Omega^{1,0}(X, End^{\hat{1}}(E)) \oplus \Omega^{1,1}(X, End^{\hat{0}}(E))$$

Definition

The *disk kernel* of the Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ determined by Ω and by the Kähler metric G is the element $\Pi_{\mathbf{a}} := \Pi_{\mathbf{a}}^{\Omega, G} \in \mathrm{PV}(X, End(E))$ defined through the relation:

$$\Pi_{\mathbf{a}}^{\Omega, G} = \frac{1}{d!} \det{}_{\Omega} A_{\mathbf{a}}^G \quad .$$

Definition

The *off-shell boundary-bulk map* of the Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ determined by Ω and by the Kähler metric G is the $\mathcal{C}^\infty(M, \mathbb{R})$-linear map $f_{\mathbf{a}}^B := f_{\mathbf{a}}^{B,\Omega,G} : \mathrm{End}_{\mathrm{DF}(X,W)}(a) \to \mathrm{PV}(X)$ defined through:

$$f_{\mathbf{a}}^{B,\Omega,G}(\alpha) \stackrel{\mathrm{def.}}{=} \mathrm{str}(\Pi_{\mathbf{a}}^{\Omega,G}\alpha) \quad , \quad \forall \alpha \in \mathrm{End}_{\mathrm{DF}(X,W)}(a) = \mathcal{A}(X, End(E)) \quad .$$

Notice that $f_{\mathbf{a}}^B$ has total $\mathbb{Z}_2$-degree μ.

Proposition

We have:

$$\delta_W \circ f_{\mathbf{a}}^B = (-1)^d f_{\mathbf{a}}^B \circ \delta_a \quad .$$

In particular, $f_{\mathbf{a}}^B$ descends to an $\mathrm{O}(X)$-linear map from $\mathrm{Hom}_{\mathrm{DF}(X,W)}(a)$ to $\mathrm{HPV}(X, W)$.

Definition

The *cohomological boundary-bulk map* of $\mathbf{a} = (E, h, D)$ is the $\mathrm{O}(X)$-linear map $f_{\mathbf{a}} := f_{\mathbf{a}}^{\Omega,G} : \mathrm{End}_{\mathrm{HDF}(X,W)}(a) \to \mathrm{HPV}(X, W)$ induced by $f_{\mathbf{a}}^{B,\Omega,G}$ on cohomology.

Definition

The *canonical off-shell bulk-boundary map* of the Hermitian holomorphic factorization $\mathbf{a} = (E, h, D)$ determined by Ω and by the Kähler metric G is the $\mathcal{C}^\infty(M, \mathbb{R})$-linear map $e_{\mathbf{a}}^{B} := e_{\mathbf{a}}^{B,\Omega,G} : \mathrm{PV}(X) \to \mathrm{End}_{\mathrm{DF}(X,W)}(a)$ defined through:

$$e_{\mathbf{a}}^{B,\Omega,G}(\omega) \stackrel{\text{def.}}{=} \Omega \lrcorner_0 \left(\omega \Pi_{\mathbf{a}}^{\Omega,G}\right) \quad , \quad \forall \omega \in \mathrm{PV}(X) \ .$$

Notice that $e_{\mathbf{a}}^{B}$ has total $\mathbb{Z}_2$-degree $\hat{0}$.

Proposition

We have:

$$\delta_a \circ e_{\mathbf{a}}^{B} = (-1)^d e_{\mathbf{a}}^{B} \circ \delta_W \ .$$

In particular, $e_{\mathbf{a}}^{B}$ descends to an $\mathrm{O}(X)$-linear map from $\mathrm{HPV}(X, W)$ to $\mathrm{Hom}_{\mathrm{DF}(X,W)}(a)$.

Definition

The *cohomological bulk-boundary map* of $\mathbf{a} = (E, h, D)$ is the $\mathrm{O}(X)$-linear map $e_{\mathbf{a}} := e_{\mathbf{a}}^{\Omega,G} : \mathrm{HPV}(X, W) \to \mathrm{End}_{\mathrm{HDF}(X,W)}(a)$ induced by $e_{\mathbf{a}}^{B,\Omega,G}$ on cohomology.

Definition

Let X be a complex manifold with $\dim_{\mathbb{C}} X = d$. We say that X is a **Stein manifold** if the following three conditions are satisfied:

- Holomorphic functions separate points of X.
- X is holomorphically convex.
- For every point $x \in X$ there exist globally-defined holomorphic functions $f_1, \ldots, f_d \in \mathrm{O}(X)$ whose differentials $\mathrm{d}f_j$ are linearly independent at x.

Examples

- $\mathbb{C}^d$ is a Stein manifold
- Every domain of holomorphy in $\mathbb{C}^d$ is a Stein manifold
- Every closed complex submanifold of a Stein manifold is a Stein manifold
- Every Stein manifold X of complex dimension d can be embedded in $\mathbb{C}^{2d+1}$ through a biholomorphic proper map
- A complex manifold is Stein iff it is biholomorphic to a closed complex submanifold of $\mathbb{C}^N$ for some N.

Cartan's theorem B

For every coherent analytic sheaf $\mathcal{F}$ on a Stein manifold X, the cohomology $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Theorem

Suppose that X is Stein. Then the spectral sequence defined previously collapses at E_2 and $\mathrm{HPV}(X, W)$ is concentrated in non-positive degrees.
For all $k = -d, \ldots, 0$, the $\mathrm{O}(X)$-module $\mathrm{HPV}^k(X)$ is isomorphic with the cohomology at position k of the following sequence of finitely-generated projective $\mathrm{O}(X)$-modules:

$$(\mathcal{P}_W) : \quad 0 \to \mathrm{H}^0(X, \wedge^d TX) \stackrel{\iota_W}{\to} \cdots \stackrel{\iota_W}{\to} \mathrm{H}^0(X, TX) \stackrel{\iota_W}{\to} \mathrm{O}(X) \to 0$$

where $\mathrm{O}(X)$ sits in position zero.

An analytic model for the bulk algebra in the Stein case

Proof: Since X is Stein, Cartan's theorem B implies $\mathbf{E}_1^{i,j} = \mathrm{H}^j_{\overline{\partial}}(\mathcal{A}(X, \wedge^{|i|} TX)) = 0$ for $j > 0$ and all $i = -d, \ldots, 0$. Thus the only non-trivial row of the page $\mathbf{E}_1$ of the spectral sequence is the bottom row $\mathbf{E}_1^{\bullet,0}$, whose nodes are given by:

$$\mathbf{E}_1^{i,0} := \mathrm{H}^0_{\overline{\partial}}(\mathcal{A}(X, \wedge^{|i|} TX)) = \mathrm{H}_{\overline{\partial}}(\mathrm{PV}^{i,0}(X)) = \Gamma(X, \wedge^{|i|} TX) = \mathrm{H}^0(\wedge^{|i|} TX)$$

Thus page $\mathbf{E}_1$ reduces to:

$$\begin{array}{ccccccc}
\mathbf{E}_1^{-d,d} = 0 & \rightarrow & \mathbf{E}_1^{-d+1,d} = 0 & \rightarrow & \mathbf{E}_1^{-d+2,d} = 0 & \cdots & \mathbf{E}_1^{0,d} = 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
\mathbf{E}_1^{-d,2} = 0 & \rightarrow & \mathbf{E}_1^{-d+1,2} = 0 & \rightarrow & \mathbf{E}_0^{-d+2,2} = 0 & \cdots & \mathbf{E}_1^{0,2} = 0 \\
\uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} \\
\mathbf{E}_1^{-d,1} = 0 & \rightarrow & \mathbf{E}_1^{-d+1,1} = 0 & \rightarrow & \mathbf{E}_1^{-d+2,1} = 0 & \cdots & \mathbf{E}_1^{0,1} = 0 \\
\uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} & & \uparrow \overline{\partial} \\
\mathbf{E}_1^{-d,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_1^{-d+1,0} & \xrightarrow{\boldsymbol{\iota}_W} & \mathbf{E}_1^{-d+2,0} & \cdots & \mathbf{E}_1^{0,0}
\end{array}$$

The spectral sequence collapses at $\mathbf{E}_2$ and we have $\mathbf{E}_\infty^k = \mathbf{E}_2^{k,0} = \mathrm{H}^k_{\iota_W}(\mathbf{E}_1^{\bullet,0}) = \mathrm{H}^k(\mathcal{P}_W)$ for all $k = -d, \ldots, 0$.

The Serre-Swan theorem for Stein manifolds implies that $(\mathcal{P}_W)$ is a sequence of finitely-generated projective $\mathrm{O}(X)$-modules.

Proposition

Suppose that X is Stein and $\dim_{\mathbb{C}} Z_W = 0$. Then $\mathrm{HPV}^k(X) = 0$ for $k \neq 0$ and there exists a natural isomorphism of $\mathrm{O}(X)$-modules:

$$\mathrm{HPV}^0(X) \simeq_{\mathrm{O}(X)} \mathrm{H}^0(Jac_W) = \mathrm{Jac}(X, W) \quad .$$

We used the following definitions:

- $\mathcal{J}_W \stackrel{\mathrm{def.}}{=} \mathrm{im}(\boldsymbol{\iota}_W : TX \to O_X)$ (the **critical sheaf** of W)
- $Jac_W \stackrel{\mathrm{def.}}{=} O_X/\mathcal{J}_W$ (the **Jacobi sheaf** of W)
- $\mathrm{Jac}(X, W) \stackrel{\mathrm{def.}}{=} \Gamma(X, Jac_W)$ (the **Jacobi algebra** of (X, W))

Definition

The holomorphic dg-category of holomorphic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category $\mathrm{F}(X, W)$ defined as follows:

- The objects are the holomorphic factorizations of W.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$, $a_2 = (E_2, D_2)$ of W:

  $$\mathrm{Hom}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom(E_1, E_2))$$

  endowed with the $\mathbb{Z}_2$-grading with homogeneous components:

  $$\mathrm{Hom}^{\kappa}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, Hom^{\kappa}(E_1, E_2)), \forall \kappa \in \mathbb{Z}_2$$

  and with the differentials $\mathfrak{d}_{a_1,a_2}$ determined uniquely by the condition:

  $$\mathfrak{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^{\kappa} f \circ D_1, \forall f \in \Gamma(X, Hom^{\kappa}(E_1, E_2)), \forall \kappa \in \mathbb{Z}_2$$

- The composition of morphisms is induced by that of $\mathrm{VB}(X)$, which is the *full* subcategory of $\mathrm{Coh}(X)$ whose objects are the locally-free sheaves of finite rank.

Theorem

Suppose that X is Stein. Then $\mathrm{HDF}(X, W)$ and the cohomological category of holomorphic factorizations $\mathrm{HF}(X, W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{F}(X, W))$ are equivalent.

Definition

An $\mathrm{O}(X)$-*supermodule* is a $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-module M endowed with a direct sum decomposition $M = M^{\hat{0}} \oplus M^{\hat{1}}$ into submodules.

$\mathrm{O}(X)$-supermodules form an $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded category $\mathrm{Mod}^s_{\mathrm{O}(X)}$ if we define the Hom space $\mathrm{Hom}(M_1, M_2)$ from a supermodule M_1 to a supermodule M_2 to be the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-module with homogeneous components:

$$\mathrm{Hom}^{\hat{0}}(M_1, M_2) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(M_1^{\hat{0}}, M_2^{\hat{0}}) \oplus \mathrm{Hom}(M_1^{\hat{1}}, M_2^{\hat{1}})$$
$$\mathrm{Hom}^{\hat{1}}(M_1, M_2) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(M_1^{\hat{0}}, M_2^{\hat{1}}) \oplus \mathrm{Hom}(M_1^{\hat{1}}, M_2^{\hat{0}}) \quad .$$

The composition is defined in the obvious manner. Given an $\mathrm{O}(X)$-supermodule M:

$$\mathrm{End}(M) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(M, M) \ .$$

Definition

An $\mathrm{O}(X)$-supermodule $M = M^{\hat{0}} \oplus M^{\hat{1}}$ is called *finitely-generated* if both of its $\mathbb{Z}_2$-homogeneous components $M^{\hat{0}}$ and $M^{\hat{1}}$ are finitely-generated over $\mathrm{O}(X)$. It is called *projective* if both $M^{\hat{0}}$ and $M^{\hat{1}}$ are projective $\mathrm{O}(X)$-modules.

Let $\mathrm{Mod}^s_{\mathrm{O}(X)}$ denote the category of $\mathrm{O}(X)$-supermodules and $\mathrm{mod}^s_{\mathrm{O}(X)}$ denote the full sub-category of finitely-generated $\mathrm{O}(X)$-supermodules.

Projective factorizations

Definition

A projective analytic factorization of W is a pair (P, D), where P is a finitely-generated projective $\mathrm{O}(X)$-supermodule and $D \in \mathrm{End}^{\hat{1}}_{\mathrm{O}(X)}(P)$ is an odd endomorphism of P such that $D^2 = W\mathrm{id}_P$.

Definition

The dg-category $\mathrm{PF}(X, W)$ of projective analytic factorizations of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dG category defined as follows:

- The objects are the projective analytic factorizations of W.
- Given two projective analytic factorizations (P_1, D_1) and (P_2, D_2) of W:

$$\mathrm{Hom}_{\mathrm{PF}(X,W)}((P_1, D_1), (P_2, D_2)) = \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2) \quad ,$$

  endowed with the $\mathbb{Z}_2$-grading and with the $\mathrm{O}(X)$-linear odd differential $\mathfrak{d} := \mathfrak{d}_{(P_1,D_1),(P_2,D_2)}$ determined uniquely by the condition:

$$\mathfrak{d}(f) = D_2 \circ f - (-1)^{\mathrm{deg} f} f \circ D_1$$

  for all elements $f \in \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$ which have pure $\mathbb{Z}_2$-degree.
- The composition of morphisms is inherited from $\mathrm{mod}^s_{\mathrm{O}(X)}$.

Projective factorizations

Definition

The cohomological category $\mathrm{HPF}(X, W)$ of analytic projective factorizations of W is the total cohomology category $\mathrm{HPF}(X, W) \stackrel{\text{def.}}{=} \mathrm{H}(\mathrm{PF}(X, W))$, which is a $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear category.

Theorem

The categories $\mathrm{HDF}(X, W)$ and $\mathrm{HPF}(X, W)$ are equivalent when X is Stein. When X is Stein and Z_W is compact, the category of topological D-branes of the B-type Landau-Ginzburg theory can be identified with $\mathrm{HPF}(X, W)$.

Tempered objects and the bulk and boundary flows

The bulk flow

Let G be a Kähler metric on X and ∇ its Levi-Civita connection. Let:

$$\mathrm{Hess}_G(\overline{W}) \stackrel{\mathrm{def.}}{=} \nabla(\mathrm{grad}_G \overline{W}) \in \Omega^1(X, TX)$$

denote the Hessian operator of $\overline{W}$ and:

$$H_G \stackrel{\mathrm{def.}}{=} \mathrm{Hess}_G^{0,1}(\overline{W}) = \nabla^{0,1}(\mathrm{grad}_G \overline{W}) = \boldsymbol{\bar\partial}_{TX}(\mathrm{grad}_G \overline{W}) \in \mathrm{PV}^{-1,1}(X)$$

denote its $(0,1)$-part. Let

$$||\partial W||_G^2 \stackrel{\mathrm{def.}}{=} \hat{h}_G(\partial W, \partial W) = h_G(\mathrm{grad}_G \overline{W}, \mathrm{grad}_G \overline{W}) = (\partial W)(\mathrm{grad}_G \bar{W}) \in \mathrm{PV}^{0,0}(X)$$

denote the squared norm of ∂W. Since H_G is nilpotent in the algebra $\mathrm{PV}(X)$, we can define its exponential. For any $\lambda \in [0, +\infty)$, we have:

$$e^{-\mathrm{i}\lambda H_G} = \sum_{p=0}^{d} \frac{1}{p!}(-\mathrm{i}\lambda)^p (H_G)^p \in \mathrm{PV}^0(X) \quad ,$$

where the expansion reduces to the first $d+1$ terms.

Definition

The *bulk flow generator* determined by the Kähler metric G is the element:

$$L_G \stackrel{\mathrm{def.}}{=} ||\partial W||_G^2 + \mathrm{i} H_G \in \mathrm{PV}^{0,0}(X) \oplus \mathrm{PV}^{-1,1}(X) \subset \mathrm{PV}^0(X) \quad .$$

L_G has degree zero with respect to the canonical $\mathbb{Z}$-grading of $\mathrm{PV}(X)$.

Proposition

We have:

$$L_G = \delta_W v_G \quad ,$$

where:

$$v_G \stackrel{\mathrm{def.}}{=} \mathrm{i}\ \mathrm{grad}_G \overline{W} \in \Gamma_\infty(X, TX) = \mathrm{PV}^{-1,0}(X) \quad .$$

Let $\widehat{L_G}$ denote the operator of left multiplication with the element L_G in the algebra $\mathrm{PV}(X)$.

Definition

The *bulk flow* determined by the Kähler metric G is the semigroup $(U_G(\lambda))_{\lambda \geq 0}$ generated by $\widehat{L_G}$. Thus $U_G(\lambda)$ is the even $\mathcal{C}^\infty(M, \mathbb{R})$-linear endomorphism of $\mathrm{PV}(X)$ defined through:

$$U_G(\lambda)(\omega) \stackrel{\mathrm{def.}}{=} e^{-\lambda L_G}\omega \quad , \quad \forall \omega \in \mathrm{PV}(X) \quad .$$

Proposition

For any $\lambda \in [0, +\infty)$*, the endomorphism* $U_G(\lambda)$ *is homotopy equivalent with* $\mathrm{id}_{\mathrm{PV}(X)}$*. In particular, we have:*

$$\delta_W \circ U_G(\lambda) = U_G(\lambda) \circ \delta_W \quad .$$

Thus $U_G(\lambda)$ *preserves the subspaces* $\ker(\delta_W)$ *and* $\mathrm{im}(\delta_W)$ *and it induces the identity endomorphism of* $\mathrm{HPV}(X, W)$ *on the cohomology of* δ_W*.*

Definition

For any $\lambda \geq 0$, the λ*-tempered trace induced by* G *and* Ω *on* $\mathrm{PV}_c(X)$ is the $\mathbb{C}$-linear map $\mathrm{Tr}^{(\lambda)} := \mathrm{Tr}^{(\lambda),\Omega,G} : \mathrm{PV}_c(X) \to \mathcal{C}^\infty(M, \mathbb{R})$ defined through:

$$\mathrm{Tr}^{(\lambda),\Omega,G} \stackrel{\mathrm{def.}}{=} \mathrm{Tr}_B^\Omega \circ U_G(\lambda) \quad .$$

This map has degree zero with respect to the canonical $\mathbb{Z}$-grading of $\mathrm{PV}_c(X)$.

Proposition

For any $\omega \in \mathrm{PV}_c^{i,j}(X)$*, we have:*

$$\mathrm{Tr}^{(\lambda)}(\omega) = 0 \ \ \text{unless} \ \ i + j = 0$$

and:

$$\mathrm{Tr}^{(\lambda)}(\omega) = \frac{(-\mathrm{i}\lambda)^{d-j}}{(d-j)!} \int_X \Omega \wedge \left(\Omega \lrcorner [(H_G)^{d-j}\omega]\right) e^{-\lambda ||\partial W||_G^2} \ \ \text{when} \ \ \omega \in \mathrm{PV}_c^{-j,j}(X)$$

Proposition

Let $\omega \in \mathrm{PV}_c(X)$*. Then the following statements hold for any* $\lambda \geq 0$*:*

1. *If* $\omega = \delta_W \eta$ *for some* $\eta \in \mathrm{PV}_c(X)$*, then* $\mathrm{Tr}^{(\lambda)}(\omega) = 0$.
2. *If* $\delta_W \omega = 0$*, then* $\mathrm{Tr}^{(\lambda)}(\omega)$ *does not depend on* λ *or* G *and coincides with* $\mathrm{Tr}_B(\omega)$*:*
$$\mathrm{Tr}^{(\lambda)}(\omega) = \mathrm{Tr}^{(0)}(\omega) = \mathrm{Tr}_B(\omega) \ \ .$$
In particular, the map induced by $\mathrm{Tr}^{(\lambda)}(\omega)$ *on* $\mathrm{HPV}_c(X, W)$ *coincides with* $\mathrm{Tr}_B(\omega)$.

Boundary flows

We have:

$$\boldsymbol{\partial}_{\mathbf{a}}^2 = 0 \quad , \quad \boldsymbol{\partial}_{\mathbf{a}} \overline{\boldsymbol{\partial}}_a + \overline{\boldsymbol{\partial}}_a \boldsymbol{\partial}_{\mathbf{a}} = [F, \cdot] \quad ,$$

where $\overline{\boldsymbol{\partial}}_a = \overline{\boldsymbol{\partial}}_{End(E)}$.

Definition

The *flow generator of* $\mathbf{a} = (E, h, D)$ *determined by the Kähler metric* G is defined through:

$$L_{\mathbf{a}}^G \overset{\text{def.}}{=} ||\partial W||_G^2 \mathrm{id}_E + H_G \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F) \in$$
$$\in \mathcal{A}^0(X, End^{\hat{0}}(E)) \oplus \mathcal{A}^1(X, End^{\hat{1}}(E)) \oplus \mathcal{A}^2(X, End^{\hat{0}}(E)) \quad .$$

Proposition

We have:

$$L_{\mathbf{a}}^G = \delta_a v_{\mathbf{a}}^G \quad ,$$

where:

$$v_{\mathbf{a}}^G \overset{\text{def.}}{=} \mathrm{grad}_G \overline{W} \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F) \in \mathcal{A}^0(X, End^{\hat{1}}(E)) \oplus \mathcal{A}^1(X, End^{\hat{0}}(E)) \quad .$$

Since $H_G \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F)$ is nilpotent, we can define its exponential. For any $\lambda \geq 0$, we have:

$$e^{-\lambda H_G \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F)} = \sum_{k=0}^{d} \frac{(-\lambda)^k}{k!} [H_G \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F)]^k \in \mathrm{End}^{\hat{0}}_{\mathrm{DF}(X,W)}(a) \quad ,$$

where the series reduces to the first $d+1$ terms. Define:

$$e^{-\lambda L^G_{\mathbf{a}}} \stackrel{\mathrm{def.}}{=} e^{-\lambda ||\partial W||^2_G} e^{-\lambda H_G \lrcorner (\boldsymbol{\partial}_{\mathbf{a}} D + F)} \in \mathrm{End}^{\hat{0}}_{\mathrm{DF}(X,W)}(a) \quad .$$

Proposition

For any $\lambda \geq 0$, we have:

$$e^{-\lambda L^G_{\mathbf{a}}} = 1 - \delta_W S^G_{\mathbf{a}}(\lambda) \quad ,$$

where:

$$S^G_{\mathbf{a}}(\lambda) \stackrel{\mathrm{def.}}{=} v^G_{\mathbf{a}} \int_0^\lambda \mathrm{d}t e^{-tL^G_{\mathbf{a}}} \in \mathrm{End}^{\hat{1}}_{\mathrm{DF}(X,W)}(a) \quad .$$

In particular, we have:

$$\delta_a(e^{-\lambda L^G_{\mathbf{a}}}) = 0 \quad .$$

Definition

The *boundary flow of* $\mathbf{a} = (E, h, D)$ determined by the Kähler metric G is the semigroup $(U_{\mathbf{a}}^G(\lambda))_{\lambda \geq 0}$ generated by $\widehat{L_{\mathbf{a}}^G}$. Thus $U_{\mathbf{a}}^G(\lambda)$ is the even $\mathcal{C}^\infty(M, \mathbb{R})$-linear endomorphism of $\mathrm{End}_{\mathrm{DF}(X,W)}(a)$ defined through:

$$U_{\mathbf{a}}^G(\lambda)(\alpha) \stackrel{\mathrm{def.}}{=} e^{-\lambda L_{\mathbf{a}}^G}\alpha \quad , \quad \forall \alpha \in \mathrm{End}_{\mathrm{DF}(X,W)}(a) \quad .$$

Proposition

For any $\lambda \geq 0$, *the endomorphism* $U_{\mathbf{a}}^G(\lambda)$ *is homotopy equivalent with* $\mathrm{id}_{\mathrm{End}_{\mathrm{DF}(X,W)}(a)}$. *In particular, we have:*

$$\delta_a \circ U_{\mathbf{a}}^G(\lambda) = U_{\mathbf{a}}^G(\lambda) \circ \delta_a \quad .$$

Hence $U_{\mathbf{a}}^G(\lambda)$ *preserves the subspaces* $\ker(\delta_a)$ *and* $\mathrm{im}(\delta_a)$ *and it induces the identity endomorphism of* $\mathrm{End}_{\mathrm{DF}(X,W)}(a)$ *on the cohomology of* δ_a.

Definition

Let $\lambda \in \mathbb{R}_{\geq 0}$. The *$\lambda$-tempered trace* of $\mathbf{a} = (E, h, D)$ induced by Ω and G is the $\mathbb{C}$-linear map $\mathrm{tr}_{\mathbf{a}}^{(\lambda)} := \mathrm{tr}_{\mathbf{a}}^{(\lambda),\Omega,G} : \mathrm{End}_{\mathrm{HDF}_c(X,W)}(a) \to \mathbb{C}$ defined through:

$$\mathrm{tr}_{\mathbf{a}}^{(\lambda),\Omega,G} \stackrel{\text{def.}}{=} \mathrm{tr}_a^{B,\Omega} \circ U_{\mathbf{a}}^G(\lambda) \quad .$$

Proposition

Let $\alpha \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$. Then the following statements hold for any $\lambda \geq 0$:

1. *If $\alpha = \delta_a \beta$ for some $\beta \in \mathrm{End}_{\mathrm{DF}_c(X,W)}(a)$, then $\mathrm{tr}_{\mathbf{a}}^{(\lambda)}(\alpha) = 0$.*
2. *If $\delta_a \alpha = 0$, then $\mathrm{tr}_{\mathbf{a}}^{(\lambda)}(\alpha)$ does not depend on λ or on the metrics G and h:*

$$\mathrm{tr}_{\mathbf{a}}^{(\lambda)}(\alpha) = \mathrm{tr}_{\mathbf{a}}^{(0)}(\alpha) = \mathrm{tr}_a^B(\alpha) \quad .$$

In particular, the map induced by $\mathrm{tr}_{\mathbf{a}}^{(\lambda)}$ on $\mathrm{End}_{\mathrm{HDF}_c(X,W)}(a)$ coincides with tr_a^c.

Proposition

Let $\mathbf{a}_1 = (E_1, h_1, D_1)$ *and* $\mathbf{a}_2 = (E_2, h_2, D_2)$ *be two Hermitian holomorphic factorizations of* W *with underlying holomorphic factorizations* $a_1 = (E_1, D_1)$ *and* $a_2 = (E_2, D_2)$. *Let* $\alpha \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_1, a_2)$ *and* $\beta \in \mathrm{Hom}_{\mathrm{DF}_c(X,W)}(a_2, a_1)$ *have pure total* $\mathbb{Z}_2$*-degree and satisfy* $\delta_{a_1,a_2}\alpha = \delta_{a_2,a_1}\beta = 0$. *Then:*

$$\mathrm{tr}_{\mathbf{a}_2}^{(\lambda),G}(\alpha\beta) = (-1)^{\deg\alpha \deg\beta}\mathrm{tr}_{\mathbf{a}_1}^{(\lambda),G}(\beta\alpha) \quad .$$

There also exist tempered versions of the bulk-boundary and boundary-bulk maps. Together with the tempered bulk and boundary traces, they provide a family of cochain-level models for the TFT datum, parameterized by $\lambda \in [0, +\infty)$.

The worldsheet Lagrangian

The **bulk action** is:

$$\tilde{S}_{bulk} = S_B + S_W + s \quad ,$$

where:

$$S_B = \int_\Sigma d^2\sigma\sqrt{g}\Big[G_{i\bar{j}}\Big(g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \mathrm{i}\varepsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \frac{1}{2}g^{\alpha\beta}\rho^i_\alpha D_\beta\eta^{\bar{j}}$$
$$-\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha D_\beta\theta^{\bar{j}} - \tilde{F}^i\tilde{F}^{\bar{j}}\Big) + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}R_{i\bar{l}k\bar{j}}\rho^i_\alpha\bar{\chi}^{\bar{l}}\rho^k_\beta\chi^{\bar{j}}\Big]$$

is the action of the B-twisted sigma model and $S_W = S_0 + S_1$ is the potential-dependent term, with:

$$S_0 = -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\bar{\chi}^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}}\Big]$$
$$S_1 = -\frac{\mathrm{i}}{2}\int_\Sigma d^2\sigma\sqrt{g}\Big[(\partial_i W)\tilde{F}^i + \frac{\mathrm{i}}{4}\varepsilon^{\alpha\beta}D_i\partial_j W\rho^i_\alpha\rho^j_\beta\Big] \quad .$$

Here:

$$s := \mathrm{i}\int_\Sigma d^2\sigma\sqrt{g}\varepsilon^{\alpha\beta}\partial_\alpha(G_{\bar{i}j}\chi^{\bar{i}}\rho^j_\beta) = \mathrm{i}\int_\Sigma d(G_{\bar{i}j}\chi^{\bar{i}}\rho^j) \quad .$$

is a correction needed to solve the so-called "Warner problem".

The **fields** involved are:

- the Grassmann even fields:
  - the scalar field $\phi : \Sigma \to X$
  - the Riemannian metric g on Σ,
  - the auxiliary fields $\tilde{F} \in \Gamma_\infty(\phi^*(\mathcal{T}_\mathbb{C} X))$
- the Grassmann odd fields:
  - $\eta, \chi, \bar{\chi} \in \Gamma_\infty(\phi^*(\bar{T}X))$, $\theta \in \Gamma_\infty(\phi^*(T^*X))$, $\rho \in \Gamma_\infty(\phi^*(TX) \otimes \mathcal{T}^*\Sigma)$

Here $\mathcal{T}X$ is the real tangent bundle of X and $\mathcal{T}_\mathbb{C} X = \mathcal{T}X \otimes \mathbb{C} = TX \oplus \bar{T}X$ is its complexification, while TX and $\bar{T}X$ are the holomorphic and antiholomorphic tangent bundles of X. $\mathcal{T}\Sigma$ is the real tangent bundle of Σ.

We define the **partition function** on an oriented Riemann surface Σ with corners by:

$$Z := \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta] e^{-\tilde{S}_{bulk}} \mathcal{U}_1 \ldots \mathcal{U}_h \quad ,$$

where h is the number of holes and the factors $\mathcal{U}_h$ have complicated expressions depending on the superconnection $\mathcal{B}$ and the fields as well as on "boundary condition changing operators" inserted at the corners of each hole. ($\mathcal{U}_1 \ldots \mathcal{U}_h = e^{-\tilde{S}_{boundary}}$)

Consider a complex superbundle $E = E^{\hat{0}} \oplus E^{\hat{1}}$ on X and a superconnection $\mathcal{B}$ on E. The bundle $End(E)$ is $\mathbb{Z}_2$-graded:

$$End^{\hat{0}}(E) := End(E^{\hat{0}}) \oplus End(E^{\hat{1}})$$
$$End^{\hat{1}}(E) := Hom(E^{\hat{0}}, E^{\hat{1}}) \oplus Hom(E^{\hat{1}}, E^{\hat{0}}) \quad .$$

In a local frame of E compatible with the grading, $\mathcal{B}$ corresponds to:

$$\mathcal{B} = \begin{bmatrix} A^{(+)} & v \\ u & A^{(-)} \end{bmatrix}$$

where $v \in \Gamma_\infty(X, Hom(E^{\hat{1}}, E^{\hat{0}}))$ and $u \in \Gamma_\infty(X, Hom(E^{\hat{0}}, E^{\hat{1}}))$, while $A^{(+)}$ and $A^{(-)}$ are connection one-forms on $E^{\hat{0}}$ and $E^{\hat{1}}$, such that $A^{(+)} \in \Omega^{(0,1)}(End(E^{\hat{0}}))$ and $A^{(-)} \in \Omega^{(0,1)}(End(E^{\hat{1}}))$.