BEREZIN TRANSFORMS ATTACHED TO LANDAU LEVEL

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Geometry and Physics Seminar

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August 10, 2016, Bucharest, Romania

Coherent states for the Harmonic oscillator: $|z\rangle$

Glauber (1951)

- CS as eigenstates of the annihilation operator a

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- The operators a and a^* satisfy $[a,a^*]=1$.
- The Hamiltonian of the harmonic oscillator: $H=a^*a+\frac{1}{2}$

IWATA (1951)

The number state expansion for the normalized CS:

$$|z> = \left(e^{|z|^2}\right)^{-\frac{1}{2}} \sum_{k \geq 0} \frac{z^k}{\sqrt{k!}} |k>.$$

Eigenstates of $\widehat{H}$ are denoted $|k>$, $k = 0, 1, 2, ...$ with the condition

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Glauber & Feynman (1951)

CS as the states produced by the action of the displacement operator

$$D(z) = \exp(za^* - z^* a)$$

on the vacuum $|0>$ as

$$|z> = D(z)|0>,$$

SCHRÖDINGER (1926)

- CS as minimum uncertainty states:

$$\langle \xi \mid z \rangle = \pi^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\xi^2 + \sqrt{2}\xi z - \frac{1}{2}z^2 - \frac{1}{2}|z|^2\right), \quad \xi \in \mathbb{R}$$

The fluctuations are

$$\begin{cases} (\Delta\widehat{p})^2 = < z|\widehat{p}^2|z > - \left(< z|\widehat{p}|z >\right)^2 = \frac{1}{2}, \\ (\Delta\widehat{x})^2 = < z|\widehat{x}^2|z > - \left(< z|\widehat{x}|z >\right)^2 = \frac{1}{2} \end{cases}$$

so that

$$\Delta p \Delta x = \frac{1}{2}.$$

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SUMMARY: CS OF THE HARMONIC OSCILLATOR

- are obtained by the 4 ways.
- satisfy the resolution of the identity: $\mathbf{1}_{L^2(\mathbb{R})} = \frac{1}{\pi}\int d^2z\, |z\rangle\langle z|$.
- form an overcomplet set.
- are not orthogonal: $\langle z|w\rangle \neq 0$. A big advantage!

Generalization "à la Gilmore-Perelomov"

CS are produced by the action T_g of the group element $g \in G$ on a reference state ϕ_0 in a representation Hilbert space as:

$$\widetilde{\Phi}_{\mathfrak{g}} = T_g \left[\phi_0 \right].$$

Some generalizations

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- **Example:** Barut-Girardello CS.

$$\widetilde{A} = K_- := K_1 - iK_2$$

K_1, K_2 and K_3 are generators of the Lie algebra $su(1,1)$.

Generalization " à la Schrödinger"

Try to "minimize" the generalized Heisenberg uncertainty relation for Hermitian operators $\widehat{A}$ and $\widehat{B}$ different from $\widehat{x}$ and $\widehat{p}$:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\left\langle\left[\widehat{A},\widehat{B}\right]\right\rangle\right|$$

looking for states for which this becomes an equality.

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Generalization "à la Iwata"

By choosing different set of coefficients $\{c_n(z)\}$ and functions $\{\varphi_n\}$ satisfying suitable conditions as

$$\widetilde{\Phi}_z = \sum_n c_n(z) \mid \varphi_n \rangle .$$

Ref: V V Dodonov 2002, *Nonclassical' states in quantum optics: a 'squeezed' review of the first 75 years*. J. Opt. B: Quantum Semiclass. Opt. **(451 Refs)**

A Formalism

Ref: A coherent states formalism starting from a measure space "*as a set of data*": J. P. Gazeau 2009, *Coherent states in quantum physics*.

- Let $X = \{x \mid x \in X\}$ be a set equipped with a measure $d\mu$ and $L^2(X, d\mu)$ the space of $d\mu-$square integrable functions on X.

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$$\mid x >:= (\mathcal{N}(x))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \overline{\Phi_j(x)} \mid \varphi_j >$$

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- where $\mathcal{N}(x) = \sum_{j=0}^{+\infty} \Phi_j(x) \overline{\Phi_j(x)} < \infty$.

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- The resolution of the identity of $\mathcal{H}$

$$\mathbf{1}_{\mathcal{H}} = \int_X \mid x >< x \mid \mathcal{N}(x)\, d\mu(x) \qquad (RI)$$

The Dirac's bra-ket notation $\mid x >< x \mid$ means the rank-one -operator

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- An isometry

$$\begin{aligned} W : \quad & \mathcal{H} \longrightarrow \mathcal{A}^2 \subset L^2(X, \mu) \\ & \varphi \mapsto W[\phi](x) = (\mathcal{N}(x))^{1/2} \langle \phi | x \rangle \end{aligned}$$

called a coherent states transform (CST).

Example I: The Landau problem on the Euclidean plane $\mathbb{C}$

- $X \equiv \mathbb{C}$,
- The measure: $e^{-|z|^2} d\mu(z)$, $d\mu$ the Lebesgue measure on $\mathbb{C}$
- $\mathcal{A}^2 \equiv \mathcal{A}^2_m(\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2} d\mu),\ \widetilde{\Delta}\varphi = m\varphi \right\}, \quad m \in \mathbb{N}.$
  The Landau Hamiltonian
$$\widetilde{\Delta} = -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}}.$$
- $|\varphi_p\rangle$: eingenstates of the harmonic oscillator.
$$\langle \xi | \varphi_p \rangle =:= \left(\sqrt{\pi} 2^p p!\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_p(\xi),\ p = 0, 1, 2, \ldots \quad \xi \in \mathbb{R}$$
  $H_p(\xi)$: Hermite polynomial.
- The coefficients
$$\Phi^m_p(z) = (-1)^{m \wedge p} (m \wedge p)! \left(\pi m! p!\right)^{-\frac{1}{2}} |z|^{|m-p|} e^{-i(m-p)\arg z} L^{(|m-p|)}_{m \wedge p}(z\overline{z})$$
$$\tilde{\triangle}\Phi^m_p(z) = m\Phi^m_p(z)$$
- The coherent states:
$$|z, m\rangle = (\mathcal{N}(z))^{-1/2} \sum_{p=0}^{\infty} \frac{\overline{\Phi^n_p(z)}}{\sqrt{\pi m! p!}} |\varphi_p\rangle$$
- The overlap function between two CS:
$$\langle z, m | w, m \rangle = (\mathcal{N}(z)\mathcal{N}(w))^{-\frac{1}{2}} e^{z\overline{w}} L^{(0)}_m(|z - w|^2)$$

THEOREM

- A closed form for these CS is

$$\phi_{z,m}(\xi) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\overline{z}^2 + \sqrt{2}\xi\overline{z} - \frac{1}{2}|z|^2 - \frac{1}{2}\xi^2} H_m\left(\xi - \frac{z+\overline{z}}{\sqrt{2}}\right)$$

Theorem

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- The coherent state transform: $B_m^{\arg} : L^2(\mathbb{R}, d\xi) \to \mathcal{A}_m^2(\mathbb{C})$

$$B_m^{\arg}[f](z) := (\mathcal{N}_m(z))^{\frac{1}{2}} \langle f, \phi_{z,m} \rangle_{L^2(\mathbb{R})}$$

Explicitly,

$$B_m^{\arg}[f](z) = \mathfrak{c}_m \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2}z^2 + \sqrt{2}\xi z - \frac{1}{2}\xi^2} H_m\left(\xi - \frac{z+\bar{z}}{\sqrt{2}}\right) d\xi$$

is a generalized Bargmann transform of index $m = 0, 1, 2, \ldots$.

- Case $m = 0$: The Bargmann transform $B_0^{\arg}$ (V. Bargmann, 1961) corresponds to the lowest Landau level LLL.

Example 2: The Landau problem on the Poincaré disk $\mathbb{D}$

- $X = \mathbb{D}$, $(1 - |z|^2)^{-2} d\mu$, $d\mu$ the Lebesgue measure on $\mathbb{D}$.

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- The coefficients

$$\Phi_k^{\nu,m}(z) = (-1)^k \left(\frac{2(\nu-m)-1}{\pi}\right)^{\frac{1}{2}} \left(\frac{k!\Gamma(2(\nu-m)-m)}{m!\Gamma(2(\nu-m)+k)}\right)^{\frac{1}{2}}$$

$$\times \left(1-|z|^2\right)^{-m} \overline{z}^{m-k} P_k^{(m-k,2(\nu-m)-1)}\left(1-2|z|^2\right)$$

$P_k^{(\alpha,\beta)}(.)$ Jacobi polynomial.

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- The vectors $|\varphi_k\rangle \in \mathcal{H}$: eigenstates of the PHO

$$\varphi_k^{\nu,m}(\xi) := \left(\frac{k!}{\Gamma(2\nu-2m+k)}\right)^{\frac{1}{2}} \xi^{\nu-m} e^{-\frac{1}{2}\xi} L_k^{(2(\nu-m)-1)}(\xi),$$

THE LANDAU PROBLEM ON THE POINCARÉ DISK $\mathbb{D}$

- The coherent states:

$$|z,\nu,m\rangle := \left(\pi^{-1}(2\nu-2m-1)\left(1-|z|^2\right)^{-2\nu}\right)^{-\frac{1}{2}}\sum_{k=0}^{+\infty}\overline{\Phi_k^{\nu,m}(z)}\,|\,\varphi_k^{\nu,m}>$$

The Landau problem on the Poincaré disk $\mathbb{D}$

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- A closed form for these CS is

$$\langle \xi | z,\nu,m\rangle = (-1)^m \left(\frac{m!}{\Gamma(2\nu-m)}\right)^{\frac{1}{2}} \frac{|1-z|^{2m}}{(1-z)^{2\nu}} \left(1-|z|^2\right)^{\nu-m} \xi^{\nu-m}$$
$$\times \exp\left(-\frac{\xi}{2}\frac{1+z}{1-z}\right) L_m^{2(\nu-m)-1}\left(\xi\frac{1-z\bar{z}}{|1-z|^2}\right)$$

Theorem — Elwassouli, Ghanmi, Intissar & Mouayn, *Ann. Henri Poincaré 2012*

- The coherent state transform: $B_m^{\mathrm{arg}} : L^2(\mathbb{R}_+^*, \xi^{-1} d\xi) \to \mathcal{A}_m^{2,\nu}(\mathbb{D})$ defined by

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Explicitly

$$B_m^{\mathrm{arg}}[f](z) = \mathfrak{c}_m (1-z)^{-2\nu} \left(\frac{1-z\overline{z}}{|1-z|^2} \right)^{-m}$$

$$\times \int_0^{+\infty} \xi^{\nu-m} \exp\left(-\frac{\xi}{2} \left(\frac{1+z}{1-z} \right) \right) L_m^{2(\nu-m)-1} \left(\xi \frac{1-z\overline{z}}{|1-z|^2} \right) f(\xi) \frac{d\xi}{\xi}.$$

- For $m = 0$, B_0^{arg} is the second Bargmann transform (V. Bargmann 1961) corresponds to the lowest hyperbolic Landau level LHLL.

COMPLETNESS THEOREM ABREU, BALAZS, DE GOSSON & MOUAYN, *Ann. Phys* 2015

The Euclidean setting

Theorem 1. *Let* $(|(x,y),\pi,m\rangle)_{(x,y)\in\mathbb{R}^2}$ *be a system of coherent states attached to the mth Landau level. Then the following holds*

- if $\omega^2 < \frac{1}{m+1}$ the system $(|(x,y),\pi,m\rangle)_{(x,y)\in\bigwedge_\omega}$ is complete
- if $\omega^2 > 1$ then the system $(|(x,y),\pi,m\rangle)_{(x,y)\in\bigwedge_\omega}$ is not complete

where $\bigwedge_\omega = \omega(\mathbb{Z} + i\mathbb{Z})$ is a square lattice of area ω^2.

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The Hyperbolic setting

Theorem 2. *Let* $\{|z, B, m\rangle\}_{z\in\mathbb{C}^+}$ *be a system of coherent states attached to the mth hyperbolic level. If the subsystem* $\{|g, \xi_0, B, m\rangle\}_{g\in G}$ *indexed by the Fuchsian group G associated with the automorphic form* F_0 *of weight* m_0*, vanishing at one point* $\xi_0 \in \mathbb{C}^+$ *is complete, then*

$$m_0 \geq \frac{1}{2}\frac{B-m}{1+m}$$

EXAMPLE 3: THE LANDAU PROBLEM ON THE RIEMANN SPHERE

- $X = \mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$
- $d\mu(z) = (1 + z\bar{z})^{-2} d\eta(z)$, $d\eta(z)$ Lebesgue measure on $\mathbb{C}$.
- $\mathcal{A}^2$: a *generalized Bergman space* on the Riemann sphere

$$\mathcal{A}_m^{\nu}\left(\mathbb{S}^2\right) = \left\{\Phi \in L^2\left(\mathbb{S}^2, d\mu(z)\right), H_{2\nu}\Phi = \lambda_{\nu,m}\Phi\right\}.$$

- The Hamiltonian of ν magnetic field

$$H_{2\nu} = -(1+z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \nu z(1+z\bar{z})\frac{\partial}{\partial z} + \nu \bar{z}(1+z\bar{z})\frac{\partial}{\partial \bar{z}} + \nu^2(1+z\bar{z}) - \nu^2$$

- Spherical Landau levels: $\lambda_{m,\nu} := (2m+1)\nu + m(m+1)$, $m = 0, 1, 2, \cdots$.
- $\mathcal{H} \equiv l^2(\Omega_{N+1})$ the Hibert space of square summable functions on the finite discrete set $\Omega_{N+1} = \{x_j = j - pN, j = 0, 1, 2, \cdots, N\}$
- The Kravchuk oscillator:

$$\mathcal{L}_{\xi}^{N} = 2p(1-p)N + \frac{1}{2} + (1-2p)\frac{\xi}{h} - \sqrt{p(1-p)}\left(\alpha(\xi)e^{h\partial_{\xi}} + \alpha(\xi - h)e^{-h\partial_{\xi}}\right),$$

$$h = \sqrt{2Np(1-p)}, \alpha(\xi) = \sqrt{((1-p)N - h^{-1}\xi)(pN + 1 + h^{-1}\xi)}$$

THEOREM Z. MOUAYN *Ann. Henri Poincaré* 2015

- We define a discrete Bargmann transform

$$\mathcal{B}_{\nu,m} : l^2(\Omega^{p,q}_{N+1}) \to \mathcal{A}^{\nu}_{m}\left(\mathbb{S}^2\right).$$

$$\mathcal{B}_{\nu,m}[f](z) := \frac{(N+1)^{\frac{1}{2}} N!}{\sqrt{m!\,(N-m)!}} \left(\frac{\sqrt{q}-\bar{z}\sqrt{p}}{\sqrt{q\,(1+z\bar{z})}}\right)^{N} \left(\frac{z\sqrt{q}+\sqrt{p}}{\bar{z}\sqrt{p}-\sqrt{q}}\right)^{m}$$

$$\times \sum_{j=0}^{N} f(j-Np)\sqrt{\frac{p^j q^{N-j}}{j!\,(N-j)!}} \left(\frac{1+\bar{z}\sqrt{\frac{q}{p}}}{1-\bar{z}\sqrt{\frac{p}{q}}}\right)^{j} {}_2F_1\left(\begin{matrix}-m,-j\\ -N\end{matrix}\;\middle|\; \frac{1+z\bar{z}}{\left|\sqrt{p}+\bar{z}\sqrt{q}\right|^2}\right);$$

  where ${}_2F_1(-m,-j,-N \mid \cdot)$: Gauss hypergeometric function (Kravchuk polynomial). Here, $0<p<1$, $q=1-p$, $N=2(\nu+m)$, $m\in\mathbb{Z}_+$ and $2\nu = 1,2,\cdots$.
- Case $m=0$: $B_{\nu,0}$ is the analytic representation (A. Chenaghlou and O. Faizy, J. Math, Phys. 2007) corresponds to the lowest spherical Landau level LSLL.

Coherent states quantization

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- The (RI) is bridge between the classical and the quantum mechanics.

$$1 \mapsto \mathbf{1}_{\mathcal{H}} = \int_X \mid x >< x \mid 1 \mathcal{N}(x) \, d\mu(x)$$

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- The Klauder-Berezin coherent states quantization consists in associating to a classical observable, that is a function $f(x)$ on X having specific properties, the operator-valued integral

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The function $f(x)$ is called upper (or contravariant) symbol of the operator A_f and is nonunique in general.

Berezin transform

$$f \mapsto A_f := \int_X |\, x >< x \,|\, f(x)\, \mathcal{N}(x)\, d\mu(x)$$

- The expectation value $\langle x \mid A_f \mid x \rangle$ of A_f with respect to the set of coherent states $\{|\, x >\}_{x \in X}$ is called lower (or covariant) symbol of A_f.
- Associating to the classical observable $f(x)$ the obtained mean value $\langle x \mid A_f \mid x \rangle$, we get the Berezin transform of this observable. That is,

$$B[f](x) := \langle x \mid A_f \mid x \rangle\,, \; x \in X.$$

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Some references:

- F. A. Berezin 1972, *Covariant and contravariant symbols of operators*, *Izv. Akad. SSSR Ser. Mat.*
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EXAMPLE 1: THE EUCLIDEAN COMPLEX PLANE $\mathbb{C}$

- $X \equiv \mathbb{C}$,
- The measure: $e^{-|z|^2} d\mu(z)$, $d\mu$ the Lebesgue measure on $\mathbb{C}$
- $\mathcal{A}^2 \equiv \mathfrak{F}(\mathbb{C}) : e^{-|z|^2} d\mu(z)$ $-$square integrable analytic functions on $\mathbb{C}$, Segal-Bargmann-Fock space

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$$|z,0\rangle = (\mathcal{N}(z))^{-\frac{1}{2}} \sum_{p=0}^{+\infty} \frac{\bar{z}^p}{\sqrt{p!}} |\varphi_p\rangle$$

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- To any $\varphi \in L^2(\mathbb{C}, d\mu)$, we associate

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$$\mathbb{E}_{\{|z,0\rangle\}}(A_\varphi) = \langle z,0| A_\varphi |z,0\rangle$$

This gives the well known Berezin transform

$$\mathcal{B}_0^{er}[\varphi](z) = \mathbb{E}_{\{|z,0\rangle\}}(A_\varphi) = \pi^{-1} \int_{\mathbb{C}} e^{-|z-w|^2} \varphi(w) d\mu(w)$$

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- The Berezin transform B_0^{er} can also be expressed as convolution

$$\mathcal{B}_0^{er}[\varphi](z) := \pi^{-1} (e^{-|w|^2} * \varphi)(z), \varphi \in L^2(\mathbb{C}, d\mu)$$

.

$\mathcal{B}_0^{er}$ AS FUNCTION OF THE EUCLIDEAN LAPLACIAN

The Berezin transform B_0^{er} can be written as

$$\mathcal{B}_0^{er} = \exp\left(\frac{1}{4}\Delta_{\mathbb{C}}\right).$$

Ref: J. Peetre 1990, J. Operator Theory

EXAMPLE 2: THE LANDAU PROBLEM ON $\mathbb{C}^n$

- $X = \mathbb{R}^{2n} = \mathbb{C}^n$
- The Hamiltonian of ν magnetic field

$$H_\nu = -\frac{1}{4}\sum_{j=1}^{n}\left(\left(\partial_{x_j} + i\nu y_j\right)^2 + \left(\partial_{y_j} - i\nu x_j\right)^2\right) - \frac{n}{2}$$

  acting on $L^2(\mathbb{R}^{2n}, d\mu)$, $d\mu$ Lebesgue measure.
- Intertwining relation

$$\widetilde{\Delta}_\nu := e^{\frac{1}{2}\nu|z|^2} H_\nu \, e^{-\frac{1}{2}\nu|z|^2}$$

- Ground state transformation

$$Q[\phi](z) := e^{\frac{1}{2}\nu|z|^2}\,\phi(z)\,, \in L^2(\mathbb{C}^n, e^{-\nu|z|^2}d\mu); \;\; z \in \mathbb{C}^n.$$

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- Explicit expression for the operator $\widetilde{\Delta}_\nu$:

$$\widetilde{\Delta}_\nu = -\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j \partial \overline{z}_j} + \nu\sum_{j=1}^{n}\overline{z}_j\frac{\partial}{\partial \overline{z}_j}.$$

- We consider the case of $\nu = 1$, $\widetilde{\Delta} := \widetilde{\Delta}_1$,

$$\widetilde{\Delta} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} + \sum_{j=1}^{n} \overline{z}_j \frac{\partial}{\partial \overline{z}_j}.$$

$C_0^\infty(\mathbb{C}^n)$ as its regular domain in the Hilbert space $L^2(\mathbb{C}^n, e^{-|z|^2} d\mu)$.

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- $\widetilde{\Delta}$ is an unbounded symmetric operator on $C_0^\infty(\mathbb{C}^n)$ which is essentially self-adjoint in the Hilbert space $L^2(\mathbb{C}^n, e^{-|z|^2} d\mu)$..

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$$\epsilon_m := m, \quad m = 0, 1, 2, \ldots.$$

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- The resolvent kernel
- The heat Kernel
- The wave kernel

Indexed by $m = 0, 1, 2, ...$, as eigenspaces of the operator $\widetilde{\Delta}$,

$$\mathcal{A}_m^2(\mathbb{C}^n) := \left\{\varphi \in L^2(\mathbb{C}^n, e^{-|z|^2} d\mu),\ \widetilde{\Delta}\varphi = m\varphi\right\}.$$

A function $f : \mathbb{C}^n \to \mathbb{C}$ belongs to $\mathcal{A}_m^2(\mathbb{C}^n)$ if and only if

$$f(z) = \sum_{p=0}^{+\infty} \sum_{q=0}^{m} {}_1F_1\left(-m+q,\ n+p+q,\ \rho^2\right) \rho^{p+q} \sum_{j=1}^{d(n,p,q)} \gamma_{p,q,j} h_{p,q}^j(\omega)$$

in $C^\infty(\mathbb{C}^n)$, $z = \rho\omega$, $\omega \in S^{2n-1}$, $\rho > 0$, ${}_1F_1$ is the confluent hypergeometric function $\gamma_{p,q} := (\gamma_{p,q,j}) \in \mathbb{C}^{d(n,p,q)}$ satisfy

$$\sum_{p=0}^{+\infty} \sum_{q=0}^{m} (m-q)!\,(p+q+n-1)!\,\Gamma(n+p+q)\ \frac{\left|\gamma_{p,q}\right|^2}{2\Gamma(n+p+m)} < +\infty$$

and $\left(h_{p,q}^j(.)\right)$, $1 \le j \le d(n,p,q)$ is an orthonormal basis of $H(p,q)$. Here $d(n,p,q) = \dim H(p,q)$

THE SPACE $\mathcal{A}_0^2(\mathbb{C}^n)$

In the case $m = 0$ the space $\mathcal{A}_0^2(\mathbb{C}^n)$ coincides with the Segal-Bargmann-Fock space $\mathfrak{F}(\mathbb{C}^n)$ of entire functions in $L^2(\mathbb{C}^n, e^{-|z|^2} d\mu)$.

Ref: N. Askour, A. Intissar & Z. Mouayn, C. R. Acad. Sci. Paris 1997

AN ORTHONORMAL BASIS OF $\mathcal{A}_m^2(\mathbb{C}^n)$

- An orthonormal basis of $\mathcal{A}_m^2(\mathbb{C}^n)$ can be written in terms of the Laguerre polynomials and the spherical harmonics polynomials $h_{p,q}^j(z, \overline{z})$

$$\Phi_{j,p,q}^m(z) := \left(\frac{2(m-q)!}{\Gamma(n+m+p)} \right)^{\frac{1}{2}} L_{m-q}^{(n+p+q-1)}\left(|z|^2\right) h_{p,q}^j(z, \overline{z})$$

where $p = 0, 1, 2, ...; q = 0, 1, 2, ..., m, j = 1, ..., d(n, p, q)$.

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where $p = 0, 1, 2, ...; q = 0, 1, 2, ..., m, j = 1, ..., d(n, p, q)$.

- These basis elements will play the role of coefficients in the following superposition:

A set of coherent states $\{|z,m\rangle\}_{z\in\mathbb{C}^n}$

- For $m = 0, 1, 2, ...,$ a class of generalized coherent states is defined by

$$| z, m >= (\mathcal{N}_m(z))^{-\frac{1}{2}} \sum_{\substack{1\leq j\leq d(n,p,q)\\ 0\leq q\leq m, 0\leq p<+\infty}}^{+\infty} \overline{\Phi^m_{j,p,q}(z)} |\varphi_{j,p,q}\rangle$$

- $\Phi^m_{j,p,q}(z)$: orthonormal basis of $A^2_m(\mathbb{C}^n)$
- $|\varphi_{j,p,q}\rangle$: orthonormal basis of another (functional) Hilbert space $\mathcal{H}$
- $\dim \mathcal{H} = \dim A^2_m(\mathbb{C}^n) = +\infty$
- $\mathcal{N}_m(z)$ is a normalization factor such that $\langle z, m|z, m\rangle_{\mathcal{H}} = 1$:

$$\mathcal{N}_m(z) = \frac{\pi^{-n}\Gamma(n+m)}{\Gamma(m+1)\Gamma(n)} e^{\langle z,z\rangle}$$

- The CS satisfy the resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_{\mathbb{C}^n} |z, m\rangle \langle z, m| \mathcal{N}_m(z)\, d\mu(z)$$

THE OVERLAP INTEGRAL BETWEEN TWO CS IN $\mathbb{C}^n$

This quantity is defined by

$$\langle z, m | w, m \rangle = \left(\mathcal{N}_m(z)\mathcal{N}_m(w)\right)^{-\frac{1}{2}} \sum_{\substack{1\leq j\leq d(n,p,q) \\ 0\leq q\leq m, 0\leq p<+\infty}}^{+\infty} \Phi^m_{j,p,q}(z)\overline{\Phi^m_{j,p,q}(w)}$$

$$= \left(\mathcal{N}_m(z)\mathcal{N}_m(w)\right)^{-\frac{1}{2}} \pi^{-n} e^{\langle z,w\rangle} L_m^{(n-1)}\left(|z-w|^2\right)$$

The overlap integral between two CS in $\mathbb{C}^n$

This quantity is defined by

$$\langle z, m | w, m \rangle = \left(\mathcal{N}_m(z) \mathcal{N}_m(w) \right)^{-\frac{1}{2}} \sum_{\substack{1 \leq j \leq d(n,p,q) \\ 0 \leq q \leq m, 0 \leq p < +\infty}}^{+\infty} \Phi^m_{j,p,q}(z) \overline{\Phi^m_{j,p,q}(w)}$$

$$= \left(\mathcal{N}_m(z) \mathcal{N}_m(w) \right)^{-\frac{1}{2}} \pi^{-n} e^{\langle z,w \rangle} L_m^{(n-1)} \left(|z-w|^2 \right)$$

This can be proved by direct calculations using an addition formula due to T. Koornwinder SIAM. J. Math. Anal. 1977 and Mourad. Ismail 2012 :

$$\exp(ixy \sin\psi) \mathcal{L}_s^{(\sigma)} \left(x^2 + y^2 - 2xy \cos\psi \right)$$

$$= \sum_{k=0}^{+\infty} \sum_{l=0}^{s} \frac{\sigma}{\sigma+k+l} \begin{pmatrix} s \\ l \end{pmatrix} \frac{(\sigma+s+1)}{k!(\sigma+l)_k (\sigma+k)_l}$$

$$\times x^{k+l} y^{k+l} \mathcal{L}_{s-l}^{(\sigma+k+l)} \left(x^2 \right) \mathcal{L}_{s-l}^{(\sigma+k+l)} \left(y^2 \right) R_{k,l}^{\sigma-1} \left(re^{i\psi} \right)$$

where $\mathcal{L}_s^{(\sigma)}(.)$ are Laguerre functions and $R_{k,l}^{\sigma}(.)$ are disk polynomials.

CS quantization and Berezin transform

- To any function $\varphi \in L^2(\mathbb{C}^n, d\mu)$ we associate the operator-valued integral

$$\varphi \mapsto A_\varphi = \int_{\mathbb{C}^n} |z, m\rangle \langle z, m| \, \varphi(z) \, \mathcal{N}_m(z) \, d\mu(z)$$

- Next define the Berezin transform of φ as the expectation value

$$\mathcal{B}_m^{er}[\varphi](z) := \langle z, m| A_\varphi |z, m\rangle$$

- After calculations using the above overlap integral $\langle z, m|w, m\rangle_{\mathcal{H}}$, we arrive at

$$\mathcal{B}_m^{er}[\varphi](z) = \frac{m!}{(n)_m \pi^n} \int_{\mathbb{C}^n} e^{-|z-w|^2} \left(L_m^{(n-1)}\left(|z-w|^2\right) \right)^2 \varphi(w) d\mu(w)$$

where $\varphi \in L^\infty(\mathbb{C}^n)$.

Berezin transform and the Euclidean Laplacian

- This transform can be written via a convolution product

$$\mathcal{B}_m^{er}[\varphi] = h_m * \varphi, \quad \varphi \in L^2(\mathbb{C}^n, d\mu)$$

involving the function

$$h_m(z) = \frac{m!}{(n)_m \pi^n} e^{-|z|^2} \left(L_m^{(n-1)}\left(|z|^2\right)\right)^2$$

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- $h_m \mapsto \widehat{h_m}$: is the Fourier transform.

THEOREM (Z. MOUAYN, J. FOURIER ANAL & APP 2012)

For $m = 0, 1, 2, \ldots$, the Berezin transform can be expressed as a function of the Laplacian $\triangle_{\mathbb{C}^n}$ as

$$\mathcal{B}_m^{er} = e^{\frac{1}{4}\Delta_{\mathbb{C}^n}} \sum_{j=0}^{2m} \gamma_j^{(m,n)} \left(\Delta_{\mathbb{C}^n}\right)^j$$

with coefficients

$$\gamma_j^{(m,n)} := \frac{2^{2m}\,(m!)^3\,(-1)^j\ {}_3F_2\left(\frac{j}{2}-m, \frac{j+1}{2}-m, j+n, j-m+1, j-m+1; 1\right)}{(n)_m\, j!\, 2^{3j}\,(2m-j)!\,\left(\Gamma\left(j-m+1\right)\right)^2}$$

given in terms of a ${}_3F_2$-sum.

- In particular for $m = 0$ we recover $\mathcal{B}_0^{er} = e^{\frac{1}{4}\triangle_{\mathbb{C}}}$.

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Idea on the proof. we use a linearization of the product of Laguerre polynomials and next we calculate some integral involving Bessel functions.

Example 3: The Landau problem on the Bergman ball $\mathbb{B}^n$

- $X = \mathbb{B}^n$
- $d\mu_n = (1 - < z, z >)^{-n-1}\, d\mu$, $d\mu$: Lebesgue measure on $\mathbb{B}^n$.

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- The Schrödinger operator with ν-magnetic field on $\mathbb{B}^n$

$$H_\nu = -4(1 - |z|^2)(\sum_{i,j=1}^{n} (\delta_{ij-} z_i \overline{z_j}) \partial_j \overline{\partial}_j + \nu \sum_{j=1}^{n} (z_j \partial_j - \overline{z_j} \overline{\partial}_j) + \nu^2) + 4\nu^2$$

provided that $\nu > n/2$. The notations $\partial_j = \partial / \partial z_j$ and $\overline{\partial}_j = \partial / \partial \overline{z_j}$

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$$H_\nu = -4(1 - |z|^2)(\sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z_j})\, \partial_j \overline{\partial}_j + \nu \sum_{j=1}^{n} (z_j \partial_j - \overline{z_j} \overline{\partial}_j) + \nu^2) + 4\nu^2$$

provided that $\nu > n/2$. The notations $\partial_j = \partial / \partial z_j$ and $\overline{\partial}_j = \partial / \partial \overline{z_j}$

- H_ν is an elliptic densely defined operator on $L^2(\mathbb{B}^n, (1 - \langle z, z \rangle)^{-(n+1)} d\mu)$ admitting a unique self-adjoint realization also denoted by H_ν.

EXAMPLE 3: THE LANDAU PROBLEM ON THE BERGMAN BALL $\mathbb{B}^n$

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- H_ν is an elliptic densely defined operator on $L^2(\mathbb{B}^n, (1-\langle z, z\rangle)^{-(n+1)} d\mu)$ admitting a unique self-adjoint realization also denoted by H_ν.
- Its spectrum: $\left[n^2, +\infty\right[$ (scattering states) and a finite number of infinitely degenerate eigenvalues (bound states):

$$\epsilon_m^{\nu,n} = 4\nu(2m+n) - 4m(m+n),\ m = 0, 1, ..., [\nu - n/2]\,.$$

called hyperbolic Landau levels on $\mathbb{B}^n$.

The generalized Bergman spaces $\mathcal{A}^{2,\nu}_m(\mathbb{B}^n)$

- Focus on the discrete part of the spectrum $\epsilon^{\nu,n}_m$, and the corresponding eigenspace

$$\mathcal{A}^{2,\nu}_m(\mathbb{B}^n) = \left\{\varphi \in L^2(\mathbb{B}^n, d\mu_n), H_\nu\varphi = \epsilon^{\nu,n}_m\varphi\right\}$$

where $m = 0, 1, ..., [\nu - n/2]$.

- An eigenfunction f of H_ν with eigenvalue $\epsilon^{\nu,n}_m$, in terms of the appropriate Fourier series in $\mathbb{B}^n$:

$$f(z) = \sum_{p,q=0}^{+\infty} \frac{(1-\rho^2)^{\frac{i\lambda+n}{2}}}{\rho^{-p-q}} \, {}_2F_1\left(\frac{i\lambda+n}{2}+\nu+p, \frac{i\lambda+n}{2}-\nu+q, p+q+n; \rho^2\right)$$
$$\times \sum_{j=1}^{d(n,p,q)} a^{\lambda,\nu}_{p,q,j} h^j_{p,q}(\theta),$$

in $C^\infty(\mathbb{B}^n)$, $z = \rho\theta$, $\rho \in [0,1[$ and $\theta \in \partial\mathbb{B}^n$, ${}_2F_1$: Gauss hypergeometric function and $(a^{\lambda,\nu}_{p,q,j}) \in \mathbb{C}^{d(n,p,q)}$. Here $\{h^j_{p,q}\}_{1\leq j\leq d(n,p,q)}$ is an orthonormal basis of $H(p,q)$.

THE SPACE $\mathcal{A}_0^{2,\nu}(\mathbb{B}^n)$

For $m = 0$, the space $\mathcal{A}_0^{2,\nu}(\mathbb{B}^n)$ is isomorphic to the weighted Bergman space of holomorphic function ψ on $\mathbb{B}^n$ satisfying

$$\int_{\mathbb{B}^n} |\psi(z)|^2 ((1 - \langle z, z\rangle)^{2\nu-n-1} d\mu(z) < +\infty. \qquad (*)$$

This fact justify why the eigenspace $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$ has been called a generalized Bergman spaces of index m.

AN ORTONORMAL BASIS OF $\mathcal{A}_m^{2,\nu}(\mathbb{B}^n)$

It can be given explicitly by

$$\Phi_{p,q}^{\nu,m,j}(z) = \kappa_{p,q}^{\nu,m,n}\left(1 - |z|^2\right)^{\nu-m} P_{m-q}^{(n+p+q-1,2[\nu-m]-n)}\left(1 - 2|z|^2\right) h_{p,q}^j(z, \overline{z})$$

with

$$\kappa_{p,q}^{\nu,m,n} = \left(\frac{\pi^n \Gamma(n+m+p)\Gamma(2\nu-n-m-q+1)}{n!2(2[\nu-m]-n)(m-q)!\Gamma(2\nu-m+p)}\right)^{-\frac{1}{2}}$$

$p = 0, 1, 2, ...,$ $q = 0, 1, ..., m$ and $j = 1, ..., d(n; p, q)$.

A SET OF COHERENT STATES $\{|z,\nu,m\rangle\}_{z\in\mathbb{B}^n}$

- $X = \mathbb{B}^n$, $x \equiv z \in \mathbb{B}^n$. $d\mu_n$,

A set of coherent states $\{|z, \nu, m\rangle\}_{z \in \mathbb{B}^n}$

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$$| z,\nu,m >:= (\mathcal{N}(z))^{-\frac{1}{2}} \sum_{\substack{0\leq q\leq m, 0\leq p<+\infty \\ 1\leq j\leq d(n,p,q)}} \overline{\Phi_{p,q,j}^{\nu,m}(z)}\varphi_{p,q,j}$$

- The normalization factor

$$\mathcal{N}_m(z)=\frac{(2(\nu-m)-n)\Gamma(2\nu-m)}{\pi^n\Gamma(2\nu-m-n+1)}\frac{\Gamma(m+n)}{m!\Gamma(n)}$$

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- The CS satisfy the resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_{\mathbb{B}^n} |z,\nu,m\rangle\langle z,\nu,m| \mathcal{N}_m(z)\, d\mu_n(z)$$

The overlap integral between two CS in $\mathbb{B}^n$

This quantity is defined by

$$\langle z,\nu,m|w,\nu,m\rangle = \left(\mathcal{N}(z)\mathcal{N}(w)\right)^{-\frac{1}{2}} \sum_{\substack{0\le q\le m, 0\le p<+\infty \\ 1\le j\le d(n,p,q)}} \Phi_{p,q,j}^{\nu,m}(z)\,\overline{\Phi_{p,q,j}^{\nu,m}(w)}$$

$$= \frac{(2[\nu-m]-n)\,\Gamma(2\nu-m)}{\pi^n\Gamma(2\nu-m-n+1)}\left(\mathcal{N}(z)\mathcal{N}(w)\right)^{-\frac{1}{2}}\left(\frac{1-\overline{\langle z,w\rangle}}{1-\langle z,w\rangle}\right)^{\nu}$$
$$\times\left(\cosh\left(d(z,w)\right)\right)^{-2(\nu-m)} P_m^{(n-1,2[\nu-m]-n)}\left(1-2\tanh^2\left(d(z,w)\right)\right)$$

$P_m^{(\alpha,\beta)}(.)$ denotes Jacobi polynomial.

CS quantization and Berezin transform

- For any $\varphi \in L^2(\mathbb{B}^n, d\mu_n)$, the operator-valued integral

$$\varphi \mapsto A_\varphi := \int_{\mathbb{B}^n} |z, \nu, m\rangle \langle z, \nu, m| \, \varphi(z) \, \mathcal{N}(z) \, d\mu_n(z)$$

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The function $\varphi(z)$ is a upper symbol of the operator A_φ.

- The Berezin transform is defined as the expectation value

$$\mathcal{B}_m^{er}[\varphi](z) \equiv \mathbb{E}_{\{|z,\nu,m\rangle\}}(A_\varphi) = \langle z,\nu,m| \, A_\varphi \, |z,\nu,m\rangle \quad (\textit{lower symbol of } A_\varphi).$$

- Using the overlap integral between two CS, we obtain:

$$\begin{aligned}\mathcal{B}_m^{er}[\varphi](z) &= \tau_{\nu,m,n} \int_{\mathbb{B}^n} \left(\frac{(1-|z|^2(1-|\xi|^2)}{|1-\langle z,\xi\rangle|^2} \right)^{2(\nu-m)} \\ &\quad \times \left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2|\xi|^2\right)\right)^2 \varphi(\xi) \, d\mu_n(\xi).\end{aligned}$$

$$\tau_{\nu,m,n} = \frac{\Gamma(n) \, m! \, (2(\nu-m)-n) \, \Gamma(2\nu-m)}{\pi^n \Gamma(n+m) \, \Gamma(2\nu-m-n+1)}$$

The Berezin transform $\mathcal{B}_0^{er}$

- For $m=0$, this transfrom is the well known Berezin transfom attached to the weighted Bergman space $\mathcal{A}_0^{2,\nu}(\mathbb{B}^n)$ of holomorphic function ψ on $\mathbb{B}^n$ satisfying the growth condition $(*)$ and given by

$$\mathcal{B}_0^{er}[\varphi](z)=\frac{(2\nu-n)\Gamma(2\nu)}{\pi^n\Gamma(2\nu-n+1)}\int_{\mathbb{B}}(\cosh d(z,\xi))^{-4\nu}\frac{\varphi(\xi)}{\left(1-|\xi|^2\right)^{n+1}}d\mu(\xi)$$

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- This have been written as a function of the Bergman Laplacian $\Delta_{\mathbb{B}^n}$ as

$$\mathcal{B}_0^{er} = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+n+1)} \left| \Gamma\left(\alpha + 1 + \frac{n}{2} + \frac{i}{2}\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right)\right|^2$$

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- The above form, involving gamma factors, was derived by Peetre so that α there occurring in the weight of the Bergman space, corresponds to $2\nu - n - 1$.

THE BEREZIN TRANSFORMS AS FUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR

THEOREM — A. GHANMI & Z. MOUAYN HOUSTON. J. MATH 2014

The transform $\mathcal{B}_m^{er}$ can be expressed as a function of the Laplace-Beltrami operator $\Delta_{\mathbb{B}^n}$ in terms of a ${}_3F_2$-sum as

$$\mathcal{B}_m^{er} = \sum_{j=0}^{2m} C_j^{\nu,n,m} \frac{\Gamma\left(2(\nu-m) - \frac{1}{2}\left(n - i\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right)\right)}{\Gamma\left(2(\nu-m) + j + \frac{1}{2}\left(n + i\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right)\right)}$$

$$\times {}_3F_2\left[\begin{matrix} \frac{1}{2}\left(n + i\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right), n+j, \frac{1}{2}\left(n + i\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right) \\ (\nu-m) + j + \frac{1}{2}\left(n + i\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right), n \end{matrix} \;\middle|\; 1\right]$$

where

$$C_j^{\nu,n,m} = \frac{(2(\nu-m)-n)\Gamma(n+m)(-1)^j\Gamma(n+j)}{m!\Gamma(2\nu-n-m+1)\Gamma(2\nu-n)}$$

$$\times \sum_{p=\max(0,j-m)}^{\min(m,j)} \frac{(m!)^2\Gamma(2\nu-m)\Gamma(2\nu-m+j-p)}{(j-p)!(m+p-j)!p!(m-p)!\Gamma(n+j-p)\Gamma(n+p)}$$

On the proof

- Consider the non-negative elliptic self-adjoint operator $-L_n := -\Delta_{\mathbb{B}^n} - n^2$. Then, for given suitable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the operator $f(-L_n)$ is defined by

$$f(-\Delta_{\mathbb{B}^n} - n^2)[\varphi](z) = \int_{\mathbb{B}^n} \left(\int_0^{+\infty} \Psi(z, w; \lambda) f(\lambda) d\lambda \right) \frac{\varphi(w)}{(1 - |w|^2)^{n+1}} d\mu(w),$$

where the spectral kernel is given by

$$\Psi(z, w; \lambda) = \frac{|\Gamma(\frac{n+i\lambda}{2})|^4}{4\pi^{n+1}\Gamma(n)|\Gamma(i\lambda)|^2} \, {}_2F_1 \left[\begin{array}{c} \frac{n+i\lambda}{2}, \frac{n-i\lambda}{2} \\ n \end{array} \Big| - \sinh^2(d(z, w)) \right]$$

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- By equating the previous integral representation to

$$\mathcal{B}_m \left[\varphi\right](z) = \int_{\mathbb{B}^n} B_m(z, w) \frac{\varphi(w)}{(1 - |w|^2)^{n+1}} d\mu(w)$$

we get

$$\int_0^{+\infty} \Psi(z, w; \lambda) f(\lambda) d\lambda = B_m(z, w).$$

On the proof

- To determine the function f we use the Fourier-Jacobi transform

$$h \in L^2\left(\mathbb{R}_+, \Delta_{\alpha,\beta}(t)dt\right) \longmapsto g \in L^2\left(\mathbb{R}_+, (2\pi)^{-1}\left|c_{\alpha,\beta}(\lambda)\right|^{-2} d\lambda\right)$$

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$$g(\lambda) = \int_0^{+\infty} h(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha.\beta}(t)dt.$$

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  1. Explicit expressions of the different quantities

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  3. A special formula of the product of two Jacobi polynomials
  4. A special integral representation for the ${}_2F_1$-sum,

- we arrive at

$$f(\lambda) = \frac{(2(\nu - m) - n)\Gamma(n+m)}{m!\Gamma(2\nu - (n+m) + 1)\Gamma(2\nu - m)} \sum_{j=0}^{2m} (-2)^j A_j$$
$$\times \mathcal{B}\left(n+j, \frac{i\lambda - n}{2} + 2(\nu - m)\right) {}_3F_2\left[\begin{array}{c} \frac{n+i\lambda}{2}, n+j, \frac{n+i\lambda}{2} \\ \frac{n+i\lambda}{2} + 2(\nu - m) + j, n \end{array} \Big| 1\right].$$

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$$\times \mathcal{B}\left(n+j, \frac{i\lambda - n}{2} + 2(\nu - m)\right) {}_3F_2\left[\begin{array}{c} \frac{n+i\lambda}{2}, n+j, \frac{n+i\lambda}{2} \\ \frac{n+i\lambda}{2} + 2(\nu - m) + j, n \end{array} \middle| 1\right].$$

- Replacing λ by $\sqrt{-\Delta_{\mathbb{B}^n} - n^2}$ and expressing the Beta function in terms of Gamma functions we obtain the announced formula.

THEOREM A. BOUSSEJRA AND Z. MOUAYN (TO APPEAR IN MOSCOW J. MATH)

The Berezin transform $\mathcal{B}_m^{er}$ can be expressed as a function of the Laplace-Beltrami operator $\Delta_{\mathbb{B}^n}$ as

$$\mathcal{B}_m^{er} = \left|\Gamma\left(2(\nu - m) - \frac{n}{2} + \frac{i}{2}\sqrt{-\Delta_{\mathbb{B}^n} - n^2}\right)\right|^2 \times \sum_{k=0}^{2m} \gamma_k^{\nu,n,m} W_k\left(\frac{-1}{4}\Delta_{\mathbb{B}^n} - \frac{n^2}{4}; 2(\nu - m) - \frac{n}{2}, 1 + \frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$$

where $W_k(.)$ are Wilson polynomials,

$$\gamma_k^{\nu,n,m} = \frac{2n!m!\Gamma(n)(2(\nu - m) - n)\Gamma(2\nu - m)(-1)^k \times A_k^{\nu,n,m}}{\pi^n\Gamma(n+m)\Gamma(2\nu - m - n + 1)k!\Gamma^2(2(\nu - m) + k)}$$

and $A_k^{\nu,n,m}$ are some coefficients.

On the proof

- Making appeal to an automorphism $g_z \in Aut(\mathbb{B}^n)$ such that $g_z.0 = z \in \mathbb{B}^n$ and expressing the distances occurring in the formula

$$\cosh^{-2} d(z,\xi) = 1 - \left|g_z^{-1}.\xi\right|^2, \tanh^2 d(z,\xi) = \left|g_z^{-1}.\xi\right|^2.$$

we get the integral transform

$$\mathcal{B}_m^{er}[\varphi](z) = c_m^{\nu,n} \int_{\mathbb{B}^n} \frac{\left(P_m^{(n-1,2(\nu-m)-n)}\left(1 - 2\left|g_z^{-1}.\xi\right|^2\right)\right)^2}{\left(1 - \left|g_z^{-1}.\xi\right|^2\right)^{-2(\nu-m)}} \varphi(\xi) \frac{d\mu(\xi)}{\left(1 - |\xi|^2\right)^{n+1}}$$

which can be viewed as "convolution product" of the φ with the radial function

$$h_m^{\nu,n}(\xi) := \left(1 - |\xi|^2\right)^{2(\nu-m)} \left(P_m^{(n-1,2(\nu-m)-n)}\left(1 - 2|\xi|^2\right)\right)^2, \xi \in \mathbb{B}^n.$$

- We have to calculate the Fourier-Helgason transform of $h_m^{\nu,n}(\xi)$, which reads

$$\mathfrak{F}^H\left[h_m^{\nu,n}\right](\lambda,\omega) := \int_{\mathbb{B}^n} h_m^{\nu,n}(\xi)\,\overline{P_\lambda(\xi,\omega)}\frac{d\mu(\xi)}{\left(1-|\xi|^2\right)^{n+1}}, \quad (\lambda,\omega)\in\mathbb{R}\times\partial\mathbb{B}^n,$$

where $P_\lambda(.,.)$ is the Poisson kernel

$$P_\lambda(\xi,\omega) = \left(\frac{1-|\xi|^2}{|1-\langle\xi,\omega\rangle|^2}\right)^{(n+i\lambda)/2}, (\xi,\omega)\in\mathbb{B}^n\times\partial\mathbb{B}^n$$

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- Denote by $d\sigma$ the Lebesgue (surface) measure on $\partial\mathbb{B}^n$ and set $\xi=\rho\theta$ with $0<\rho<1$,$\theta\in\partial\mathbb{B}^n$, and use polar coordinates

$$\int_{\mathbb{B}^n}\Phi(\xi)\,d\mu(\xi) = 2n\int_0^1 \rho^{2n-1}d\rho\int_{\partial\mathbb{B}^n}\Phi(\rho\theta)\,d\sigma(\theta).$$

- Then, the integral $\mathfrak{F}^{H}\left[h_{m}^{\nu,n}\right](\lambda)$ takes the form

$$\mathfrak{F}^{H}\left[h_{m}^{\nu,n}\right](\lambda)=2n\int_{0}^{1}\frac{\rho^{2n-1}\overline{\mathcal{S}_{\lambda,\omega}^{n}(\rho)}}{\left(1-\rho^{2}\right)^{n+1-2(\nu-m)}}\left(P_{m}^{(n-1,2(\nu-m)-n)}\left(1-2\rho^{2}\right)\right)^{2}d\rho$$

- Then, the integral $\mathfrak{F}^H\left[h_m^{\nu,n}\right](\lambda)$ takes the form

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- After calculations using many transformations, we get

$$\mathfrak{F}^H\left[h_m^{\nu,n}\right](\lambda,\omega)=2n\int_0^1\frac{\rho^{2n-1}}{(1-\rho^2)^{n+1-2(\nu-m)}}\left(P_m^{(n-1,2(\nu-m)-n)}(1-2\rho^2)\right)^2$$

$$\times{}_2F_1\left(\frac{n+i\lambda}{2},\frac{n-i\lambda}{2},n;\frac{\rho^2}{\rho^2-1}\right)d\rho$$

Linearization of the square of Jacobi polynomial as a Clebsh-Gordon type formula (Chaggara, H. and Koepf, W , *App. Math. Lett.* 2010):

$$P_s^{(\kappa,\epsilon)}(u)\, P_l^{(\tau,\eta)}(u) = \sum_{k=0}^{s+l} A_{s,l}(k)\, P_k^{(\alpha,\delta)}(u)$$

In our setting, the linearization coefficients $A_{s,l}(k)$ are of the form

$$A_k^{\nu,n,m} = \frac{(2(\nu-m)+n)_k\,(n)_{2m}\,(2k+2(\nu-m)+n)\,(-1)^k\,(2m)!\,\left((2(\nu-m))_{2m}\right)^2}{(n)_k\,(2(\nu-m)+n)_{2m+k+1}\,(m!)^2\,(2m-k)!\,\left((2(\nu-m))_m\right)^2}$$

$$\times F_{2:1}^{2:2}\left(\begin{array}{c} -2m+k, -2\nu-k-n : -m, -n-m+1; -m, -m-n+1 \\ -2m, -2m-n+1 : 1-2\nu, 1-2\nu \end{array}\middle| 1, 1\right)$$

Here $F_{l:l'}^{p:p'}(.)$ denotes the Kampé de Fériet double hypergeometric function :

$$F_{l:l'}^{p:p'}\left(\begin{array}{c} (a_p) : (b_{p'}), (c_{p'}) \\ (d_l) : (\kappa_{l'}), (\varrho_{l'}) \end{array}\middle| x, y\right) = \sum_{q,s=0}^{+\infty} \frac{[a_p]_{q+s}\,[b_{p'}]_q\,[c_{p'}]_s}{[d_l]_{q+s}\,[\kappa_{l'}]_q\,[\varrho_{l'}]_s}\,\frac{x^q}{q!}\,\frac{y^s}{s!}$$

where $[a_p]_s = \prod_{j=1}^p (a_j)_s$ in which $(x)_s = x(x+1)\ldots(x+s-1)$ is the Pochhammer symobol.

Inserting

$$\left(P_m^{(n-1,2(\nu-m)-n)}\left(1-2\rho^2\right)\right)^2=\sum_{k=0}^{2m}A_k^{\nu,n,m}P_k^{(n-1,2(\nu-m))}\left(1-2\rho^2\right)$$

into

$$\mathfrak{F}^H\left[h_m^{\nu,n}\right](\lambda)=\sum_{k=0}^{2m}A_k^{\nu,n,m}\mathfrak{I}_k^{\nu,n,m}(\lambda)$$

where the last term in this sum is

$$\begin{aligned}\mathfrak{I}_k^{\nu,m}(\lambda) &= \int_0^1\frac{2n\rho^{2n-1}}{\left(1-\rho^2\right)^{n+1-2(\nu-m)}}P_k^{(n-1,2(\nu-m))}\left(1-2\rho^2\right)\\ &\times{}_2F_1\left(\frac{1}{2}(n+i\lambda),\frac{1}{2}(n-i\lambda),n;\frac{\rho^2}{\rho^2-1}\right)d\rho.\end{aligned}$$

By the change of variable $\rho=\tanh t$,

$$\begin{aligned}\mathfrak{I}_k^{\nu,m}(\lambda) &= \int_0^{+\infty}2n(\sinh t)^{2n-1}P_k^{(n-1,2(\nu-m))}\left(1-2\tanh^2 t\right)\\ &\times(\cosh t)^{-4(\nu-m)+1}\;.{}_2F_1\left(\frac{n+i\lambda}{2},\frac{n-i\lambda}{2},n;-\sinh^2 t\right)dt.\end{aligned}$$

Using the result established (Koorwinder, Lecture Notes in Math. 1985):

$$\int_0^{+\infty} (\cosh t)^{-\alpha+\beta-\delta-\mu'-1} (\sinh t)^{2\alpha+1} P_k^{(\alpha,\delta)}\left(1-2\tanh^2 t\right)$$

$$\times {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}, \alpha+1; -\sinh^2 t\right) dt$$

$$= \frac{\Gamma(\alpha+1)(-1)^k \Gamma\left(\frac{1}{2}(\delta+\mu'+1+i\lambda)\right)\Gamma\left(\frac{1}{2}(\delta+\mu'+1-i\lambda)\right)}{k!\Gamma\left(\frac{1}{2}(\alpha+\beta+\delta+\mu'+2)+k\right)\Gamma\left(\frac{1}{2}(\alpha-\beta+\delta+\mu'+2)+k\right)}$$

$$\times W_k\left(\frac{1}{4}\lambda^2; \frac{1}{2}(\delta+\mu'+1), \frac{1}{2}(\delta-\mu'+1), \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha-\beta+1)\right)$$

where $\beta, \delta, \lambda \in \mathbb{R}$, $\alpha, \delta > -1$, $\delta + \Re(\mu)' > -1$ and $W_k(.)$ is the Wilson polynomial given in terms of the ${}_4F_3$-sum as :

$$W_k\left(x^2, a, b, c, d\right) := (a+b)_k (a+c)_k (a+d)_k$$

$$\times {}_4F_3\left(\begin{matrix} -k, k+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \middle| 1\right)$$

For the special values $\alpha = n-1, \delta = 2(\nu - m) - n, \beta = 0$ and $\mu' = 2(\nu - m) - n - 1$, we find that

$$\mathfrak{I}_k^{\nu,m}(\lambda) = \frac{2n\Gamma(n)(-1)^k}{k!\Gamma^2(2(\nu-m)+k)} \left|\Gamma\left(2(\nu-m) - \frac{n}{2} + i\frac{\lambda}{2}\right)\right|^2$$

$$\times W_k\left(\frac{1}{4}\lambda^2; 2(\nu-m) - \frac{n}{2}, 1 + \frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$$

Finally, replacing λ by $\sqrt{-\Delta_{\mathbb{B}^n} - n^2}$, we arrive at the announced result.

Example 4: The Landau problem on the complex projective space $\mathbb{C}\mathbb{P}^n$

- $X = \mathbb{C}\mathbb{P}^n$
- $d\mu_n = (1 + < z, z >)^{-n-1} \, d\mu$, $d\mu$: Lebesgue measure on $\mathbb{C}^n$.

EXAMPLE 4: THE LANDAU PROBLEM ON THE COMPLEX PROJECTIVE SPACE $\mathbb{CP}^n$

- $X = \mathbb{CP}^n$
- $d\mu_n = (1+ < z, z >)^{-n-1} \, d\mu$, $d\mu$: Lebesgue measure on $\mathbb{C}^n$.
- The Schrödinger operator with ν-magnetic field on $\mathbb{CP}^n$

$$H_\nu = 4(1 + |z|^2)\Big(\sum_{i,j=1}^{n} (\delta_{ij} + z_i\overline{z_j})\,\partial_j\overline{\partial}_j + \nu \sum_{j=1}^{n}(z_j\partial_j - \overline{z_j}\overline{\partial}_j) - \nu^2\Big) + 4\nu^2$$

provided that $2\nu \in \mathbb{Z}^+$. The notations $\partial_j = \partial/\partial z_j$ and $\overline{\partial}_j = \partial/\partial\overline{z_j}$.

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- The associated discrete spectrum

$$\epsilon_m^{\nu,n} = -4\nu(m+\nu)(m+\nu+n) + 4\nu^2, \; m = 0, 1, 2, ..., \infty$$

called spherical Landau levels on $\mathbb{CP}^n$.

THE EIGENSPACES $\mathcal{A}_m^{\nu}(\mathbb{CP}^n)$

- The corresponding eigenspaces

$$A_m^{\nu}(\mathbb{CP}^n) = \{F : \mathbb{C}^n \to \mathbb{C},\ \Delta_\nu F = \epsilon_m^{\nu,n} F \text{ and } \sup_{\rho>0} \int_{S^{2n-1}} |F(\rho z)|^2 d\mu < +\infty\}$$

- Any function $F(z)$ in $\mathcal{A}_m^{\nu}(\mathbb{CP}^n)$ can be written in the form

$$F(z) = \left(1+|z|^2\right)^{-(m+\nu)} \sum_{0\leq p\leq m; 0\leq q\leq m+2\nu} {}_2F_1\left(p-m, q-m-2\nu, n+p+q; -|z|^2\right) h_{p,q}(z$$

${}_2F_1$ is the Gauss hypergeometric function and

$$\lim_{r\to\infty} F(r\omega) = \sum_{0\leq p\leq m} (-1)^{m-p} \frac{\Gamma(m-p+1)\Gamma(n+2p+2\nu)}{\Gamma(m+n+p+2\nu)} h_{p,p+2\nu}(\omega,\overline{\omega})$$

for $z = z\omega,\ r > 0, \omega \in \mathbb{S}^{2n-1}$ and $h_{p,q}(z,\bar{z}) \in \mathcal{H}(p,q)$.

- The dimension of $\mathcal{A}_m^{\nu}(\mathbb{CP}^n)$ is

$$dim(\mathcal{A}_m^{\nu}(\mathbb{CP}^n)) := (2m+n+2\nu)\frac{\Gamma(m+n)\Gamma(m+n+2\nu)}{n\Gamma^2(n)\Gamma(m+1)\Gamma(m+2\nu+1)}$$

A SET OF COHERENT STATES $\{|z,m\rangle\}_{z\in\mathbb{C}^n}$

- For $m = 0, 1, 2, ...,$ a class of generalized coherent states is defined by

$$| z, m >= (\mathcal{N}_m(z))^{-\frac{1}{2}} \sum_{\substack{1\le j\le d(n,p,q)\\ 0\le q\le m+2\nu, 0\le p<m}} \overline{\Phi^m_{j,p,q}(z)}\, |\varphi_{j,p,q}\rangle$$

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- $\mathcal{N}_m(z)$ is a normalization factor such that $\langle z,m|z,m\rangle_{\mathcal{H}} = 1$:

$$\mathcal{N}_m(z) = \frac{2(2m+2\nu+n)\Gamma(m+n+2\nu)}{Vol(\mathbb{S}^{2n-1})\Gamma(m+2\nu+1)} \frac{\Gamma(m+n)}{\Gamma(m+1)(\Gamma(n))^2}$$

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- The CS satisfy the resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_{\mathbb{C}^n} |z, m\rangle \langle z, m| \, \mathcal{N}_m(z) \, d\mu_n(z)$$

THE OVERLAP INTEGRAL BETWEEN TWO CS IN $\mathbb{C}^n$

This quantity is defined by

$$\langle z, m | w, m \rangle = \left(\mathcal{N}_m(z)\mathcal{N}_m(w)\right)^{-\frac{1}{2}} \sum_{\substack{1 \le j \le d(n,p,q) \\ 0 \le q \le m+2\nu, 0 \le p < m}} \Phi^m_{j,p,q}(z)\,\overline{\Phi^m_{j,p,q}(w)}$$

$$= \frac{2(2m+2\nu+n)\Gamma(m+n+2\nu)}{Vol(\mathbb{S}^{2n-1})\Gamma(n)\Gamma(m+2\nu+1)} \left[\frac{|1+\langle z|w\rangle|^2}{(1+|z|^2)(1+|w|^2)}\right]^{\nu} P_n^{(n-1,2\nu)}(\cos 2d(z,w)),$$

where $d(z,\omega)$ is the Fubini-Study distance given by

$$\cos^2 d(z,w) = \frac{|1+\langle z,w\rangle|^2}{(1+|z|^2)(1+|w|^2)}$$

CS quantization and Berezin transform

- For any $\varphi \in L^2(\mathbb{C}^n, d\mu_n)$, the operator-valued integral

$$\varphi \mapsto A_\varphi := \int_{\mathbb{C}^n} |z, \nu, m\rangle \langle z, \nu, m| \, \varphi(z) \, \mathcal{N}_m(z) \, d\mu_n(z)$$

The function $\varphi(z)$ is a upper symbol of the operator A_φ.

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The function $\varphi(z)$ is a upper symbol of the operator A_φ.

- The Berezin transform is defined as the expectation value

$$\mathcal{B}_m^{er}[\varphi](z) \equiv \mathbb{E}_{\{|z,\nu,m\rangle\}}(A_\varphi) = \langle z,\nu,m| A_\varphi |z,\nu,m\rangle \quad (\textit{lower symbol of } A_\varphi).$$

- Using the overlap integral between two CS, we obtain:

$$\mathcal{B}_m[\varphi](z) = c_m^{\nu,n} \int_{\mathbb{C}^n} \left(\frac{|1+ <z,w>|^2}{(1+|z|^2)(1+|w|^2)} \right)^{2\nu} \left(P_m^{(n-1,2\nu)}(\cos 2d(z,w)) \right)^2 \varphi(w) d\mu_n(w)$$

with

$$c_m^{\nu,n} = \frac{2(m+2\nu+n)(m+2\nu+n-1)}{\pi^n (m+2\nu)!}$$

THE BEREZIN TRANSFORMS AS FUNCTIONS OF $\triangle_{FS}$

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$$f(\lambda_k) = 4\pi(n)_{2B}\frac{(2B+m)!}{(m!(2B)!)^2}(m+2B+n)(m+2B+n-1)$$

$$\frac{(-2B)_k k!}{(n)_k(n+k)_{2B}(n+k-1)!}\sum_{j=0}^{m}\frac{(-m)_j(2B+m+n)_j(n)_j}{j!(2B-k+1)_j(2B+k+1+n)_j}$$

$${}_4F_3\left(\begin{matrix}-m\ ,\ 2B+m+n\ ,\ 2B+1+j\ ,\ n+j\\ 2B-k+1+j\ ,2B+k+1+n+j,2B+1\end{matrix}\middle|\,1\right)$$

On the proof

- The Berezin kernel

$$B_m(z,w) = c_m^{\nu,n}\left(\cos^2 d(z,w)\right)^{2\nu}\left(P_m^{(n-1,2\nu)}(\cos^2 d(z,w))\right)^2$$

- The kernel of $f(-\triangle_{FS})$:

$$K(z,w) = \sum_{k=0}^{+\infty} f(\lambda_k)\psi_n(k;z,w),\ \lambda_k = k(k+n)\ \text{ eigenvalues of } -\triangle_{FS}$$

- The spectral function

$$\psi_n(k;z,w) = \frac{d(k,n+1)}{Vol(\mathbb{S}^{2n+1})}\frac{P_k^{(n-1,0)}\left(2|\langle z,w\rangle|^2-1\right)}{P_k^{(n-1,0)}(1)}$$

- f : the unknown function?
- By equating $B_m(z,w) = K(z,w)$

- Next, we identify with the formula

$$t^{\mu} P_{m_1}^{(\alpha_1,\beta_1)}(1-2x_1 t) P_{m_2}^{(\alpha_2,\beta_2)}(1-2x_2 t)$$

$$= (\alpha+1)_{\mu} \binom{\alpha_1+m_1}{m_1} \binom{\alpha_2+m_2}{m_2} \sum_{k=0}^{+\infty} \frac{(\alpha+\beta+2k+1)(-\mu)_k}{(\alpha+1)_k(\alpha+\beta+k+1)_{\mu+1}} P_k^{(\alpha,\beta)}(1-2t)$$

$$F_{2:1,1}^{2:2,2}\left[\begin{matrix} \mu+1,\ \alpha+1 & : -m_1\ ,\alpha_1+\beta_1+m_1+1\ , -m_2\ ,\alpha_2+\beta_2+m_2+1 \\ \mu-k+1\ , \quad \alpha+\beta+\mu+2+k & : \quad \alpha_1+1\ , \quad \alpha_2+1 \end{matrix}\ \middle|\ x_1, x_2\right]$$

References:

- Srivastava, H. M. (Oct. 1987). *Some Clebsch-Gordan type linearization relations and other polynomial expansions associated with a class of generalized multiple hypergeometric series arising in physical and quantum chemical applications*.

- After isolating the $\psi_n(k, z, w)$ part in this formla and calculations, we arrive at $f(\lambda_k)$

THE BEREZIN TRANSFORM $\mathcal{B}_0^{er}$ ON $\mathbb{CP}^1$

- A set of coherent states

$$\phi_{z,\nu,m}(\xi) = \sqrt{\frac{(2\nu+2m)!}{(2\nu+m)!m!}} \left(\frac{(\xi-\bar{z})(1+z\xi)}{1+z\bar{z}} \right)^m \left(\frac{(1+z\xi)^2}{1+z\bar{z}} \right)^\nu$$

- For lowest Landau level $m = 0$

$$\phi_{z,\nu,0}(\xi) = \left(\frac{(1+z\xi)^2}{1+z\bar{z}} \right)^\nu \; in\ L^2(\mathbb{C}, (1+z\bar{z})^{-2\nu} d\mu_1(z))$$

- Use $\phi \mapsto (1+z\bar{e})^{-\nu}\phi$ then $\tilde{\phi}_{z,\nu,0}(\xi) = (1+z\zeta)^{2\nu}$ *in* $L^2(\mathbb{C}, d\mu_1(z))$
- The Laplace-Beltrami operator $\Delta := -(1+z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}$

$$\mathfrak{B}_0^{1/\hbar} = \prod_{n=1}^{+\infty} \left(1 + \nu^{-2} \frac{\Delta}{(1+n\nu^{-1})\,(1+(n+1)\,\nu^{-1})} \right)$$

here $1/\hbar = 2\nu$.

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Special Thanks